Dynamics of Continuous, Discrete and Impulsive Systems **2 (1996)** 397-409

MEASURABLE AND DIRECTIONALLY CONTINUOUS SELECTIONS FOR THE CONTROL OF UNCERTAIN SYSTEMS [1]

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Abstract. We use different results on the existence of selections of multi-valued maps to control a dynamical system affected by deterministic uncertainty. This control system is described by a system of differential inclusions.

AMS subject classification: 93B52, 93C15

Key words: nonlinear uncertain systems, tracking, feedback controls.

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[1] Research supported, in part, by the C.N.R. and by the National project M.U.R.S.T. (40%) "Teoria del controllo dei sistemi dinamici".

1. Introduction

We consider a differential inclusion of the form

$$\dot{x} \in F(t, x, u) \qquad t \in [0, +\infty], u \in U \subset \mathbb{R}^m, \ x \in \mathbb{R}^n, \tag{1}$$

which models a nonlinear controlled dynamics affected by a deterministic uncertainty. The multivalued map F has closed and convex values, it is t-measurable and (x, u)-continuous in the Hausdorff metric, U is a compact set. A system dynamics f of F is a function t-measurable and (x, u)-continuous (i.e. f is a Carathéodory map) such that $f(t, x, u) \in F(t, x, u)$ for almost all (a.a.) $t \in [0, +\infty)$ and any $(x, u) \in \mathbb{R}^n \times U$.

Together with (1) we consider a reference model for (1) which is represented by a nonlinear control system of the form

$$\dot{y} = g(t, y, v) \qquad t \in [0, +\infty), v \in V \subset \mathbb{R}^l, \ y \in \mathbb{R}^n, \tag{2}$$

where g is a Carathéodory function and V is a compact set.

The initial state $x(0) = x_0$ of (1) is also uncertain, but bounded, namely $x_0 \in B(0, r)$ for some $r > 0$. The initial states $y(0) = y_0$ for the reference model (2) are taken in a ball $B(0, \rho) \subset \mathbb{R}^n$.

Throughout this paper we assume that both F and g satisfy conditions ensuring that solutions extend to $[0, +\infty)$.

Fix any state-control pair $(y(t), v(t))$, $t \in [0, +\infty)$, of the model. The aim of this paper is to solve the following problem.

- Given $\alpha \in \mathbb{R}_+$, to determine a state feedback control $u = u(t, x)$, not necessarily continuous in the state x, in such a way that for any system dynamics $f(t, x, u(t, x)) \in F(t, x, u(t, x))$ and for any initial condition $x_0 \in B(0, r)$ the corresponding solutions $x = x(t)$ of the Krasowskiĭ's regularization

$$\begin{cases} \dot{x} \in \mathcal{K}(\varphi)(t, x(t)), & t \in [0, +\infty) \\ x(0) = x_0, \end{cases} \tag{3}$$

where $\varphi(t, x) := f(t, x, u(t, x))$ and $\mathcal{K}(\varphi)(t, x) = \bigcap_{\delta > 0} \overline{\text{co}}\, \varphi(t, B(x, \delta))$ satisfy the inequality

$$|y(t) - x(t)| \leq \beta e^{-\alpha t}, \quad t \in [0, +\infty), \quad \text{for some } \beta > 0. \tag{4}$$

Here $B(x, \delta) = \{z \in \mathbb{R}^n : |z - x| < \delta\}$.

The approach we use to solve this problem is similar to that employed in ([8] and the references therein) to obtain the asymptotic linearization of a nonlinear uncertain system. This approach is based on the theory of variable structure systems and the use of a suitable (non differentiable) Liapunov function.

In Section 2 we will use a selection theorem due to Bressan ([3], Theorem 1) to prove the existence of a directionally continuous selection of a multivalued map $U(t,x)$, called the regulation map, suitably defined by means of the dynamics F and g. Such a selection is sufficiently smooth to allow us to solve the proposed problem under the assumption that F is bounded in a neighborhood of the reference trajectory (Theorem 1). In [8] we used the Michael's selection theorem to prove the existence of a continuous selection from U, this requires more restrictive conditions on the uncertain dynamics F.

