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Viability for Feedback Control Systems in Banach Spaces via Caratheodory Closed-loop Controls*

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Abstract

A constrained feedback nonlinear control system in infinite dimensional spaces is considered. By means of a selection theorem we prove the existence of a Carathéodory-type control law of the multivalued constraint control map. Using this result we prove the existence of viable solutions. A comparison theorem for the trajectories of the system is also proved together with the existence of extremal, periodic and stationary solutions.

1 Introduction

In this paper we consider a feedback control system (f, U) of the form

$$x'(t) = f(t, x(t), u(t, x(t))) \qquad \text{for almost all } (a.a.)\, t \in [0, T] \tag{1}$$

$$u(t, x(t)) \in U(t, x(t)) \qquad t \in [0, T] \tag{2}$$

where the state $x = x(t)$ belongs to a separable Banach space X and the control $u = u(t, x)$ belongs to a separable Banach space Z. Given a nonempty, closed subset $K \subset X$ we study the following viability problem for system (f, U): to find conditions under which for every initial state $x_0 \in K$ there exists a trajectory

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$x = x(t)$, $t \in [0, T]$, of system (f, U), starting at x_0 and such that $x(t) \in K$ for all $t \in [0, T]$, i.e. x is viable. In the past few years different aspects of viability theory have been extensively studied, see e.g. monographs [1], [2], [3], [9] and papers [4], [8], [11], [13], [14], [15].

In this paper, under suitable assumptions on the nonempty closed set K and on the multivalued map $U : [0, T] \times K \multimap Z$, we will show the existence of a single-valued control $u(t, x) \in U(t, x)$, which will be called a closed-loop control or single-valued feedback, such that there exists a solution $x = x(t)$ of (1) which is viable. Specifically, assuming that K is a sleek subset of X (see Definition 2.1) and conditions (U1)-(U3) on the multivalued map U, by using a selection theorem due to Rybinski([17]) we can prove the existence of a closed-loop control of Carathéodory type. If $f(t, x, u) = f(x, u)$, $U(t, x) = U(x)$ and X, Z are finite dimensional vector spaces then the problem of finding viable solutions corresponding to continuous closed-loop controls was presented and solved in ([1], Chapter 6). In fact, in this case our Theorem 3.6 in Section 3 reduces to Proposition 6.6.1 of [1]. Furthermore, we obtain a comparison result for the trajectories of the control system, and we prove also the existence of extremal, periodic and stationary solutions.

2 Preliminaries

Let X, Y be topological spaces, a multivalued map (multimap) $F : X \multimap Y$ is said to be:

(a) lower semicontinuous (l.s.c.) if the set $F^{-1}(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$ is open in X for every open set $V \subset Y$;

(b) upper semicontinuous (u.s.c.) if the set $F^{-1}(W)$ is closed in X for every closed set $W \subset Y$;

(c) continuous if F is both l.s.c. and u.s.c.;

(d) closed if its graph $\mathrm{Gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is a closed subset of $X \times Y$.

A map $\varphi : X \to Y$ is said to be a selection of a multimap F if $\varphi(x) \in F(x)$ for any $x \in X$. Let K be a nonempty, closed subset of a Banach space X. We recall that the Bouligand contingent cone to K at $x \in K$ is defined as the set $T_K(x) = \left\{ y \in X : \liminf_{h \to 0^+} \frac{d_K(x+hy)}{h} = 0 \right\}$ where $d_K(v) = \mathrm{dist}\,(v, K) = \inf_{z \in K} \|v - z\|$ (see e.g., [1],[2],[3]). It is easy to verify that the sets $T_K(x)$ are closed cones in X.

Definition 2.1. (cf. [1], [3]). A nonempty closed subset $K \subset X$ is said to be *sleek* if the multimap $T_K : K \multimap X$ is l.s.c.

It is known (see [3], Theorem 4.2.2) that any nonempty, closed, convex set K is sleek and that the multimap T_K on a sleek subset is convex-valued ([3], Theorem 4.1.8). Let us note that an arbitrary closed set is not sleek in general. (see an example in [4]). To exhibit another important class of sleek sets we need the following definition (cf. [4], [11], [15]).

