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A SLIDING MANIFOLD APPROACH TO THE FEEDBACK CONTROL OF RIGID ROBOTS

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SUMMARY

In this paper we propose a sliding manifold approach to the control of rigid robotic manipulators. We design a PD feedback controller via singular perturbation theory which guarantees the tracking of a reference trajectory. The control signal tends during a fast transient to the well-defined equivalent control with fast nonoscillating modes and then it remains close to this in the uniform topology. Then the resulting closed-loop system does not have the drawbacks of high-gain feedback systems, even if it retains robustness properties with respect to disturbances and plant parameter uncertainties. An application of the proposed procedure to a robotic system which includes actuator dynamics and tachometers is presented.

KEY WORDS singularly perturbed systems; sliding manifold

INTRODUCTION

The dynamics of rigid robotic manipulators is described by a set of coupled highly nonlinear second-order differential equations and the derivation of such equations is not an easy task for manipulators with large degrees of freedom. Moreover, the model is affected by uncertainties due to variable payloads which must be handled, link parameter uncertainties and unknown static and dynamic friction forces. For these reasons, the problem of designing control laws for rigid robotic manipulators which guarantee asymptotic trajectory tracking with boundedness of all internal signals has enticed researchers for many years.

In the last ten years a large number of nonlinear control strategies have been proposed. The first contributions were based on exact compensation of nonlinearities. Such control techniques rely on the property that the equations of motion for a rigid manipulator may be exactly linearized and decoupled by static nonlinear state feedback. These control strategies are called computed torque or inverse dynamic based control techniques.[1,2]

Since the Erzberger conditions for perfect model matching are satisfied for a rigid robot, a model reference approach has been proposed in Reference 3 and nonlinear adaptive model following control techniques or techniques based on variable structure system theory have been proposed in References 4–9. The proposed techniques lead to control laws which are

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discontinuous on sliding surfaces. The adaptive ones do not incorporate any parameter identification algorithms, but the adaptation is realized through signal synthesis in a fixed parameter structure. Such control systems have very good properties, they exhibit stable behaviour, accurate tracking, and insensitivity with respect to disturbances and variation of manipulator parameters.

The main drawback of these control systems is the so-called chattering phenomenon. In the robotic literature some modifications of these control schemes have been proposed[10–12] in order to avoid such a phenomenon. They assure ultimate boundedness of the tracking error with a trade-off between tracking accuracy and absence of the chattering.

As concerns adaptive control algorithms for rigid manipulators which explicitly incorporate parameter estimation in the control law, we refer to Reference 13 and the references therein. Some of these algorithms require both measurement of the joint acceleration and boundedness of the inverse of the estimated inertia matrix. Although these hypotheses can be removed by using passivity-based control methods, open problems remain to be investigated such as unmodelled dynamics, transient performance and that asymptotic stability of the closed-loop system has not been proved to be uniform.

In this paper we propose a sliding manifold approach for the control of a rigid robotic manipulator. Specifically, given a reference trajectory and an initial configuration of the manipulator, we can design a sliding manifold for which the equivalent control is well defined. Symmetry and positive definiteness properties of the inertia matrix allows us to introduce, by resorting to the singular perturbation theory,[14] a linear PD feedback controller, which turns out to be the solution of a differential equation containing a small positive parameter. The main features of this controller are the following.

First, it can be proved, by using the concept of logarithmic norm of a matrix, that the control signal, for sufficiently small values of the parameter, after a rapid transient, depending only on its initial value, in which it approaches the equivalent control with fast nonoscillating modes, remains close to the latter in the uniform topology. It turns out that if we choose the initial condition for the control signal close to the initial condition of the equivalent control, then practically there is not any transient. Then the resulting closed-loop system does not have the drawbacks of high-gain systems such as peaking phenomenon.

Second, the corresponding response of the system describing the dynamics of the manipulator is in a given neighbourhood of the sliding manifold for any time.

Third, as in the VSS theory, the control system has the approximability property.[15,16] Such a property permits the control system to be robust with respect to a given class of perturbations on the state variable measurements.

The proposed control algorithm is then applied to the robotic system proposed in Reference 17 as a benchmark problem for control system design. Such a problem includes actuator dynamics in the model formulation. The presence of a tachometer for the angular velocity measurement is also considered in order to test the approximability property.

DYNAMICS OF RIGID ROBOT MANIPULATORS

The manipulator dynamical model can be written as follows

$$\mathbf{B}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathbf{e}(\boldsymbol{\theta}) = \boldsymbol{\tau} \tag{1}$$

where $\boldsymbol{\theta}$ is the vector of generalized coordinates, $\boldsymbol{\tau}$ is the vector of generalized forces applied at various joints, $\mathbf{B}(\boldsymbol{\theta})$ is the inertia matrix, the vector $\mathbf{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ takes into account Coriolis and centrifugal torques, $\mathbf{e}(\boldsymbol{\theta})$ is the vector of gravitational torques.

In order to rewrite equation (1) in state variable form for a manipulator with m degrees of freedom it is convenient to use the following notation.

$$x_i = \theta_i,\ x_{i+m} = \dot{\theta}_i,\ u_i = \tau_i, \qquad i = 1, \ldots, m \tag{2}$$

and hence $\mathbf{x} = (\boldsymbol{\theta}^{\mathrm{T}}, \dot{\boldsymbol{\theta}}^{\mathrm{T}})^{\mathrm{T}}$, and $\mathbf{u} = (u_1, \ldots, u_m)^{\mathrm{T}}$. Let moreover

$$\bar{\mathbf{B}}(\mathbf{x}) = \mathbf{B}^{-1}(\boldsymbol{\theta}) \tag{3}$$

Now equation (1) can be rewritten as follows

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{4}$$

where

$$\mathbf{A}(\mathbf{x}) = \begin{pmatrix} 0 & \mathbf{I}_m \\ \mathbf{A}_{21}(\mathbf{x}) & \mathbf{A}_{22}(\mathbf{x}) \end{pmatrix} \tag{5}$$

and

$$\mathbf{B}(\mathbf{x}) = \begin{pmatrix} 0 \\ \bar{\mathbf{B}}(\mathbf{x}) \end{pmatrix} \tag{6}$$

$\bar{\mathbf{B}}(\mathbf{x})$ is positive definite, bounded and symmetric and all the elements of $\mathbf{A}(\mathbf{x})$ are bounded.

THE SLIDING MANIFOLD APPROACH

We now propose a *sliding manifold* approach to the control of the class of nonlinear dynamical systems described by equations (4)–(6) via the classical singular perturbation theory. A general framework of this approach has been outlined in Reference 18, with the purpose of eliminating the chattering phenomenon in variable structure systems control.