In fact, using a directionally continuous selection $u(t,x)$ it is possible to prove (see [4]) that the solution set of the Cauchy problem

$$\begin{cases} \dot{x} = f(t,x,u(t,x)) \\ x(0) = x_0 \end{cases} \tag{5}$$

is the same of that of the differential inclusion (3) for any selection of system dynamics f from (1). By a solution of (5) we mean an absolutely continuous function $x(t)$ defined in $[0,+\infty)$ which satisfies (5) for a.a. $t \in [0,+\infty)$. As we will see, to prove Theorem 1 it is essential to deal with the solutions of (5) instead of those of (3). Observe that the solution set of the Filippov's regularization, (see [2]), of (5) is contained in the solution set of (3).

Finally, in Section 3, under different assumptions, we will prove a result (Theorem 2) which solves our problem for the Filippov's regularization of (5). Specifically, we will define a regulation map $\widetilde{U}(t,x)$ under less restrictive conditions than those of Section 2. These conditions only allow us to prove the existence of a selection $\widetilde{u}(t,x)$ which is measurable in both the variables. On the other hand, we will assume that for the corresponding Filippov's regularizations of any system dynamics, $f(t,x,\widetilde{u}(t,x))$ is contained in $F(t,x,\widetilde{u}(t,x))$, or a similar condition on $\widetilde{u}(t,x)$ and the regulation map $\widetilde{U}(t,x)$.

We would like to point out that there are other methods for dealing with the problem considered in this paper. We mention here [6] and the references therein. In fact, by using Bouligand tangent cones and viability theory, (see [1]), in [6] one can establish an invariance property of a given closed set in $\mathbb{R}^n$ with respect to any system dynamics of an uncertain control system. Then one can use this invariance property to solve tracking problems like those considered here.

2. Assumptions and Preliminaries

Let $D = [0, +\infty) \times \mathbb{R}^n \times U$. We assume the following conditions on the multivalued map $F : D \multimap \mathbb{R}^n$:

(H_1) $F(t, x, u)$ is a nonempty, bounded, closed, convex set for a.a. $t \in [0, +\infty)$ and any $(x, u) \in R^n \times U$. F is t-measurable and (x, u)-continuous in the Hausdorff metric.

Observe that, by (H_1) and the Michael's selection theorem, (see e.g. [2]), it is easy to see that F has the Carathéodory selection property. That is, for any $p_0 \in F(t_0, x_0, u_0)$ with $(t_0, x_0, u_0) \in D$, there exists a Carathéodory map f such that $f(t, x, u) \in F(t, x, u)$ for a.a. $t \in [0, \infty)$ and any $(x, u) \in \mathbb{R}^n \times U$ and $p_0 = f(t_0, x_0, u_0)$, i.e. f is a system dynamics. In fact, the multivalued map $G : D \multimap \mathbb{R}^n$ defined as follows:

$$G(t, x, u) = \begin{cases} F(t, x, u) & \text{if } (t, x, u) \neq (t_0, x_0, u_0) \\ p_0 & \text{if } (t, x, u) = (t_0, x_0, u_0), \end{cases}$$

satisfies the assumptions of the Michael's selection theorem for a.a. $t \in [0, \infty)$, (see Lemma 3 in the sequel).

The function $g : E \to \mathbb{R}^n$, where $E = [0, +\infty) \times \mathbb{R}^n \times V$, which represents the dynamics of the reference model, is assumed to be Carathéodory.

Fix any state-control pair $(y(t), v(t))$, with $t \in [0, +\infty)$, of the reference model. Given $\alpha \in \mathbb{R}_+$, let C be a $n \times n$ matrix such that $\mathrm{Re}\, \lambda(C) \leq -\alpha$, where $\lambda(C)$ denotes any eigenvalue of C.

Define a function $s : [0, +\infty) \times \mathbb{R}^n \times B(0, r) \to \mathbb{R}^n$ as follows:

$$s(t, x, x_0) = y(t) - x - e^{Ct} c_0 \tag{6}$$

where $c = c_0(x_0, y_0)$ is a continuous function from $B(0, r) \times B(0, \rho)$ to $\mathbb{R}^n$ and $y_0 = y(0)$.