Definition 2.2. A nonempty, closed subset $K \subset X$ is said to be a *proximate retract* if there exist an open neighbourhood $\mathcal{U}$ of K in X and a continuous map $p : \mathcal{U} \to K$ (metric retraction) having the following properties: $p(x) = x$ for all $x \in K$; $\|p(u) - u\| = d_K(u)$ for all $u \in \mathcal{U}$.

Proposition 2.3. *Any compact C^2-manifold K in a Hilbert space X is a proximate retract.*

To prove this fact we can choose as $\mathcal{U}$ a tubular neighbourhood of K in X (see [12]).

Adapting the methods employed by S. Plaskacz in [15] to the infinite-dimensional case we can obtain the following result.

Proposition 2.4. *Every compact proximate retract K in a Hilbert space X is a sleek subset.*

3 A Closed-loop Control Problem on Sleek Subsets

Let X and Z be separable Banach spaces representing the state space and the control space respectively. The dynamics of the system (f, U) under consideration is described by the map $f : [0,T] \times X \times Z \to X$ and the feedback multimap $U : [0,T] \times X \multimap Z$ of feasible controls. Therefore, the evolution of the control system (f, U) is governed by the relations

$$x'(t) = f(t, x(t), u(t, x(t))) \qquad \text{for } a.a.\ t \in [0,T] \tag{1}$$

$$u(t, x(t)) \in U(t, x(t)) \qquad t \in [0,T]. \tag{2}$$

For a given nonempty, closed subset $K \subset X$ we can now formulate the viable control problem in the following way:

(P) " Does there exist a control map $u : [0,T] \times K \to Z$ such that the control system (f, U) with any initial condition $x(0) = x_0 \in K$ has a viable trajectory $x : [0,T] \to X$, i.e. $x(t) \in K$ for any $t \in [0,T]$?"

To deal with this problem we need the following notion.

Definition 3.1. The control system (f, U) is tangent to K provided that for all $x \in K$ and for $a.a.\ t \in [0,T]$ there exists $u \in U(t,x)$ such that $f(t,x,u) \in T_K(x)$. Therefore if the control system (f, U) is tangent to K, the regulation multimap $R : [0,T] \times K \multimap Z$, given by $R(t,x) = \{u \in U(t,x) : f(t,x,u) \in T_K(x)\}$ is well-defined. (At points $t \in [0,T]$ where the tangency condition is not verified we may assume $R(t,x) = U(t,x)$.) We will solve the closed-loop control problem (P) by finding suitable selections of the regulation multimap R. Assume that the interval $[0,T]$ is endowed with the Lebesgue measure generating the σ-algebra $\mathcal{A}$ of measurable subsets. Let $\mathcal{B}(K)$ denote the σ-algebra of Borel subsets of K. Then $\mathcal{A} \times \mathcal{B}(K)$ is the product σ-algebra on $[0,T] \times K$. Let Y be a separable Banach space. Let us recall that a multimap $G : [0,T] \times K \multimap Y$ with nonempty closed values is said to be measurable if $G^{-1}(V) \in \mathcal{A} \times \mathcal{B}(K)$ for every open set $V \subset Y$. It is known (see, e.g. [7], [16]) that the above definition of measurability is equivalent to the following property: there exists a sequence $\{g_n\}_{n\in\mathbf{N}}$, $g_n : [0,T] \times K \to Y$ of measurable selections of G such that for every $(t,x) \in [0,T] \times K$ $G(t,x) = \overline{\{g_n(t,x)\}_{n\in\mathbf{N}}}$ (the Castaing representation). We will need in the sequel the following results. The first is a version of [17, Theorem 2].

Proposition 3.2. *Suppose that the multimap $G : [0,T] \times K \multimap Y$ with nonempty, closed, convex values is measurable and $G(t,\cdot) : K \multimap Y$ is l.s.c. for a.a. $t \in [0,T]$. Then there exists a Carathéodory selection of G, i.e. a map $g : [0,T] \times K \to Y$ such that*

(a) *$g(t,\cdot)$ is continuous for a.a. $t \in [0,T]$ and $g(\cdot,x)$ is measurable for all $x \in K$;*

(b) *$g(t,x) \in G(t,x)$ for $(t,x) \in [0,T] \times K$.*

The set of all Caratheodory selections of a multimap G will be denoted by $\mathcal{C}(G)$.