We assume the following condition.

(F1) the two maps $\mathbf{x} \to \mathbf{A}(\mathbf{x})$ and $\mathbf{x} \to \mathbf{B}(\mathbf{x})$ satisfy a Lipschitz condition on $\mathbb{R}^n$.

It is easy to verify that in the case of the mechanical system under consideration such a condition is satisfied.

As direct consequence of **(F1)** we have that $\forall \rho > 0$, $\exists\ \gamma_\rho > 0$ such that for any $(\mathbf{x}, \mathbf{u})$ with $|\mathbf{x}| + |\mathbf{u}| < \rho$ it follows that

$$|\mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}| < \gamma_\rho \tag{7}$$

where the symbols $|\cdot|$ and $\|\cdot\|$ denote the Euclidean norm of a vector in $\mathbb{R}^n$ and the induced norm of a matrix respectively.

Let $s: \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^m$, $m \leqslant n$, be a continuously differentiable map. Define the related *sliding manifold* $\mathcal{S}$ as follows

$$\mathcal{S} = \{(\mathbf{x}, t) \in \mathbb{R}^n \times [0, \infty) : \mathbf{s}(\mathbf{x}, t) = 0\} \tag{8}$$

We make the following assumptions on $\mathcal{S}$.

(S1) For any $t \in [0, \infty)$ there exists $\mathbf{x} \in \mathbb{R}^n$ such that

$$(\mathbf{x}, t) \in \mathcal{S} \tag{9}$$

For any $\varepsilon > 0$, consider the following system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{10}$$

$$\varepsilon\dot{\mathbf{u}} = \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \tag{11}$$

where

$$\mathbf{g}(\mathbf{x}, \mathbf{u}, t) = \frac{\partial \mathbf{s}(\mathbf{x}, t)}{\partial t} + \frac{\partial \mathbf{s}(\mathbf{x}, t)}{\partial \mathbf{x}}(\mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}) \tag{12}$$

(S2) There exists a neighbourhood $\mathcal{J}$ of the manifold $\mathcal{S}$ such that, for any fixed $(\mathbf{x}, t) \in \mathcal{J}$, the map

$$\mathbf{u} \rightarrow \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \tag{13}$$

is one-to-one on $\mathbb{R}^m$ and its range contains zero.

In our case, a necessary and sufficient condition for **(S2)** is that the matrix

$$\frac{\partial \mathbf{s}(\mathbf{x}, t)}{\partial \mathbf{x}} \mathbf{B}(\mathbf{x}) \tag{14}$$

is not singular for any $(\mathbf{x}, t) \in \mathcal{J}$.

Definition 1

For any $(\mathbf{x}, t) \in \mathcal{J}$, the unique solution (if any) $\mathbf{u}^*(\mathbf{x}, t)$ of the algebraic equation $\mathbf{g}(\mathbf{x}, \mathbf{u}, t) = 0$ is called the *equivalent control* for the variable structure system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \\ \mathbf{s}(\mathbf{x}, t) &= 0 \end{aligned} \tag{15}$$

If we assume that the matrix (14) is not singular we obtain

$$\mathbf{u}^*(\mathbf{x}, t) = -\left(\frac{\partial \mathbf{s}(\mathbf{x}, t)}{\partial \mathbf{x}} \mathbf{B}(\mathbf{x})\right)^{-1}\left(\frac{\partial \mathbf{s}(\mathbf{x}, t)}{\partial t} + \frac{\partial \mathbf{s}(\mathbf{x}, t)}{\partial \mathbf{x}} \mathbf{A}(\mathbf{x})\mathbf{x}\right) \tag{16}$$

for any $(\mathbf{x}, t) \in \mathcal{J}$. Moreover, for any $(\mathbf{x}, t) \in \mathcal{J}$ and any $\mathbf{v} \in \mathbb{R}^m$ with $\mathbf{v} \neq \mathbf{u}^*(\mathbf{x}, t)$ we have $\mathbf{g}(\mathbf{x}, \mathbf{v}, t) \neq 0$.

(S3) For any pair $(\mathbf{x}_0, \mathbf{u}_0) \in \mathbb{R}^n \times \mathbb{R}^m$ and any $\varepsilon > 0$, the system (10)–(11) has an unique solution $(\mathbf{x}(t, \varepsilon), \mathbf{u}(t, \varepsilon))$ defined on $[0, \infty)$ such that $(\mathbf{x}(0, \varepsilon), \mathbf{u}(0, \varepsilon)) = (\mathbf{x}_0, \mathbf{u}_0)$.

Condition **(F1)**, together with the condition that the map $\mathbf{x} \rightarrow \mathbf{g}(\mathbf{x}, \mathbf{u}, t)$ satisfies a Lipschitz condition uniformly in $\mathbf{u}$ and t, guarantee that **(S3)** is satisfied.

(S4) The equilibrium point $\bar{\mathbf{u}}^* = \mathbf{u}^*(\bar{\mathbf{x}}, \bar{t})$ of the equation (11) is asymptotically stable, for any $(\bar{\mathbf{x}}, \bar{t}) \in \mathcal{J}$. That is the solution $\mathbf{z} = \mathbf{z}(s)$ of the Cauchy problem

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{g}(\bar{\mathbf{x}}, \mathbf{z}, \bar{t}) \\ \mathbf{z}(0) &= \bar{\mathbf{u}} \end{aligned} \tag{17}$$

satisfies $\lim_{s\to\infty} \mathbf{z}(s) = \bar{\mathbf{u}}^*$, whenever $\bar{\mathbf{u}}$ is sufficiently close to $\bar{\mathbf{u}}^*$ and $(\bar{\mathbf{x}}, \bar{t}) \in \mathcal{S}$. Furthermore, we suppose that the asymptotic stability is uniform in $(\bar{\mathbf{x}}, \bar{t}) \in \mathcal{S}$.

(S5) The origin $\mathbf{x} = 0$ of the system (10)–(11) corresponding to $\varepsilon = 0$, the so-called reduced system, is exponentially stable.

Definition 2

A point $(\bar{\mathbf{x}}, \bar{\mathbf{u}}, \bar{t})$ such that the solution of the Cauchy problem (17) converges asymptotically to $\mathbf{u}^*(\bar{\mathbf{x}}, \bar{t})$ is said to belong to the *domain of influence* of $\mathbf{u}^*(\bar{\mathbf{x}}, \bar{t})$.