Consider now the function $V : \mathbb{R}^n \to \mathbb{R}_+$ given by:

$$V(s) = \sum_{i=1}^{n} |s_i|.$$

Clearly, V is not a differentiable function in $\mathbb{R}^n$, thus we consider the Clarke's generalized gradient of $V(s)$, denoted by $\partial V(s)$.

We have the following Lemma (see [10]).

Lemma 1 *For an arbitrary absolutely continuous function $x = x(t)$ in $[0, +\infty)$, consider $\sigma(t) = y(t) - x(t) - e^{Ct} c_0$. We have the following properties:*

(i) *if* $\frac{d}{dt}V(\sigma(t))$ *exists then* $\frac{d}{dt}V(\sigma(t)) = \zeta \cdot \dot{\sigma}(t)$ *for any* $\zeta \in \partial V(\sigma(t))$*;*

(ii) *the function* $V(\sigma(t))$ *is absolutely continuous in* $[0, +\infty)$*;*

(iii) *if* $\frac{d}{dt}V(\sigma(t)) \leq -k^2 < 0$ *for a.a.* $t \in \{t \in [0, +\infty) : \sigma(t) \neq 0\}$, *then there exists* $T > 0$ *such that* $\sigma(t) = 0$ *for* $t \geq T$.

Remark 1 Observe that property (i) implies that for any $t \in [0, \infty)$ at which $\frac{d}{dt}V(\sigma(t))$ exists, if $\sigma_i(t) = 0$ for some $1 \leq i \leq n$ then $\frac{d}{dt}\sigma_i(t) = 0$.

Consider the set

$$A = \{(t, x) \in [0, +\infty) \times \mathbb{R}^n : s(t, x, x_0) \neq 0 \; for\, any \; x_0 \in B(0, r)\}.$$

Given $k \neq 0$, we define the function $m : A \times U \to \mathbb{R}$ as follows:

$$m(t, x, u) = \inf_{x_0 \in B(0,r)} \inf_{\zeta \in \partial V(s(t,x,x_0))} \inf_{w \in F(t,x,u)} [\zeta \cdot w - \zeta \cdot (\dot{y}(t) - Ce^{Ct}c_0)] - k^2$$

where $c_0 = c_0(x_0, y_0)$.

Let $(\bar{t}, \bar{x}) \in \overline{A} \backslash A$ for any $u \in U$ we define

$$m(\bar{t}, \bar{x}, u) = \liminf_{(t,x) \to (\bar{t}, \bar{x})} m(t, x, u).$$

We have the following result.

Lemma 2 *The function* $m : \overline{A} \times U \to \mathbb{R}$ *is* t*-measurable,* x*-lower semicontinuous and* u*-continuous.*

Proof. Given $k \neq 0$, define the function $m_1 : \overline{A} \times B(0, r) \times U \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$m_1(t, x, x_0, u, \zeta) = \inf_{w \in F(t,x,u)} [\zeta \cdot w - \zeta \cdot (\dot{y}(t) - Ce^{Ct}c_0)] - k^2.$$

By (H_1), m_1 is t-measurable and (x, x_0, u, ζ)-continuous.

Let $m_2 : \overline{A} \times B(0, r) \times U \to \mathbb{R}$ be the function defined by:

$$m_2(t, x, x_0, u) = \inf_{\zeta \in \partial V(s(t,x,x_0))} m_1(t, x, x_0, u, \zeta).$$

Since the multivalued map $(t, x, x_0) \multimap \partial V(s(t, x, x_0))$ is t-measurable, (x, x_0)-upper semicontinuous with compact, convex values , we have that m_2 is t-measurable, (x, x_0)-lower semicontinuous and u-continuous. (see [2], Theorem 5, p. 52). This concludes the proof.

□

Consider now the multivalued map $U : \overline{A} \multimap \mathbb{R}^m$ defined by:

$$U(t,x) = \{u \in U : m(t,x,u) \geq 0\}.$$

U is called the *regulation map.* We assume:

(H_2) for any $(t,x) \in \overline{A}$ the set $U(t,x)$ is nonempty.

It is easy to verify that the multivalued map $(t,x) \multimap U(t,x)$ is t-measurable and x-lower semicontinuous with nonempty, closed values. Moreover the following property holds.