Proposition 3.3. *Let K be a nonempty, closed subset of a separable Banach space X. Assume that the map $g : [0,T] \times K \to X$ satisfies the following conditions:*

(a) *$g(t,\cdot)$ is continuous for a.a. $t \in [0,T]$ and $g(\cdot,x)$ is measurable for all $x \in K$;*

(b) *$g(t,x) \in T_K(x)$ for a.a. $t \in [0,T]$ and all $x \in K$.*

Assume that at least one of the following conditions is satisfied:

(i) *$\|g(t,x)\| < c(1 + \|x\|)$ in $[0,T] \times K$ for some $c > 0$ and $\lim_{\tau\to 0^+} \alpha(g(J_{t,\tau} \times D)) \leq k(t)\alpha(D)$ for all bounded $D \subset K$ and some $k(\cdot) \in L^1_+$, where $J_{t,\tau} = [t-\tau, t+\tau] \cap [0,T]$ and α is the Kuratowski measure of noncompactness (see, e.g. [9]);*

(ii) *$\|g(t,x)\| \leq \beta(t)$ on $[0,T] \times K$ for some $\beta(\cdot) \in L^1_+$ and K is boundedly compact, i.e. its intersection with every bounded subset is relatively compact.*

Then the problem $x'(t) = g(t, x(t))$, *for a.a.* $t \in [0,T]$, $x(0) = x_0 \in K$ $x(t) \in K$, $t \in [0,T]$, *has a solution for any initial value* $x_0 \in K$.

Proof. Part (i) can be directly derived from ([13], Lemma 2). Taking into account that the Bouligand cone T_K and the Clarke cone C_K coincide on sleek sets ([3], Theorem 4.1.8) we can obtain part (ii) from ([14], Theorem 2).

We set now the assumptions under which we consider problem (P). We suppose that the dynamics map $f : [0,T] \times K \times Z \to X$ and the feedback multimap $U : [0,T] \times K \multimap Z$ satisfy the following conditions:

(f1) the map $f(\cdot,\cdot,u) : [0,T] \times K \to X$ is measurable for every $u \in Z$;
(f2) the map $f(t,\cdot,\cdot) : K \times Z \to X$ is continuous for $a.a.$ $t \in [0,T]$;
(f3) the map $f(t,x,\cdot) : Z \to X$ is affine for $a.a.$ $t \in [0,T]$ and $x \in K$;
(U1) the multimap $U : [0,T] \times K \multimap Z$ has convex, closed values for $a.a.$ $t \in [0,T]$ and $x \in K$;
(U2) the multimap U is measurable;
(U3) the multimap $U(t,\cdot) : K \multimap Z$ is l.s.c. for $a.a.$ $t \in [0,T]$.
Furthermore, assume the tangency condition in the following form:
(fU) for $a.a.$ $t \in [0,T]$ and all $x \in K$ there exist $\gamma > 0$, $\delta > 0$ and $r > 0$ such that for any $\tilde{x} \in K$, $\|\tilde{x} - x\| < \delta$, we have $\gamma B_X \subset f(t,\tilde{x}, U(t,\tilde{x}) \cap rB_Z) - T_K(\tilde{x})$, where B_X, B_Z denote the unit balls centered at the origin in the spaces X and Z respectively.

We can now formulate the following result.

Theorem 3.4. *Let* $K \subset X$ *be a sleek set. Then under the conditions (f1)-(f3), (U1)-(U3) and (fU) the regulation map* $R : [0,T] \times K \multimap Z$ *has the following properties:*

(i) $R(t,x)$ *has nonempty, convex, closed values for a.a.* $t \in [0,T]$ *and* $x \in K$;

(ii) R *is measurable;*

(iii) *the multimap* $R(t,\cdot) : K \multimap Z$ *is l.s.c. for a.a.* $t \in [0,T]$.