Remark 1

Observe that, if $(\tilde{\mathbf{x}}(t), \tilde{\mathbf{u}}(t))$ is a solution of system (10)–(11) corresponding to $\varepsilon = 0$, then

$$\mathbf{g}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{u}}(t), t) = \frac{\mathrm{d}\mathbf{s}(\tilde{\mathbf{x}}(t), t)}{\mathrm{d}t} = 0 \tag{18}$$

where $\tilde{\mathbf{u}}(t) = \mathbf{u}^*(\tilde{\mathbf{x}}(t), t)$. Therefore if there exists $\bar{t} \in [0, \infty)$ such that $\mathbf{s}(\tilde{\mathbf{x}}(\bar{t}), \bar{t}) = 0$, then $\mathbf{s}(\tilde{\mathbf{x}}(t), t) = 0$ for any $t \in [0, \infty)$. Consequently, the graph of $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(t)$ lies on the manifold $\mathcal{S}$.

From the classical theory of the singular perturbation on an infinite time interval[19] we have the following result.

Theorem 1

Assume that **(F1)** and **(S1)**–**(S5)** are satisfied. Let $(\mathbf{x}_0, \mathbf{u}_0, 0)$ be a point in the domain of influence of $\mathbf{u}^*(\mathbf{x}_0, 0)$ with $(\mathbf{x}_0, 0) \in \mathcal{S}$. Then the solution $(\mathbf{x}(t, \varepsilon), \mathbf{u}(t, \varepsilon))$ of the Cauchy problem

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \qquad & \mathbf{x}(0) &= \mathbf{x}_0 \\ \varepsilon\dot{\mathbf{u}} &= \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \qquad & \mathbf{u}(0) &= \mathbf{u}_0 \end{aligned} \tag{19}$$

has the following properties

$$\begin{aligned} \lim_{\varepsilon\to 0} \mathbf{x}(t, \varepsilon) &= \tilde{\mathbf{x}}(t) \qquad \text{uniformly in } [0, \infty) \\ \lim_{\varepsilon\to 0} \mathbf{u}(t, \varepsilon) &= \tilde{\mathbf{x}}(t) \qquad \text{uniformly in } [t_1, \infty) \end{aligned} \tag{20}$$

for any $t_1 > 0$ and $(\tilde{\mathbf{x}}(t), \tilde{\mathbf{u}}(t))$ is the solution of the reduced system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ 0 &= \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \end{aligned} \tag{21}$$

THE TRACKING PROBLEM

Let $\hat{\mathbf{x}}(t)$ be a desired differentiable state trajectory which the state $\mathbf{x}(t)$ of the system (10)–(12) is required to follow in $[0, \infty)$, and assume that $|\hat{\mathbf{x}}(t)| < M \ \forall t \in [0, \infty)$ for some positive constant M. Owing to the structure of the matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ the reference trajectory $\hat{\mathbf{x}}(t)$

will satisfy the condition

$$\dot{\hat{x}}_{i+m} = \hat{x}_i \qquad i = 1, \ldots, m \tag{22}$$

We can formulate the tracking problem in the following way.

— Given $\beta > 0$, $\delta > 0$ and $\mathbf{x}_0 \in \mathbb{R}^n$, it is required to design a feedback control such that, for $\varepsilon > 0$ sufficiently small, the solution $\mathbf{x}(t, \varepsilon)$ of system (10)–(12) for which $\mathbf{x}(0, \varepsilon) = \mathbf{x}_0$ satisfies the inequality

$$|\hat{\mathbf{x}}(t) - \mathbf{x}(t, \varepsilon)| \leq \delta + A\,\mathrm{e}^{-\beta t} \tag{23}$$

for any $t \in [0, \infty)$, where A is a constant depending on the data.

In order to solve this problem we define a function $\mathbf{s}: \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^m$, $n = 2m$, as follows

$$\mathbf{s}(x, t) = \mathbf{H}(\hat{\mathbf{x}}(t) - \mathbf{x} - \mathrm{e}^{\mathbf{C}t}(\hat{\mathbf{x}}_0 - \mathbf{x}_0)) \tag{24}$$

where $\hat{\mathbf{x}}_0 = \hat{\mathbf{x}}(0)$, $\mathbf{H} = (\mathbf{H}_1\ \mathbf{H}_2)$ with $\mathbf{H}_1$, $\mathbf{H}_2$ $m \times m$ matrices to be specified, the $\mathbf{C} \in \mathbb{R}^{n \times n}$ is a symmetric matrix which will be chosen in Theorem 2 in a convenient way. Observe that $\mathbf{s}(\mathbf{x}_0, 0) = 0$, i.e. $(\mathbf{x}_0, 0) \in \mathcal{S}$.

Remember that for a symmetric matrix $\mathbf{C}$, if we choose the Euclidean norm for vectors and the induced norm for matrices then

$$\lambda_{\max}(\mathbf{C}) = \mu(\mathbf{C}) \tag{25}$$

Here $\mu(\cdot)$ denotes the logarithmic norm, or measure, of a matrix.[20,21]

By (24) the equations (10)–(11) can be rewritten as follows

$$\begin{pmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{I}_m \\ \mathbf{A}_{21}(\mathbf{x}) & \mathbf{A}_{22}(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{\mathbf{B}}(\mathbf{x}) \end{pmatrix} \mathbf{u}, \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m \tag{26}$$

$$\varepsilon \dot{\mathbf{u}} = (\mathbf{H}_1\ \ \mathbf{H}_2) \left\{ \begin{pmatrix} \dot{\hat{\mathbf{x}}}_1 \\ \dot{\hat{\mathbf{x}}}_2 \end{pmatrix} - \begin{pmatrix} 0 & \mathbf{I}_m \\ \mathbf{A}_{21}(\mathbf{x}) & \mathbf{A}_{22}(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} - \begin{pmatrix} 0 \\ \bar{\mathbf{B}}(\mathbf{x}) \end{pmatrix} \mathbf{u} - \mathbf{C}\,\mathrm{e}^{\mathbf{C}t}(\hat{\mathbf{x}}_0 - \mathbf{x}_0) \right\} \tag{27}$$

Let $\Gamma(\mathbf{x}) \coloneqq \mathbf{H}_2 \bar{\mathbf{B}}(\mathbf{x})$. We have the following result.

Proposition 1

Let $\bar{\mathbf{B}}(\mathbf{x})$ be symmetric and positive definite for any $\mathbf{x} \in \mathbb{R}^n$, then for any compact set $K \in \mathbb{R}^n$ we have

$$\alpha_K \coloneqq \min_{\mathbf{x} \in K} \lambda_{\min}(\bar{\mathbf{B}}(\mathbf{x})) > 0 \tag{28}$$

Proposition 2

Let $\mathbf{H}_2 \in \mathbb{R}^{m \times m}$ be a matrix such that $\Gamma(\mathbf{x})$ is symmetric and positive definite for any $\mathbf{x} \in \mathbb{R}^n$. Then, for any compact set $K \subset \mathbb{R}^n$ we have that the linear system

$$\dot{\mathbf{z}} = -\Gamma(\mathbf{x})\mathbf{z} \tag{29}$$

is asymptotically stable uniformly with respect to $\mathbf{x} \in K$.