Lemma 3 *For any $\epsilon > 0$ there exists $T_\epsilon \subseteq [0,+\infty)$ closed with $\mu([0,+\infty)\backslash T_\epsilon) < \epsilon$ such that $U|_{T_\epsilon \times \mathbf{R}^n \cap \overline{A}}$ is lower-semicontinuous.*

Proof. Proposition 3.1 of [7] guarantees that the result holds for a finite interval time $[0,b]$. Therefore, for any $m \in \mathbb{N}$ there exists a compact set $T_m \subset [m-1,m]$ such that $\mu([m-1,m]\backslash T_m) < \frac{\epsilon}{2^m}$, and $U|_{T_m \times \mathbf{R}^n \cap \overline{A}}$ is lower-semicontinuous. From this we see that

$$\mu\left([0,\infty)\backslash \bigcup_{m \in \mathbf{N}} T_m\right) < \epsilon.$$

Moreover, it is easy to see that the set $\bigcup_{m \in \mathbf{N}} T_m$ is closed.

□

Following Bressan [3] we introduce now the notion of directionally continuous function with respect to a given cone. Let $M > 0$ and consider the cone

$$\Gamma^M := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : |x| \leq Mt\}.$$

We say that a map $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is directionally Γ^M-continuous at a point $(\bar{t},\bar{x})$ if and only if $\psi(t_n,x_n) \to \psi(\bar{t},\bar{x})$ for every sequence $(t_n,x_n) \to (\bar{t},\bar{x})$ with $(t_n - \bar{t}, x_n - \bar{x}) \in \Gamma^M$ for $n \geq 1$. Moreover, we say that it is Γ^M-continuous on a set $Q \subset \mathbb{R}^{n+1}$ if it is Γ^M-continuous at every point $(\bar{t},\bar{x}) \in Q$.

We say that a map $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is Scorza-Dragoni Γ^M-continuous, if for every $\epsilon > 0$ we can find $T_\epsilon \subseteq [0,+\infty)$ closed such that $\mu([0,+\infty)\backslash T_\epsilon) < \epsilon$ and ψ is Γ^M-continuous on $T_\epsilon \times \mathbb{R}^n$.

Assume that:

(H_3) there exists $M > 0$ such that $|F(t,x,u)| \leq M$ for any $(t,x,u) \in D$, where $|F(t,x,u)| = \sup\{|y| : y \in F(t,x,u)\}$.

Remark 2 Observe that if the reference trajectory $y = y(t), 0 \leq t < \infty$, is bounded and (4) holds for all the solutions $x(t)$ of (5), then there exists a compact set K containing all the trajectories $x = x(t)$, $t \in [0, +\infty)$, of (5) with $x_0 \in B(0, r)$.

Remark 2 suggests that (H_3) could be replaced by:

(H'_3) the function $y = y(t)$ is bounded on $[0, +\infty)$ and F is bounded on $[0, +\infty) \times K \times U$ for any compact set K of $\mathbb{R}^n$.

We need the following.

Lemma 4 ([4], Corollary 2.2) *Let B be a closed subset of $\mathbb{R} \times \mathbb{R}^n$ and let $G : B \multimap \mathbb{R}^n$ be lower semicontinuous with nonempty, closed values. Then, for any $M > 0$, G admits a Γ^M-continuous selection defined in B.*

An immediate consequence of Lemmas 3 and 4 is the following proposition.

Proposition 1 *Under assumptions (H_1), (H_2) and either (H_3) or (H'_3), the multivalued map $(t, x) \multimap U(t, x)$ has a Scorza-Dragoni Γ^M-continuous selection $u = u(t, x)$ defined in $\overline{A}$.*

This proposition guarantees that any dynamics $\varphi(t, x) := f(t, x, u(t, x)) \in F(t, x, u(t, x))$ is Scorza-Dragoni Γ^M-continuous in $\overline{A}$.

Observe that the Krasowskiĭ's regularization $\mathcal{K}(\varphi)(t, x)$ is not necessarily contained in $F(t, x, u(t, x))$. We have the following result.