Proof. Part (i) follows from (f2), (f3), (U1) and (fU). To prove (ii), let us consider the multimap $D : [0,T] \times K \multimap Z$, defined by $D(t,x) = \{u \in Z : f(t,x,u) \in T_K(x)\}$. It is clear that D has nonempty, convex, closed values. Now for any $(t,x) \in [0,T] \times K$, let $\mathcal{F}_{(t,x)} : X \multimap Z$ be the multimap defined by $\mathcal{F}_{(t,x)} = f^{-1}(t,x,\cdot)$ and consider the multimap $Q : [0,T] \times K \multimap X \times Z$ given by $Q(t,x) = \operatorname{Gr} \mathcal{F}_{(t,x)}$. If $\{u_n\}_{n\in\mathbf{N}}$ is a dense subset of Z, then from (f1) it follows that $\{q_n\}_{n\in\mathbf{N}}$, where $q_n : [0,T] \times K \to X \times Z$ is given by $q_n(t,x) = (f(t,x,u_n), u_n)$, is a Castaing representation for Q and hence Q is measurable. Now the multimap D can be written as follows $D(t,x) = \mathcal{F}_{(t,x)}(T_K(x))$. Let $V \subset Z$ be any open set. Following the arguments of [16] we can represent

V as the union of a sequence of closed sets $\{W_n\}_{n\in\mathbf{N}}$. Then it is easy to verify that the multimaps $T_n : [0,T] \times K \multimap X \times Z$ given by $T_n(t,x) = T_K(x) \times W_n$ are measurable. Then

$$\begin{aligned} D^{-1}(V) &= \{(t,x) \in [0,T] \times K : \mathcal{F}_{(t,x)}(T_K(x)) \cap V \neq \emptyset\} \\ &= \bigcup_{n=1}^{\infty} \{(t,x) \in [0,T] \times K : \mathcal{F}_{(t,x)}(T_K(x)) \cap W_n \neq \emptyset\} \\ &= \bigcup_{n=1}^{\infty} \{(t,x) \in [0,T] \times K : Q(t,x) \cap T_n(t,x) \neq \emptyset\}. \end{aligned}$$

Since Q and T_n are measurable, following the lines of ([16], Theorem 1M), it can be proved that each set of the latter union is measurable. Therefore, the multimap $R = D \cap U$ is measurable as the intersection of measurable multimaps (see [16]). Finally, (iii) follows from conditions (f2), (f3), (U1), (U3), (fU) and ([1], Theorem 6.3.1).

From Theorem 3.4 and Proposition 3.2 we can derive the following

Corollary 3.5. *Under the conditions of Theorem 3.4 the regulation multimap $R(t,x)$ has a Carathéodory selection $u : [0,T] \times K \to Z$.*

We are now in the position to formulate the main result of this section.

Theorem 3.6. *Let $K \subset X$ be a sleek subset. Assume that the control system (f,U) satisfies conditions (f1)-(f3), (U1)-(U3), (fU) and at least one of the following two conditions:*

(F_R) the multimap $F_R : [0,T] \times K \multimap X$ defined by $F_R(t,x) = f(t,x,R(t,x))$ is such that $\|F_R(t,x)\| = \sup\{\|y\| : y \in F_R(t,x)\} \leq c(1+\|x\|)$ on $[0,T] \times K$ for some $c > 0$ and $\lim_{\tau\to 0^+} \alpha(F_R(J_{t,\tau} \times D)) \leq k(t)\alpha(D)$ for all bounded $D \subset K$ and some $k(\cdot) \in L^1_+$;

(F_RK) $\|F_R(t,x)\| \leq \beta(t)$ on $[0,T] \times K$ for some $\beta(\cdot) \in L^1_+$ and K is boundedly compact.

Then problem (P) has a solution.

Proof. Corollary 3.5 ensures the existence of a map $u : [0,T] \times K \to Z$ such that $u \in \mathcal{C}(R)$. Then, it follows that the map $g : [0,T] \times K \to X$, $g(t,x) = f(t,x,u(t,x))$ satisfies all the conditions of Proposition 3.3.