Proof. By proposition 1 we have that for any compact set K there exists a constant $\alpha_K > 0$ such that $\lambda_{\min}(\Gamma(\mathbf{x})) \geqslant \alpha_K > 0$ for any $\mathbf{x} \in K$.

Let $\mathbf{x} \in K$, for any $\mathbf{u}_0 \in \mathbb{R}^m$, consider the solution of (35) satisfying $\mathbf{z}(0) = \mathbf{u}_0$

$$\mathbf{z}(s) = \mathrm{e}^{-\Gamma(\mathbf{x})s}\mathbf{u}_0 \tag{30}$$

By using the logarithmic norm of the symmetric matrix $-\Gamma(\mathbf{x})$ and its properties,[21] we get

$$|\mathbf{z}(s)| \leqslant \varepsilon^{\alpha_K s}|\mathbf{u}_0| \tag{31}$$

for any $s \geqslant 0$ and for any $\mathbf{x} \in K$. □

Remark 2

Proposition 2 guarantees a sufficient condition in order that (**S4**) is verified.

We are now in the position of proving the following result.

Theorem 2

Let $\beta > 0$, $\gamma > 0$, $\delta > 0$ and $\mathbf{x}_0 \in \mathbb{R}^n$ be given. Consider the system (26)–(27). Assume that (**F1**) and the conditions of Proposition 2 are satisfied. Moreover assume that

$$\operatorname{Re}\lambda_{\min}(\mathbf{H}_2^{-1}\mathbf{H}_1) \geqslant \beta + \gamma \tag{32}$$

and

$$\mu(\mathbf{C}) < -\operatorname{Re}\lambda_{\min}(\mathbf{H}_2^{-1}\mathbf{H}_1) + \gamma \tag{33}$$

where $\operatorname{Re}\lambda_{\min}(\mathbf{H}_2^{-1}\mathbf{H}_1)$ denotes the minimum of the real parts of all the eigenvalues of the matrix $\mathbf{H}_2^{-1}\mathbf{H}_1$.

Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the solution $(\mathbf{x}(t,\varepsilon), \mathbf{u}(t,\varepsilon))$ to (26)–(27) satisfying $(\mathbf{x}(0,\varepsilon), \mathbf{u}(0,\varepsilon)) = (\mathbf{x}_0, \mathbf{u}_0)$, whenever $\mathbf{u}_0 \in \mathbb{R}^m$, is such that

$$|\hat{\mathbf{x}}(t) - \mathbf{x}(t,\varepsilon)| \leqslant \delta + A_1\,\mathrm{e}^{\mu(\mathbf{C})t} + A_2\,\mathrm{e}^{-\beta t} \tag{34}$$

$$\mathbf{u}(t,\varepsilon) = \frac{1}{\varepsilon}\,\mathbf{H}(\hat{\mathbf{x}}(t) - \mathbf{x}(t,\varepsilon) - \mathrm{e}^{\mathbf{C}t}(\hat{\mathbf{x}}_0 - \mathbf{x}_0)) + \mathbf{u}_0 \tag{35}$$

where $t \in [0, \infty)$ and A_1, A_2 are positive constants depending on $\mathbf{H}$, $\mathbf{C}$, $\hat{\mathbf{x}}_0$ and $\mathbf{x}_0$.

Proof. Let $(\tilde{\mathbf{x}}(t), \tilde{\mathbf{u}}(t))$ be the solution of system (26)–(27) corresponding to $\varepsilon = 0$ and define

$$\tilde{\mathbf{e}} = \hat{\mathbf{x}} - \tilde{\mathbf{x}} \tag{36}$$

we show that $\tilde{\mathbf{e}} = 0$ is an exponentially stable equilibrium point of the equation

$$\dot{\tilde{\mathbf{e}}} = \phi(t, \tilde{\mathbf{e}}) \tag{37}$$

where

$$\phi(t,\tilde{\mathbf{e}}) = \dot{\hat{\mathbf{x}}}(t) - \mathbf{A}(\hat{\mathbf{x}}(t) - \tilde{\mathbf{e}}) + \mathbf{B}(\hat{\mathbf{x}}(t) - \tilde{\mathbf{e}})\mathbf{u}^*(t, \hat{\mathbf{x}}(t) - \tilde{\mathbf{e}}) \tag{38}$$

In fact, since by Remark 1 we have $\mathbf{s}(\tilde{\mathbf{x}}(t), t) = 0$, for any $t \in [0, \infty)$, the solution of (37) satisfies the algebraic equation

$$\mathbf{H}\tilde{\mathbf{e}}(t) = 0 \quad \text{for any } t \in [0, \infty) \tag{39}$$

By (4)–(6) and (22) we have

$$\dot{\tilde{\mathbf{e}}}_1 = \tilde{\mathbf{e}}_2 \tag{40}$$

Thus (39) can be rewritten in the following form

$$\dot{\tilde{\mathbf{e}}}_1 = -\mathbf{H}_2^{-1}\mathbf{H}_1\tilde{\mathbf{e}}_1 \tag{41}$$

and so if $\tilde{\mathbf{e}}_1(0) = 0$ then $\tilde{\mathbf{e}}_1(t) = 0$ for any $t \in [0, \infty)$. By (40) we obtain $\tilde{\mathbf{e}}(t) = 0$ on $[0, \infty)$. Therefore $\tilde{\mathbf{e}} = 0$ is an equilibrium point of (37).

On the other hand, if $\tilde{\mathbf{e}}(0) = \hat{\mathbf{x}}_0 - \mathbf{x}_0 \neq 0$ we have

$$\mathbf{H}\tilde{\mathbf{e}}(t) = \mathbf{H}\,\mathrm{e}^{\mathbf{C}t}\tilde{\mathbf{e}}(0) \tag{42}$$

for any $t \in [0, \infty)$ and by previous consideration we get

$$\lim_{t\to\infty} \tilde{\mathbf{e}}(t) = 0 \tag{43}$$

exponentially.