Proposition 2 *Under assumptions (H_1), (H_2) and either (H_3) or (H'_3), any solution of the Cauchy problem*

$$\begin{cases} \dot{x} \in \mathcal{K}(\varphi)(t, x) \\ x(0) = x_0, \end{cases}$$

where $x_0 \in B(0, r)$ is a solution of

$$\begin{cases} \dot{x} = \varphi(t, x) \\ x(0) = x_0. \end{cases}$$

Proof. We use the arguments of [4] and [7] which combine the Γ^M-continuity with standard techniques. Let $x(t)$, $t \in [0, \infty)$, an absolutely continuous function such that

$$\begin{cases} \dot{x}(t) \in \mathcal{K}(\varphi)(t, x(t)) \text{ for a.a. } t \in [0, \infty) \\ x(0) = x_0. \end{cases}$$

where $x_0 \in \mathbb{R}^n$. By means of Lusin's Theorem and Proposition 1 there exists $Q_n \subset [0,\infty)$, $n \in \mathbb{N}$, measurable sets such that $\dot{x}$ restricted to Q_n is continuous, φ is Γ^M continuous in $Q_n \times \mathbb{R}^n$, $\dot{x}(t) \in \mathcal{K}(\varphi)(t, x(t))$ for any $t \in Q_n$ and $\mu([0,\infty) \setminus \cup_{n\in\mathbb{N}} Q_n) = 0$. Moreover the Lebesgue's density theorem (see [9]) ensures, for any $n \in \mathbb{N}$ the existence of a measurable set $N_n \subseteq Q_n$, with $\mu(N_n) = 0$ such that any point $t \in Q_n \setminus N_n$ is a density point for Q_n. Let $t \in Q_n \setminus N_n$, we can find a sequence $\{t_m\} \subset Q_n \setminus N_n$ with $t_m > t$ and $t_m \to t$ decreasing as $m \to \infty$.

Hence $\dot{x}(t_m) \to \dot{x}(t)$. On the other hand $|x(t) - x(s)| \leq M|t-s|$ for all $t, s \in [0,\infty)$, since $|\mathcal{K}(\varphi)(t,x(t))| \leq M$ by (H_3) (or (H'_3)), the fact that φ is a selection of F and the definition of $\mathcal{K}(\varphi)$. Let $\epsilon > 0$. We have $\dot{x}(t_m) \in \mathcal{K}(\varphi)(t_m, x(t_m)) \subseteq \varphi(t_m, x(t_m)) + \frac{\epsilon}{2}B_1$ for any $m \in \mathbb{N}$. But φ is Γ^M-continuous in $(Q_n \backslash N_n) \times \mathbb{R}^n$, hence there exists $m_0(\epsilon) \in \mathbb{N}$ such that for any $m \geq m_0$ we have $|\varphi(t_m, x(t_m)) - \varphi(t, x(t))| < \epsilon/2$, that is

$$\varphi(t_m, x(t_m)) \in \varphi(t, x(t)) + \frac{\epsilon}{2}B_1.$$

Therefore for $m \geq m_0$ we have $\dot{x}(t_m) \in \varphi(t, x(t)) + \epsilon B_1$ and so $\dot{x}(t) \in \varphi(t, x(t)) + \epsilon B_1$.

For $\epsilon \to 0$ we get $\dot{x}(t) = \varphi(t, x(t))$, $x(0) = x_0$ for all $t \in \widehat{Q} = \cup_{n\in\mathbb{N}}(Q_n \setminus N_n)$ with $\mu([0,\infty) \setminus \widehat{Q}) = 0$. This concludes the proof. □

3. Main Result

Theorem 1 *Let $(y(t), v(t))$ be a state-control pair for the reference model (2) defined on $[0,+\infty)$. Assume (H_1), (H_2) and either (H_3) or (H'_3). Then for any system dynamics φ and any $x_0 \in B(0,r)$ every solution $x(t)$ of (3) satisfies the inequality $|y(t) - x(t)| \leq \beta e^{-\alpha t}$ for some $\beta > 0$ and any $t \in [0,+\infty)$.*