4 A First Application: Extremal Solutions of Control Systems

In this section, using Theorem 3.6 and developing some methods of K. Deimling (see [8], [9]), we first prove a comparison result for (1)-(2). Then, by this result we show the existence of extremal trajectories of the control system. Let X be a separable Banach space. Let $K \subset X$ be a boundedly compact cone and $Z = \mathbf{R}^m$ the space of controls. We assume that the dynamics map $f : [0,T] \times X \times Z \to X$ and the feedback multimap $U : [0,T] \times X \multimap Z$ satisfy conditions (f1)-(f3), (U1)-(U3) (with K replaced by X) and the conditions (F) the multimap $F : [0,T] \times X \multimap X$ defined by $F(t,x) = f(t,x,U(t,x))$ is integrably bounded on $[0,T] \times X$ by a function $\beta(\cdot) \in L^1_+$; (fu_0) there exists a control map $u_0 \in \mathcal{C}(U)$ such that $F(t,x) \subseteq f(t,x,u_0(t,x)) + K$ for $a.a.$ $t \in [0,T]$ and any $x \in X$. Moreover, we assume that f, u_0 and U satisfy the following quasi-monotonicity condition: (Q) for $a.a.$ $t \in [0,T]$, any $y \in X$ and any $x \in K$ there exist $\gamma > 0$, $\tau > 0$ and $r > 0$ such that for any $\widetilde{x} \in K$, $\|\widetilde{x} - x\| < \tau$ we have $\gamma B_X \subset f(t,y,u_0(t,y)) - f(t, y - \widetilde{x}, U(t, y - \widetilde{x}) \cap rB_Z) - T_K(\widetilde{x})$. Now we can prove the following comparison result.

Theorem 4.1. *Under conditions (f1)-(f3), (U1)-(U3), (F), (fu_0) and (Q) let* $\overline{x} : [0,T] \to X$ *be a solution of the problem*

$$\begin{cases} x'(t) \in F(t,x(t)) + K \qquad \text{for } a.a.\ t \in [0,T] & (3) \\ x(0) \in x_0 + K. & (4) \end{cases}$$

Then the problem

$$\begin{cases} x'(t) = f(t,x(t),u(t,x(t))) \ \text{for } a.a.\ t \in [0,T] & (5) \\ u(t,x(t)) \in U(t,x(t)) & (6) \\ x(0) = x_0 & (7) \end{cases}$$

has a solution $(\underline{x},\underline{u})$ *such that* $\underline{u} \in \mathcal{C}(U)$ *and* $\underline{x} \le \overline{x}$ *(i.e.* $\overline{x}(t) \in \underline{x}(t) + K$ *for any* $t \in [0,T]$*).*

Proof. Consider the map $\widehat{f} : [0,T] \times K \times Z \to X$ defined by $\widehat{f}(t,x,u) = \overline{x}'(t) - f(t, \overline{x}(t) - x, u)$ and the multimap $\widehat{U} : [0,T] \times K \multimap Z$ given by $\widehat{U}(t,x) = U(t, \overline{x}(t) - x)$. It is clear that $\widehat{f}$ satisfies conditions (f2), (f3), (F_RK) and $\widehat{U}$ satisfies (U1) and (U3). Let us show that $\widehat{U}$ satisfies condition (U2). First, we represent any open set $V \subset Z$ as a countable union of compact sets W_n, $n \in \mathbf{N}$. Then we consider the multimaps $\Gamma_n : [0,T] \times K \multimap [0,T] \times X \times Z$ given by $\Gamma_n(t,x) = \{(t, \overline{x}(t) - x)\} \times W_n$. Γ_n is continuous for any $n \in \mathbf{N}$ (see, e.g.[5]). The graph $\mathrm{Gr}\,U$ of the multimap U belongs to the σ-algebra $\mathcal{A} \times \mathcal{B}(X) \times \mathcal{B}(Z)$ (see [7]), then the set $\widehat{U}^{-1}(W_n) = \Gamma_n^{-1}(\mathrm{Gr}\,U)$ belongs to $\mathcal{A} \times \mathcal{B}(K)$ and thus $\widehat{U}^{-1}(V) \in \mathcal{A} \times \mathcal{B}(K)$. Analogously, we can show that $\widehat{f}$ satisfies property (f1). Now, let $t \in [0,T]$, $y = \overline{x}(t)$ and $x \in K$. From (fu_0) it follows that the equation $\overline{x}'(t) \in F(t,\overline{x}(t)) + K \subseteq f(t,\overline{x}(t), u_0(t,\overline{x}(t))) + K$ can be rewritten as $\overline{x}'(t) = f(t,\overline{x}(t), u_0(t,\overline{x}(t))) + \xi(t)$ where $\xi : [0,T] \to K$ is a measurable function. Then, by (Q), for any $\widetilde{x} \in K$, $\|\widetilde{x} - x\| < \delta$ we have