Consider now a neighbourhood $\mathcal{I}$ of $\mathcal{S}$ for which the set

$$\{\mathbf{x} \in \mathbb{R}^n\colon (t, \mathbf{x}) \in \mathcal{I} \quad \text{for some } t \in [0, \infty)\}$$

is bounded. Observe that such a neighbourhood exists since the reference trajectory is bounded in $\mathbb{R}^n$.

Therefore, for $\varepsilon > 0$ sufficiently small, under our assumptions one can easily show that assumptions **(S1)**–**(S5)** of Theorem 1 are verified in $\bar{\mathcal{I}}$, the closure of $\mathcal{I}$.

In conclusion, Theorem 1 applies to the system

$$\begin{pmatrix}\dot{\hat{\mathbf{x}}}_1\\ \dot{\hat{\mathbf{x}}}_2\end{pmatrix} - \begin{pmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{x}}_2\end{pmatrix} = \begin{pmatrix}\dot{\hat{\mathbf{x}}}_1\\ \dot{\hat{\mathbf{x}}}_2\end{pmatrix} - \begin{pmatrix}0 & \mathbf{I}_m\\ \mathbf{A}_{21}(\mathbf{x}) & \mathbf{A}_{22}(\mathbf{x})\end{pmatrix}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix} - \begin{pmatrix}0\\ \bar{\mathbf{B}}(\mathbf{x})\end{pmatrix}\mathbf{u} \tag{44}$$

$$\varepsilon\dot{\mathbf{u}} = (\mathbf{H}_1 \;\; \mathbf{H}_2)\left\{\begin{pmatrix}\dot{\hat{\mathbf{x}}}_1\\ \dot{\hat{\mathbf{x}}}_2\end{pmatrix} - \begin{pmatrix}0 & \mathbf{I}_m\\ \mathbf{A}_{21}(\mathbf{x}) & \mathbf{A}_{22}(\mathbf{X})\end{pmatrix}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix} - \begin{pmatrix}0\\ \bar{\mathbf{B}}(\mathbf{x})\end{pmatrix}\mathbf{u} - \mathbf{C}\,\mathrm{e}^{\mathbf{C}t}(\hat{\mathbf{x}}_0 - \mathbf{x}_0)\right\}$$

Thus for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ we have

$$|\mathbf{e}(t, \varepsilon) - \tilde{\mathbf{e}}(t)| \leq \delta \tag{45}$$

for any $t \in [0, \infty)$ or equivalently

$$|\tilde{\mathbf{x}}(t) - \mathbf{x}(t, \varepsilon)| \leq \delta \tag{46}$$

for any $t \in [0, \infty)$. On the other hand

$$|\hat{\mathbf{x}}(t) - \mathbf{x}(t, \varepsilon)| \leq |\tilde{\mathbf{x}}(t) - \mathbf{x}(t, \varepsilon)| + |\hat{\mathbf{x}}(t) - \tilde{\mathbf{x}}(t)| < \delta + |\hat{\mathbf{x}}_1(t)| + |\hat{\mathbf{x}}_2(t) - \hat{\mathbf{x}}_2(t)| \tag{47}$$

Using our assumptions, equations (40)–(41) and Reference 20 (Proposition 3, p. 4), by standard calculations we obtain

$$|\hat{\mathbf{x}}(t) - \mathbf{x}(t, \varepsilon)| < \delta + A_1\,\mathrm{e}^{\mu(\mathbf{C})t} + A_2\,\mathrm{e}^{(-\mathrm{Re}\lambda_{\min}(\mathbf{H}_2^{-1}\mathbf{H}_1)+\gamma)t} \tag{48}$$

where

$$A_1 = \|\mathbf{H}_2^{-1}\mathbf{H}\|\,|\hat{\mathbf{x}}(0) - \tilde{\mathbf{x}}(0)| \tag{49}$$

$$A_2 = L(1 + \|\mathbf{H}_2^{-1}\mathbf{H}_1\|)\left(|\hat{\mathbf{x}}_1(0) - \tilde{\mathbf{x}}_1(0)| + \frac{\|\mathbf{H}_2^{-1}\mathbf{H}\|\,|\hat{\mathbf{x}}(0) - \mathbf{x}(0)|}{|\mu(\mathbf{C}) + \mathrm{Re}\lambda_{\min}(\mathbf{H}_2^{-1}\mathbf{H}_1) - \gamma|}\right) \tag{50}$$

and $L = L(\gamma) > 0$ is the constant of Proposition 3 in Reference 20. From this, being $\mu(\mathbf{C})$ and $-\mathrm{Re}\,\lambda_{\min}(\mathbf{H}_2^{-1}\mathbf{H}_1) + \gamma$ less or equal to $-\beta$, we obtain (23). □

Remark 3

If in Theorem 2 the matrix $\mathbf{H}_1$ is such that $\mathbf{H}_2^{-1}\mathbf{H}_1$ is symmetric, then we can choose $\gamma = 0$ and $L = 1$. In fact, in this case we can use the estimate

$$\|e^{-\mathbf{H}_2^{-1}\mathbf{H}_1 t}\| \leqslant e^{\mu(-\mathbf{H}_2^{-1}\mathbf{H}_1)t} \tag{51}$$

for any $t \geqslant 0$, where $\mu(-\mathbf{H}_2^{-1}\mathbf{H}_1) = -\lambda_{\min}(\mathbf{H}_2^{-1}\mathbf{H}_1)$.

Remark 4

The presence of the term $e^{Ct}(\hat{\mathbf{x}}_0 - \mathbf{x}_0)$ in (24) allows us to bound the amplitude of the control $\mathbf{u}(t, \varepsilon)$ in the presence of the mismatch between the initial state of the reference trajectory and the initial condition of the actual state of the system. In fact, by Theorem 2, the control signal $\mathbf{u}(t, \varepsilon)$ belongs to a neighbourhood of the equivalent control $\tilde{\mathbf{u}}(t)$ in the time interval $[t_1, \infty)$, where $t_1 = t_1(\varepsilon) > 0$, and there is only a fast transient in which the control $\mathbf{u}(t, \varepsilon)$, depending on its initial condition $\mathbf{u}_0$, is out of this neighbourhood. More specifically

$$\begin{aligned} \forall \varepsilon > 0\, \exists t_1, \sigma > 0 : |\mathbf{u}(t, \varepsilon) - \tilde{\mathbf{u}}(t)| &\geqslant \sigma, \quad \forall t \in [0, t_1] \\ |\mathbf{u}(t, \varepsilon) - \tilde{\mathbf{u}}(t)| &< \sigma, \quad \forall t \in (t_1, \infty) \end{aligned} \tag{52}$$

while by Proposition 2 and Remark 2, $|\mathbf{u}(t, \varepsilon) - \tilde{\mathbf{u}}(t)|$ decreases exponentially in $[0, t_1]$. Moreover $t_1 \searrow 0$ as $\varepsilon \to 0$. Observe that if we choose as initial condition for the control $\mathbf{u}(t, \varepsilon)$, $\varepsilon > 0$, the initial condition $\mathbf{u}^*(0, \mathbf{x}_0)$ of the equivalent control (or one close enough to it) then there is not any transient for the control $\mathbf{u}(t, \varepsilon)$ to get the σ-neighbourhood, in fact we have, in this case, $|\mathbf{u}(t, \varepsilon) - \tilde{\mathbf{u}}(t)| < \sigma$, $\forall t \in [0, \infty)$.