Proof. From (H_3) or (H'_3) it follows that F has a Scorza-Dragoni Γ^M-continuous selection $u = u(t,x)$ in $\overline{A}$ or in $([0,+\infty)\times K)\cap\overline{A}$. For any system dynamics $\varphi(t,x) = f(t,x,u(t,x)) \in F(t,x,u(t,x))$ and for any $x_0 \in B(0,r)$ consider the Cauchy problem

$$\begin{cases} \dot{x} \in \mathcal{K}(\varphi)(t,x) \\ x(0) = x_0. \end{cases}$$

Let $x(t)$, $t \in [0,+\infty)$, be any solution of this problem, i.e.

$$\dot{x}(t) \in \bigcap_{\delta>0} \overline{\mathrm{co}}\{f(t,z,u(t,z)),\ z \in B(x(t),\delta)\}$$

for a.a. $t \in [0, +\infty)$. By Proposition 2 we have that $x(t)$ also solves the Cauchy problem

$$\begin{cases} \dot{x} = \varphi(t, x) \\ x(0) = x_0. \end{cases}$$

Therefore

$$\dot{x} = f(t, x(t), u(t, x(t)) \in F(t, x(t), u(t, x(t)))$$

for a.a. $t \in [0, +\infty)$.

Fix $t \in [0, +\infty) \cap \{t : \sigma(t) \neq 0\}$, and let ζ be any vector of $\partial V(\sigma(t))$. Then

$$\zeta \cdot \dot{x}(t) \geq \inf_{w \in F(t, x(t), u(t, x(t)))} \zeta \cdot w \geq \zeta \cdot (\dot{y}(t) - Ce^{Ct} c_0) + k^2.$$

From this, by the properties (i) and (ii) of Lemma 1 it follows that

$$\frac{d}{dt} V(\sigma(t)) = \zeta \cdot (\dot{y}(t) - \dot{x}(t) - Ce^{Ct} c_0) \leq -k^2.$$

Therefore by (iii) of Lemma 1 there exists $T > 0$ such that $\sigma(t) = 0$ for any $t \geq T$. It is possible to give an upper bound of T in terms of the data. Indeed, from the inequality

$$\frac{d}{dt} V(\sigma(t)) \leq -k^2,$$

integrating on $[0, T]$ we obtain

$$k^2 T \leq V(\sigma(0)) = \sum_{i=1}^{n} |\sigma_i(0)| \leq n(r + \rho + \gamma)$$

where $\gamma = \sup\{c_0(x_0, y_0) : (x_0, y_0) \in B(0, r) \times B(0, \rho)\}$. Hence $T \leq \dfrac{n(r + \rho + \gamma)}{k^2}$. Furthermore,

$$|y(t) - x(t)| \leq e^{-\alpha t} c_0$$

for any $t \geq T$ and thus there exists $\beta > 0$ such that

$$|y(t) - x(t)| \leq \beta e^{-\alpha t} \qquad \text{for any } t \in [0, +\infty).$$

This concludes the proof.

□

4. A Further Result

In this section we define the regulation map in a less restrictive way and we replace (H_3) and (H_3') by different conditions which allow us to prove an analogue of Theorem 1 for the solutions of the Filippov's regularization of (5). Specifically, given $k \neq 0$, we define the function $M : A \times U \to \mathbb{R}$ as follows:

$$M(t,x,u) = \inf_{x_0 \in B(0,r)} \sup_{\zeta \in \partial V(s(t,x,x_0))} \inf_{w \in F(t,x,u)} [\zeta \cdot w - \zeta \cdot (\dot{y}(t) - Ce^{Ct} c_0)] - k^2$$

where $c_0 = c_0(x_0, y_0)$.

Let $\widetilde{U} : A \multimap \mathbb{R}^n$ be the multivalued map defined by:

$$\widetilde{U}(t,x) = \{u \in U : M(t,x,u) \geq 0\}.$$

We replace (H_2) with the following:

(H_2^*) for any $(t,x) \in A$, the set $\widetilde{U}(t,x)$ is nonempty.

Lemma 5 *Assume* (H_1) and (H_2^*). *Then there exists a measurable selection* $\widetilde{u}$ *of* $\widetilde{U}$.