$$\begin{aligned} \gamma B_X &\subset f(t, \overline{x}(t), u_0(t, \overline{x}(t))) - f(t, \overline{x}(t) - \widetilde{x}, U(t, \overline{x}(t) - \widetilde{x}) \cap rB_Z) - T_K(\widetilde{x}) \\ &= \overline{x}'(t) - \xi(t) - f(t, \overline{x}(t) - \widetilde{x}, U(t, \overline{x}(t) - \widetilde{x}) \cap rB_Z) - T_K(\widetilde{x}) \\ &= \widehat{f}(t, \widetilde{x}, \widehat{U}(t, \widetilde{x}) \cap rB_Z) - T_K(\widetilde{x}) \end{aligned}$$

since $K \subset T_K(\widetilde{x})$. Hence for the control system $(\widehat{f}, \widehat{U})$, condition (fU) is also satisfied. Now from Theorem 3.5 it follows that the problem
$x'(t) = \widehat{f}(t, x(t), u(t, x(t)), u(t, x(t)) \in \widehat{U}(t, x(t)), x(0) = \overline{x}(0) - x_0 \in K$,
has a solution (x, u) such that u is Carathéodory and x is viable. Setting $\underline{x} \doteq \overline{x} - x \leq \overline{x}$ and $\underline{u}(t, x) = u(t, \overline{x}(t) - x)$ we obtain the desired solution $(\underline{x}, \underline{u})$ of the problem (5)-(7). The proof is complete.

Now we apply Theorem 4.1 to prove the existence of an extremal solution for the problem (1)-(2).

Theorem 4.2. *Let $X = \mathbf{R}^n$, $K = \mathbf{R}^n_+$, $Z = \mathbf{R}^m$ and the control system (f, U) satisfies the conditions of Theorem 4.1. In addition, let $U(t, \cdot)$ be u.s.c. for a.a. $t \in [0, T]$ and the pair (f, u_0) satisfies the following global monotonicity condition: for a.a. $t \in [0, T]$,* ***any*** *$y \in X$ and any absolutely continuous function $v : [0, T] \to K$ we have*

$$f(t, y + v(t), u_0(t, y + v(t))) - f(t, y, u_0(t, y)) \geq v'(t). \tag{8}$$

Then, for a given initial value $x(0) = x_0$, there exists a solution (x_*, u_*) of problem (1)-(2) such that $u_* \in C(U)$ and x_* is minimal, i.e. $x_* \leq x$ for any trajectory x of problem (1)-(2) with the same initial value.

Proof. Consider the Cauchy problem

$$\begin{cases} x'(t) \in G(t, x(t)) & \text{for a.a. } t \in [0, T] \quad (9) \\ x(0) = x_0 & \quad (10) \end{cases}$$

where $G(t, x) = (F(t, x) + K) \cap 2\beta(t)B_X = (f(t, x, u_0(t, x)) + K) \cap 2\beta(t)B_X$ and $\beta(\cdot)$ is the function from the condition (F). It is clear that G has compact, convex values and is integrably bounded. From the properties of multimaps (see, e.g. [5]) it follows that $G(\cdot, x)$ is measurable for every $x \in X$ and $G(t, \cdot)$ is u.s.c. for $a.a.$ $t \in [0, T]$. It is well known then that under these conditions the solution set Σ of the problem (9)-(10) is nonempty and compact (see, e.g.,[2],[5],[9]).
For x, $y \in C([0, T]; X)$, define $x \wedge y \in C([0, T]; X)$ as $(x \wedge y)_i(t) = \min\{x_i(t), y_i(t)\}$, $i = 1, 2, \ldots, n$. If x, $y \in \Sigma$, then $x \wedge y$ is Lipschitz, hence absolutely continuous. Denoting $v(t) = x(t) - (x \wedge y)(t) \in K$ and using property (8) we have for

$a.a.\ t \in [0,T]$

$$\begin{aligned}(x \wedge y)'(t) &= x'(t) - v'(t) \in f(t, x(t), u_0(t, x(t))) + K - v'(t) \\ &\in f(t, (x \wedge y)(t), u_0(t, (x \wedge y)(t))) + v'(t) + K - v'(t) \\ &= F(t, (x \wedge y)(t)) + K.\end{aligned}$$