Therefore this transient does not depend on the mismatch $(\hat{\mathbf{x}}_0 - \mathbf{x}_0)$ as it usually happens with high-gain controls. This is due to the different definition of our dynamical control which depends also on the initial condition $\mathbf{x}_0$ and which drives the tracking by means of the term $e^{Ct}(\hat{\mathbf{x}}_0 - \mathbf{x}_0)$.

APPROXIMABILITY PROPERTY

Let $M \in \mathbb{L}^1([0, \infty), \mathbb{R}^+)$ be a given function. Let $\{\mathbf{p}_\eta : \eta > 0\}$ be a one-parameter family of functions $\mathbf{p}_\eta \in \mathbb{L}^1([0, \infty), \mathbb{R}^m)$, $\eta > 0$. Assume that

$$|\mathbf{p}_\eta(t)| \leqslant M(t) \tag{53}$$

for any $\eta > 0$ and for almost all $t \in [0, \infty)$ and that

$$\sup\left\{\left|\int_0^t \mathbf{p}_\eta(s)\,\mathrm{d}s\right| : t \in [0, \infty)\right\} \to 0 \text{ as } \eta \to 0_+ \tag{54}$$

Denote by $\mathcal{H}_1$ the set of all one-parameter families $\{\mathbf{p}_\eta: \eta > 0\}$ which satisfy (53) and (54).
We give the following.

Definition 3

We say that system (15) fulfils the *approximability property* if and only if the following conditions hold

(i) the hypothesis (**S2**) is satisfied;
(ii) the set $\mathcal{H}_1$ is not empty;
(iii) if $\{\mathbf{p}_\eta\} \subset \mathcal{H}_1$ and if $(\mathbf{x}(t, \varepsilon, \eta), \mathbf{u}(t, \varepsilon, \eta))$ is a solution to

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \\ \varepsilon\dot{\mathbf{u}} &= \mathbf{g}(\mathbf{x}, \mathbf{u}, t) + \mathbf{p}_\eta \end{aligned} \tag{55}$$

such that $\mathbf{s}(\mathbf{x}(0, \varepsilon, \eta), 0) \to 0$ as $\eta \to 0$ and if $\tilde{\mathbf{x}}(t)$ is a solution in $[0, \infty)$ of the system (21) with the condition $\mathbf{s}(\tilde{\mathbf{x}}(0), 0) = 0$ then the condition

$$\lim_{\eta \to 0} \mathbf{x}(0, \varepsilon, \eta) = \tilde{\mathbf{x}}(0) \tag{56}$$

implies that

$$\lim_{\varepsilon \to 0} \lim_{\eta \to 0} \mathbf{x}(t, \varepsilon, \eta) = \tilde{\mathbf{x}}(t) \tag{57}$$

uniformly in $[0, \infty)$. Here the function $\mathbf{g}(\mathbf{x}, \mathbf{u}, t)$ is defined by (12) and (24). It is easy to prove the following.

Proposition 3

Under the assumption of Theorem 2, system (26)–(27) fulfils the *approximability property*.

APPLICATION TO ROBOTIC SYSTEM

In this section we apply the above technique to a robotic manipulator to show the effectiveness of the proposed control strategy with respect to initial condition mismatch, exogenous disturbances and corrupted measures. The model of the robot considered in this paper has been presented in Reference 17 and is depicted in Figure 1.

With reference to (1) the generalized force τ_i, $i = 1, 2, 3$, is expressed as $\tau_i = \nu_i(\tau_{E,i} - \tau_{F,i})$, where ν_i, is the gear ratio, $\tau_{E,i}$, is the electromagnetic moment acting on the rotor R_i of the motor; $\tau_{F,i}$, is the moment of friction reduced to the axis of the rotor R_i.

The control inputs are the control voltages $u_{C,i}$ of the actuators. The controlled outputs are the joint angles θ_i and the relative joint angular velocities $:\dot{\theta}_i$ $(i = 1, 2, 3)$. The ith motor is mounted on the $(i - 1)$th link and drives θ_i.

Expressions for the entries of matrices $\mathbf{B}(\boldsymbol{\theta})$, $\mathbf{e}(\boldsymbol{\theta})$, $\mathbf{c}(\boldsymbol{\theta}, :\dot{\boldsymbol{\theta}})$ and τ_F can be found in Reference 17.

The motors are described by the following equations

$$\tau_{i,E} = K_i i_{A,i} \tag{58}$$

$$L_{A,i} \frac{\mathrm{d}i_{A,i}}{\mathrm{d}t} + R_{A,i} i_{A,i} = u_{A,i} - K_i \nu_i \dot{\theta}_i \tag{59}$$

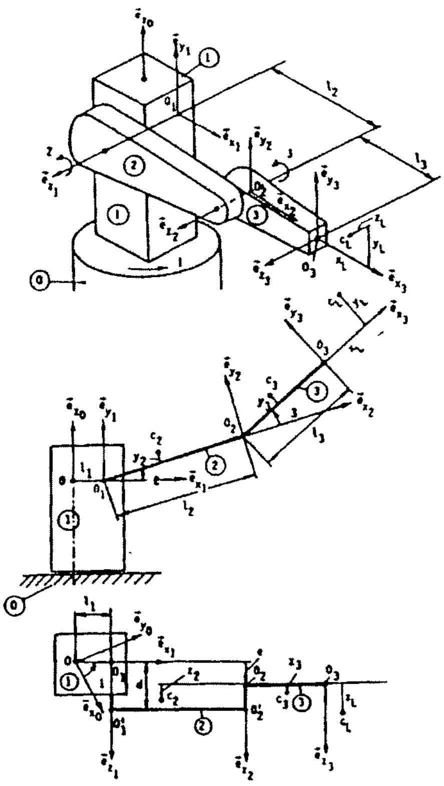

Figure 1.The robot

$$T_{St,i}\frac{\mathrm{d}u_{A,i}}{\mathrm{d}t}+u_{A,i}=K_{St,i}u_{C,i} \tag{60}$$

where

K_i motor constant (N m/A)
$L_{A,i}$ armature inductance (V s/A)
$i_{A,i}$ armature current (A)
$R_{A,i}$ armature resistance (Ω)
$u_{A,i}$ armature voltage (V)
$T_{St,i}$ time constant of the motor (s)
$K_{St,i}$ gain
$u_{C,i}$ control voltage (V)

Finally three tachometers have been considered to sense angular velocities. Their behaviour is described by the relation

$$\omega_{m,i}=k_i\dot{\theta}_i\cos\left\{\nu_i\dot{\theta}_i t-\left[2\,\mathrm{int}\left(\frac{\nu_i n\dot{\theta}_i t}{2\pi}\right)+\mathrm{sgn}(\dot{\theta}_i)\right]\frac{\pi}{n}\right\} \tag{61}$$

where $\omega_{m,i}$ is the measured angular velocity, k_i is the tachometer constant and n is the number of commutator bars.