Proof. Given $k \neq 0$, for any $(t,x,x_0) \in A \times B(0,r)$ consider the set

$$U(t,x,x_0) = \{u \in U \ : \ \sup_{\zeta \in \partial V(s(t,x,x_0))} \inf_{w \in F(t,x,u)} \zeta \cdot w \geq \geq \zeta \cdot (\dot{y}(t) - Ce^{Ct} c_0) + k^2\}.$$

By Corollary 1Q of [11] we have that the multivalued map $(t,x,x_0) \multimap U(t,x,x_0)$ is measurable with closed values. On the other hand

$$\widetilde{U}(t,x) = \bigcap_{x_0 \in B(0,r)} U(t,x,x_0)$$

thus $(t,x) \multimap \widetilde{U}(t,x)$ is measurable with closed values, moreover $\widetilde{U}(t,x) \neq \emptyset$ by (H_3^*).

□

Obviously, for any system dynamics $\widetilde{\varphi}(t,x) := f(t,x,\widetilde{u}(t,x))$ we cannot expect, in general, any kind of continuity with respect to the state x. Therefore we must consider a regularization of $\widetilde{\varphi}$. Here we consider the Filippov's regularization of $\widetilde{\varphi}$.

$$\begin{cases} \dot{x} \in \mathcal{F}(\widetilde{\varphi})(t,x), \\ x(0) = x_0, \end{cases} \tag{7}$$

where $\mathcal{F}(\widetilde{\varphi})(t,x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \widetilde{\varphi}(t, B(x,\delta) - N)$ and μ denotes Lebesgue measure in $\mathbb{R}^n$, or equivalently

$$\mathcal{F}(\widetilde{\varphi})(t,x) = \text{co}\,\{\lim_{n\to\infty} \varphi(t,x_n),\ x_n \to x,\ x_n \in N_{\widetilde{\varphi}} \cup N\}$$

with $\mu(N_{\widetilde{\varphi}}) = 0$.

We consider here Filippov's solutions of (5) instead of Krasowskiĭ's solutions since the Filippov's regularization is smaller than the Krasowskiĭ's regularization, making less restrictive the following conditions.

(H_3^*) For any system dynamics $\widetilde{\varphi}$ and any $(t,x) \in A$ we have

$$\mathcal{F}(\widetilde{\varphi})(t,x) \subseteq F(t,x,\widetilde{u}(t,x)).$$

(H_3^{**}) For any $(t,x) \in A$ we have

$$\widetilde{\mathcal{U}}(t,x) \subseteq \widetilde{U}(t,x)$$

where $\widetilde{\mathcal{U}}(t,x) = \mathcal{F}(\widetilde{u})(t,x)$.

We can now prove the following result.

Theorem 2 *Let $(y(t), v(t))$ be a state-control pair of the reference model (2) defined in $[0,+\infty)$. Assume (H_1), (H_2^*) and either (H_3^*) or (H_3^{**}). Then for any system dynamics $\widetilde{\varphi}$ and any $x_0 \in B(0,r)$ every solution $x(t)$ of (7) satisfies the inequality $|y(t) - x(t)| \leq \beta e^{-\alpha t}$ for some $\beta > 0$ and any $t \in [0,+\infty)$.*

Proof. Let $x(t)$ be a Filippov's solution of

$$\begin{cases} \dot{x} = \widetilde{\varphi}(t,x) \\ x(0) = x_0, \end{cases}$$

where $x_0 \in B(0,r)$, that is, for a.a. $t \in [0,+\infty)$

$$\dot{x}(t) \in \mathcal{F}(\widetilde{\varphi})(t, x(t)).$$

Consider $t \in [0,+\infty) \cap \{t : \sigma(t) \neq 0\}$ for which $\frac{d}{dt}V(\sigma(t))$ exists. We have

$$\frac{d}{dt}V(\sigma(t)) = \zeta_0 \cdot (\dot{y}(t) - \dot{x}(t) - Ce^{Ct}c_0)$$

where $\zeta_0 \in \partial V(\sigma(t))$ is such that

$$\sup_{\zeta \in \partial V(\sigma(t))} \inf_{w \in F(t,x(t),\widetilde{u}(t,x(t)))} [\zeta \cdot w - \zeta \cdot (\dot{y}(t) - Ce^{Ct}c_0)] - k^2 =$$
$$= \inf_{w \in F(t,x(t),\widetilde{u}(t,x(t)))} [\zeta_0 \cdot w - \zeta_0 \cdot (\dot{y}(t) - Ce^{Ct}c_0)] - k^2.$$