It is easy to see that $\|(x \wedge y)'(t)\| \leq 2\beta(t)$ for $a.a.\ t \in [0,T]$, hence $x \wedge y \in \Sigma$. Now, let $\{\sigma_n\}_{n \in \mathbf{N}}$ be a dense subset of Σ. Defining $x_1 = \sigma_1$ and $x_{n+1} = x_n \wedge \sigma_{n+1}$ we obtain a decreasing sequence $\{x_n\}_{n \in \mathbf{N}} \subset \Sigma$ converging to a minimal element $x_{min} \in \Sigma$, since x_{min} is a solution of the problem $x'(t) \in F(t, x(t)) + K$, $x(0) = x_0$. The application of Theorem 4.1 yields the existence of a solution $(\underline{x}, \underline{u})$ of the problem (1)-(2) with $\underline{x}(0) = x_0$, $\underline{u} \in \mathcal{C}(U)$ and such that $\underline{x} \leq x_{min}$. But since $\underline{x} \in \Sigma$ then $x_* = \underline{x} = x_{min}$, $u_* = \underline{u}$ is the desired solution.

5 A Second Application: Viable Periodic and Stationary Solutions of Control Systems

In this section we assume the following condition (K) K is a compact proximate retract in an Hilbert space X. For the control system (f, U), $f : [0,T] \times K \times Z \to X$, $U : [0,T] \times K \multimap Z$, where Z is a separable Banach space where the controls u take their values, consider the following periodic problem

$$\begin{cases} x'(t) = f(t, x(t), u(t, x(t))) \quad \text{for } a.a.\ t \in [0,T] & (11) \\ u \in \mathcal{C}(U) & (12) \\ x(t) \in K \qquad t \in [0,T] & (13) \\ x(0) = x(T). & (14) \end{cases}$$

To analyze this problem we need some additional definitions and results.

Definition 5.1. (see, e.g. [6]) A metric space M is an ANR-space (absolute neighbourhood retract) if there exists an open set V of a normed space E and continuous maps $p : V \to M$, $s : M \to V$ such that $p \circ s = id_M$. If we can take $V = E$ in this definition, M is called an AR-space (absolute retract).
It is clear that every proximate retract is an ANR-space.

Definition 5.2. If M is a compact ANR-space, then its Euler characteristic $\chi(M)$ may be defined as the Lefschetz number $\lambda(id_M)$ of the identity map id_M. Let $\{H_n\}_{n \geq 0}$ be the Čech homology functor with compact carriers and coefficients in the field of rational numbers $\mathbf{Q}$ (see, e.g.[10]).

Definition 5.3. A nonempty, compact space A is said to be *acyclic*, provided

$H_0(A) = \mathbf{Q}$ and $H_n(A) = 0$ for every $n > 0$.

If M is an acyclic ANR-space (in particular, a compact AR-space), then $\chi(M) = 1$.

Definition 5.4. Let P, Q be metric spaces. An u.s.c. multimap $\Gamma : P \multimap Q$ is said to be *acyclic* if its values $\Gamma(x)$, $x \in P$ are acyclic sets. We will say that a multimap $\Gamma : P \multimap Q$ is *quasiacyclic* provided there exist a metric space S, an acyclic multimap $\Gamma' : P \multimap S$ and a continuous map $q : S \to Q$ such that $\Gamma = q \circ \Gamma'$. We will use here the following special case of the Lefschetz-Eilenberg-Montgomery fixed point theorem (cf. [10]).