The output of the tachometer can be split into two components: a mean value whose expression, by supposing the angular velocity constant over a period, is

$$\bar{\omega}_{m,i} = \frac{n}{\pi} k_i \dot{\theta}_i \sin(\pi/n) \tag{62}$$

and a ripple.

The joint angles reference trajectory is depicted in Figure 2.

Moreover it is required not to exceed limits on angular rates, armature voltages and currents

$$|\dot{\theta}_{i,\max}| \leq \dot{\theta}_i^*, |i_{A,i}| \leq I_{A,i}^*, |u_{A,i}| \leq V_{A,i}^* \tag{63}$$

The control law is given by (35) and is rewritten as follows

$$\mathbf{u}_C(t) = \frac{1}{\varepsilon} \mathbf{H}\left(\hat{\mathbf{x}}(t) - \mathbf{x}(t) - e^{Ct}(\hat{\mathbf{x}}_0 - \mathbf{x}_0)\right) \tag{64}$$

where $\mathbf{x} = (\boldsymbol{\theta}^T, \dot{\boldsymbol{\theta}}^T)^T$ and $\hat{\mathbf{x}} = (\hat{\boldsymbol{\theta}}^T, \dot{\hat{\boldsymbol{\theta}}}^T)^T$.

In the Appendix the numerical values for the present case are reported.

The robot arm is initially kept in the starting position $\boldsymbol{\theta}_0 = (0\ \pi/4\ -\pi/4)^T$ by means of mechanical brakes. Then the brakes are suddenly released, and the actuators generate torques to steer the arms to the desired initial position $\hat{\boldsymbol{\theta}}_0 = (0\ \pi/2\ -\pi/2)^T$.

In Figure 3 the transient behaviour of the norm of the electromagnetic torques acting on the rotors is shown. The norm of the equivalent control is represented by the dashed line (A) and that of the actual control by the solid line (B). This simulation has been performed on a simplified model of the system, neglecting the actuator dynamics and the sensor ripple (i.e. $\mathbf{u} = \tau$). Owing to the selected value of the scalar ε and the matrix $\mathbf{H}$, the norm of the control signal reaches the value of the norm of the equivalent control in less than 5 ms. Note the absence of any peaking phenomenon.

Now we consider the effects of the actuator dynamics.

Since the armature inductance can be neglected, we consider the time constant of the motor only. Equation (29) must be rewritten as

$$\varepsilon \ddot{\mathbf{u}}_c + \varepsilon \mathbf{T}_{st}^{-1} \dot{\mathbf{u}}_c + \mathbf{T}_{st}^{-1} \Gamma(\mathbf{x}) \mathbf{u}_c = 0 \tag{65}$$

where $\mathbf{T}_{st}$ is the diagonal matrix of the motor time constant.

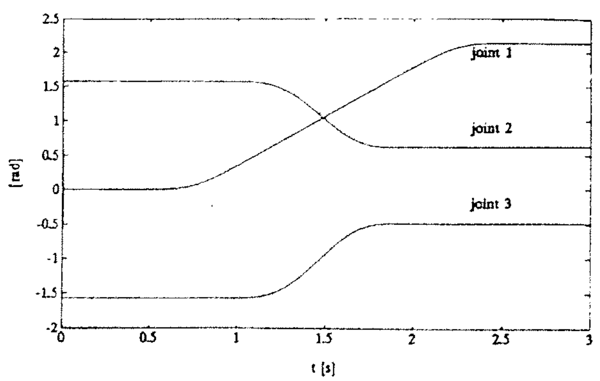

Figure 2. Reference angular trajectory $\hat{\theta}$

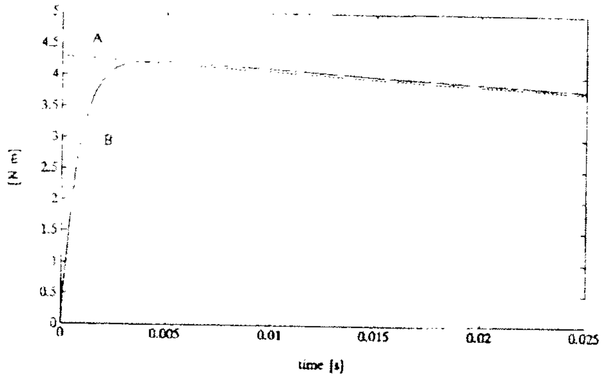

Figure 3. Transient behaviour of the norm of the control signal

By Proposition 2 we have that for any $K \in \mathbb{R}^n$ compact, there exists $a_K > 0$ such that

$$-\mu(-\Gamma(\mathbf{x})) = \lambda_{\min}(\Gamma(\mathbf{x})) \geqslant a_K > 0 \tag{66}$$

Let $\mathbf{x} \in K$ be fixed, then the eigenvalues of the asymptotic stable system (65) depend on the values of ε, $T_{\mathrm{st},i}$ and on the eigenvalues of the symmetric and positive definite matrix $\Gamma(\mathbf{x})$. It is routine to verify, by using (66), that the asymptotic stability of the system (65) is uniform with respect to $\mathbf{x} \in K$.

However, in this case, as one can easily verify by using the parameter values in the Appendix, the actual control tends to the equivalent control with oscillating modes, but the overshoots are negligible.

Now consider the model (1), (58)–(60) with the control law (64). Suppose that after the time $t = \pi/10$ s the mass of the load varies with a sinusoidal law: $m_{\mathrm{L}} = 40 + 10\sin(10t)$ kg, where m_{L} denotes the mass of the load and of the end-effector. Figure 4 shows the norms of the equivalent control (A) and of the actual control (B).

Finally we consider the effect of the tachometer ripple.