We want to prove that $\frac{d}{dt}V(\sigma(t)) \leq -\frac{k^2}{2}$. For this, we know (see [5]) that there exists N_0, depending on t, such that

$$\mathcal{F}(\widetilde{\varphi})(t,x(t)) = \bigcap_{\delta>0} \{\widetilde{\varphi}(t,z) : z \in B(x(t),\delta) - N_0\}.$$

Therefore, for $\zeta_0 \in \partial V(\sigma(t))$ and for any $n \in \mathbb{N}$ there exists $z_n \in B(x(t),1/n) - N_0$ with $s(t,z_n,x_0) \neq 0$ such that

$$\zeta_0 \cdot \dot{x}(t) \geq \zeta_0 \cdot \widetilde{\varphi}(t,z_n) - \frac{k^2}{2}.$$

Taking the limit as $n \to \infty$, we obtain

$$\zeta_0 \cdot \dot{x}(t) \geq \zeta_0 \cdot \widetilde{\varphi}_0 - \frac{k^2}{2}$$

where $\widetilde{\varphi}_0 \in \mathcal{F}(\widetilde{\varphi})(t,x(t))$. By ($H_3^*$) we have $\mathcal{F}(\widetilde{\varphi})(t,x(t)) \subseteq F(t,x(t),\widetilde{u}(t,x(t)))$ and so

$$\begin{aligned}\zeta_0 \cdot \dot{x}(t) &\geq \zeta_0 \cdot \widetilde{\varphi}_0 - \frac{k^2}{2} \geq \inf_{w \in F(t,x(t),\widetilde{u}(t,x(t)))} \zeta_0 \cdot w - \frac{k^2}{2} \geq \\ &\geq \zeta_0 \cdot (\dot{y}(t) - Ce^{Ct}c_0) + \frac{k^2}{2}.\end{aligned}$$

Therefore $\frac{d}{dt}V(\sigma(t)) \leq -\frac{k^2}{2}$. It is easy to see that the same conclusion holds even if we replace (H_3^*) by (H_3^{**}). Property (iii) of Lemma 1 concludes the proof.

□

5. References

[1] J.P. Aubin, Viability theory, Ed. Birkhäuser-Verlag, Berlin, 1992.

[2] J.P. Aubin and A. Cellina, Differential inclusions, Ed. Springer-Verlag, 1984.

[3] A. Bressan, Directionally continuous selections and differential inclusions, *Funk. Ekv.* **31**, (1988) 459–470.

[4] A. Bressan, Upper and lower semicontinuous differential inclusions: a unified approach, in Nonlinear Controllability and Optimal Control, H. Sussmann Ed., M. Dekker, 1988, 21–31.

[5] A.F. Filippov, Differential equations with discontinuous right hand side, *Am. Math. Soc. Trans.* **42**, (1964), 199–231.

[6] L. Górniewicz and P. Nistri, An invariance problem for control systems with deterministic uncertainty, to appear in Banach Center Publications, Polish Academy of Sciences.

[7] S. Hu and N.S. Papageorgiou, On the existence of periodic solutions for nonconvex valued differential inclusions in $\mathbb{R}^n$, Technical Report, to appear in *Proc. Am. Math. Soc.*

[8] J.W. Macki, P. Nistri and P. Zecca, A tracking problem for uncertain vector systems, *Nonlinear Analysis T.M.A.* **14**, (1990) 319–328; corrigendum, *Nonlinear Analysis T.M.A.* **20**, (1993) 191–192.

[9] J. Oxtoby, Measure and Category, Ed. Springer-Verlag, New York, 1971.

[10] B.E. Paden and S.S. Sastry, A calculus for computing Filippov's differential inclusion with applications to the variable structure control of robot manipulators, *IEEE Trans. Circ. Syst. CAS* **34**, (1987) 73–82.

[11] R.T. Rockafellar, Integral functional, normal integrands and measurable selection, in Nonlinear Operators and the Calculus of Variations, Lecture Notes in Mathematics 543, Springer Verlag 1976, 157–207.

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Received for publication November 1995.