Theorem 5.5. *Let M be a compact ANR-space with $\chi(M) \neq 0$ and $\Gamma : M \times [0, 1] \multimap M$ a quasiacyclic multimap. If $\Gamma(\cdot, 0) = id_M$, then $\Gamma(\cdot, 1) : M \multimap M$ has a fixed point, i.e. $x \in \Gamma(x, 1)$.*

Now, let us return to the viable periodic problem (11)-(14). We need the following

Lemma 5.6. *Let K be a compact proximate retract in the Hilbert space X. Assume that the map $g : [0, T] \times K \to X$ satisfies the following conditions:*

(a) *$g(t, \cdot)$ is continuous for a.a. $t \in [0, T]$ and $g(\cdot, x)$ is measurable for all $x \in K$;*

(b) *g is integrably bounded;*

(c) *$g(t, x) \in T_K(x)$ for a.a. $t \in [0, T]$ and all $x \in K$.*

Then the solution map $\Sigma : K \multimap C([0, T]; K)$ defined as
$\Sigma(x) = \{y : y$ is a viable solution of $y' = g(t, y(t))$ with the initial condition $y(0) = x \}$, is an acyclic multimap.

Proof. Let $\mathcal{U}$ be an open neighbourhood of K in X for which there exists a metric retraction $p : \mathcal{U} \to K$. Let $\eta : X \to [0, 1]$ be an Urysohn function such that $\eta|_M \equiv 1$ and $\eta|_{X \setminus \mathcal{U}} \equiv 0$. Define a map $\widetilde{g} : [0, T] \times X \to X$ by

$$\widetilde{g}(t, x) = \begin{cases} \eta(x)\, g(t, p(x)) & \text{if } (t, x) \in [0, T] \times \mathcal{U} \\ \{0\} & \text{if } (t, x) \in [0, T] \times (X \setminus \mathcal{U}). \end{cases}$$

It is easy to see that $\widetilde{g}$ is the extension of g on $[0, T] \times X$ satisfying conditions (a)-(c). From ([1], Theorem 5.2.1) it follows that every solution of the problem $y'(t) = \widetilde{g}(t, y(t))$, $y(0) = x \in K$, is viable. Therefore, for $x \in K$, the solution sets $\Sigma_g(x)$ and $\Sigma_{\widetilde{g}}(x)$ coincide. But from [19] it follows that every $\Sigma_{\widetilde{g}}(x)$ is an R_δ-set and hence, acyclic, and from [18] we know that the solution multimap $\Sigma_{\widetilde{g}}$ is u.s.c., which proves the lemma.

We can now prove the following periodicity result.

Theorem 5.7. *Assume $x(k) \neq 0$, and that the conditions (K), (f1)-(f3), (U1)-(U3), (fU) and $(F_R K)$ are satisfied. Then for every closed-loop control $u \in C(R)$*

the periodic problem (11)-(14) has a solution.

Proof. From Lemma 5.6 it follows that for every closed-loop control $u \in \mathcal{C}(R)$, the initial problem $x'(t) = f(t, x(t), u(t, x(t)))$, $for a.a.\ t \in [0, T]$, $u(t, x(t)) \in U(t, x(t))$, $x(0) = x \in K$, defines an acyclic solution multimap $\Sigma : K \multimap C([0, T]; K)$. Consider now the evaluation map $e : C([0, T]; K) \times [0, 1] \to K$ defined as follows $e(y(\cdot), \mu) = y(\mu T)$. It is clear that a multimap $\Gamma : K \times [0, 1] \multimap K$ defined by $\Gamma(x, \mu) = e(\Sigma(x), \mu)$ is quasiacyclic, $\Gamma(\cdot, 0) = id_K$ and $\Gamma(x, 1) = P_T(x) = \{y(T) : y \in \Sigma(x)\}$ is the translation operator along the trajectories of the control system (f, U), corresponding to a chosen closed-loop control.
From Theorem 5.5 it follows that a multimap P_T has at least a fixed point x_0 which represents the initial value of the periodic problem (11)-(14).

Corollary 5.8. *If the control system (f, U) is autonomous, i.e. $f(t, x, u) \equiv f(x, u)$; $U(t, x) \equiv U(x)$, then for every closed-loop control $u(\cdot)$ there exists a stationary trajectory $x(t) \equiv x_* \in K$.*

Proof. From Theorem 5.7 it follows that for every $n \in \mathbf{N}$ there exists a trajectory $x_n(\cdot)$ of the control system (f, U) corresponding to a closed-loop control u and such that $x_n(0) = x_n\left(\frac{T}{n}\right) = \cdots = x_n\left(\frac{(n-1)T}{n}\right) = x_n(T)$. The limit point x_* of a sequence $\{x_n\}_{n \in \mathbf{N}}$ is the desired stationary trajectory.

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