Since all the assumptions of Theorem 2 are verified, the closed-loop system fulfils the *approximability property*. In particular $\eta = 1/n$, $\mathbf{p}_\eta(t) = \mathbf{H}_2\,\mathrm{d}(\omega_{\mathrm{m}} - \bar{\omega}_{\mathrm{m}})/\mathrm{d}t$ and standard computations show that the norm (54) is smaller than $\|\mathbf{H}_2\mathbf{K}\|\ |\omega|\ \ |\cos(\pi\eta) - \sin(\pi\eta)/(\pi\eta)|$,

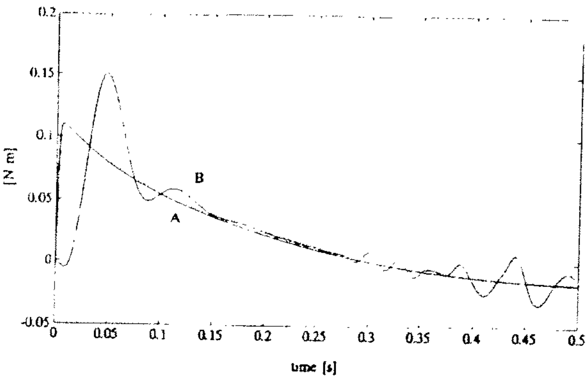

Figure 4. Norm of the control signal (with actuators)

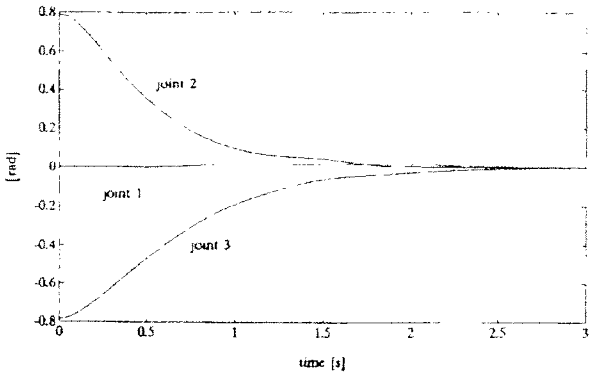

Figure 5. Joint angle tracking errors

where **K** denotes the diagonal matrix of the tachometer constants and ω is the vector of components $\sup\{|\dot{\theta}_i(t)|: t \in [0, \infty)\}$, $i = 1, 2, 3$. The standard number of commutator bars is $9 \div 18$. By using tachometers with $n = 15$, we have in the worst case (i.e. at the maximum velocity) that the norm (54) is less than 0.15.

The resulting tracking error $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ is depicted in Figure 5.

CONCLUSIONS

In this paper a sliding manifold approach to the control of the class of dynamical systems described by a set of coupled second-order nonlinear differential equations has been presented. Specifically, the proposed procedure has been applied to the robotic system proposed as benchmark problem by the IFAC. Such an approach allows us to solve the tracking problem without using discontinuous control laws as in VSS design, or adaptive control algorithms. By means of a suitable definition of the *sliding manifold* for which the *equivalent control* is well defined, we design a controller which turns out to be the solution of a differential equation containing a small parameter. The approach is different from that of high-gain feedback systems, because fast modes occur only in the control dynamics, and the control signal tends to a neighbourhood of the equivalent control with fast nonoscillating modes and remains in this neighbourhood at all times. The uniform asymptotic stability of the control scheme has been proved also in the presence of actuator dynamics. The closed-loop system has the approximability property which allows us to state that it is robust with respect to a given class of perturbations on the state variable measurements. The presence of tachometers to sense the angular joint velocities has been considered in order to test such a property.

APPENDIX

With reference to Figure 1 the data for the arm are

$$\begin{array}{lllll} l_1 = 0{\cdot}35\ \text{m} & m_2 = 100\ \text{kg} & x_2 = -0{\cdot}60\ \text{m} & y_2 = 0 & z_2 = 0{\cdot}20\ \text{m} \\ l_2 = 0{\cdot}45\ \text{m} & m_3 = 25\ \text{kg} & x_3 = -0{\cdot}30\ \text{m} & y_3 = 0 & z_3 = 0 \\ l_3 = 0{\cdot}20\ \text{m} & m_L = 40\ \text{kg} & x_L = \ \ 0{\cdot}20\ \text{m} & y_L = 0 & z_L = 0 \\ e = 0 & & & & \end{array} \tag{67}$$

where m_2, m_3 are the masses of links 2 and 3 respectively, including actuators.
The data for the motors are

$$\begin{aligned} K_i &= 0{\cdot}167 \text{ V s} \\ L_{A,i} &= 8 \times 10^{-5} \text{ V s/A} \\ R_{A,i} &= 1{\cdot}50\,\Omega \\ T_{st,i} &= 0{\cdot}012 \text{ s} \\ K_{st,i} &= 1 \end{aligned} \tag{68}$$

with $i = 1, 2, 3$.
The gear ratios are

$$\nu_1 = 220,\ \nu_2 = 150,\ \nu_3 = 130 \tag{69}$$

The data for the tachometers are

$$\begin{aligned} k_i &= 1{\cdot}007\,348\,4, \quad i = 1, 2, 3 \\ n &= 15 \end{aligned} \tag{70}$$

The data for the control law (64) are

$$\varepsilon = 10^{-3}$$

$$\mathbf{H} = \begin{pmatrix} 8 & 0 & 0 & 4 & 0 & 0 \\ 0 & 6 & 0 & 0 & 4 & 0 \\ 0 & 0 & 8 & 0 & 0 & 4 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -42 & 0 & 0 & -13 & 0 & 0 \\ 0 & -15 & 0{\cdot}06 & 0 & -8 & 0{\cdot}017 \\ 0 & 0{\cdot}06 & -8 & 0 & 0{\cdot}017 & -6 \end{pmatrix} \tag{71}$$

The matrix C has been selected as to satisfy a 'congruence condition', i.e. there exists a matrix $\mathbf{K}(\mathbf{x})$ such that $\mathbf{C} = \mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{K}(\mathbf{x})$ for any $\mathbf{x}$.
The limits on angular rates and armature voltages are those reported in Reference 17.

$$\dot{\theta}_i^* = 314/\nu_i, \quad V_{A,i}^* = 65 \text{ V} \tag{72}$$

The limits on $i_{A,i}$, derived from (72) and (59) considering the extreme conditions, are

$$I_{A,1}^* = 8 \text{ A}, \quad I_{A,2}^* = 15 \text{ A}, \quad I_{A,3}^* = 12 \text{ A} \tag{73}$$

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