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## STABILITY OF NONHOMOGENEOUS AGING VISCOELASTIC BODIES UNDER DYNAMIC LOADING

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### 0. INTRODUCTION

One important problem in the mechanics of continua is the stability of a deformable solid body. This problem may be formulated as following: to find restrictions on external forces which guarantee that small perturbations of load cause small perturbations of stress-strain in the body.

The stability of a deformable body is a nonlinear problem due to the geometrical nonlinearity of dynamic equations and the physical nonlinearity of material properties [1-3].

There are two fundamental approaches to the study of the stability of deformable bodies. The first approach may be called "bifurcation". It is used for quasistatic loading processes when the forces of inertia are neglected. Through this approach, the state of the body is stable if this state minimizes the Helmholtz free energy of the system. Analysis of stability consists of the evaluation of the second variation of free energy, and testing conditions on external forces which ensure this functional is positive definite. Some conditions of stability were formulated using the method in [1, 4]. Note that the bifurcation approach can be applied to viscoelastic bodies only because of the definition of specific free energy for viscoelastic or viscoelastoplastic materials.

The second approach may be applicable to quasistatic and dynamic processes of loading. It consists of the linearization of the equations of movement and constitutive equations, and the direct analysis of the perturbation equations using, e.g. techniques of integral estimates. Linearization in the nonlinear theory of elasticity was developed in [1], whilst the method of integral estimates for determination of critical loads was considered in [4].

Applications of these methods to real engineering problems meet great difficulties because of the variety of types of loading and material properties at finite strains. Hence, we need to introduce some assumptions which simplify the problem of stability.

In this paper our interest is focused on the analysis of the stability of nonhomogeneous aging viscoelastic bodies. The model for such materials was formulated in [5] for infinitesimal strains and in [5-7] for the case of finite strains.

Aging of materials means that the mechanical properties depend on time because of physical or chemical transformations.

### 1. STATEMENT OF THE PROBLEM

Let us consider a viscoelastic body, which in natural configuration [2] occupies a star domain  $\Omega$  with connected Lipschitz boundary  $\Gamma$ . Introduce the coordinate system  $\xi = (\xi_i)$  in the

domain  $\Omega$ . At time  $t = 0$  the body forces  $F(t, \xi)$  and the boundary forces  $f(t, \xi)$  are applied to the body. Here  $F$  and  $f$  are column vectors. Suppose that the displacement of the domain  $\Omega$  as a rigid body under external forces vanishes. The functions  $F(t, \xi)$  and  $f(t, \xi)$  are quadratic integrable at the domain  $\Omega$  and its boundary for fixed  $t \geq 0$

$$|F(t)|^2 = \int_{\Omega} F'(t, \xi)F(t, \xi) dv < \infty,$$

$$|f(t)|^2 = \int_{\Gamma} f'(t, \xi)f(t, \xi) ds < \infty.$$

Here and below  $dv$  is the element of volume of the domain  $\Omega$ , and  $ds$  is the element of area of the surface  $\Gamma$ , whilst the prime denotes transposition.

Assume that for any  $\xi$  there exist the derivatives  $\dot{F} = \partial F/\partial t$  and  $\dot{f} = \partial f/\partial t$  and the functions  $\dot{F}(t, \xi)$ ,  $\dot{f}(t, \xi)$  are quadratic integrable in  $\xi$  for any  $t \geq 0$ . Suppose that the functions  $|F(t)|$ ,  $|\dot{F}(t)|$ ,  $|f(t)|$ ,  $|\dot{f}(t)|$  are bounded and integrable in  $t$ . Denote

$$\|F\| = \sup_t |F(t)| + \int_0^{\infty} |\dot{F}(t)| dt,$$

$$\|f\| = \sup_t |f(t)| + \int_0^{\infty} |\dot{f}(t)| dt.$$

Represent the functions  $F, f$  in the form

$$F = F_0(t, \xi) + \Delta F(t, \xi), \quad f = f_0(t, \xi) + \Delta f(t, \xi),$$

where  $(F_0, f_0)$  is the basic load and  $(\Delta F, \Delta f)$  is its perturbation.

For the estimation of the nonperturbed stress state suppose that

$$\|F_0\|_1 = \|F_0\| + \sup_t |\dot{F}_0(t)| < \infty, \quad \|f_0\|_1 = \|f_0\| + \sup_t |\dot{f}_0(t)| < \infty.$$

Denote by  $u(t, \xi)$  the displacement vector, by  $\varepsilon(t, \xi)$  the strain tensor and by  $\sigma(t, \xi)$  the stress tensor. Let

$$u = u_0 + \Delta u, \quad \varepsilon = \varepsilon_0 + \Delta \varepsilon, \quad \sigma = \sigma_0 + \Delta \sigma,$$

where  $u_0, \varepsilon_0, \sigma_0$  are displacement, strain and stress caused by the basic load and  $\Delta u, \Delta \varepsilon, \Delta \sigma$  are perturbations of the corresponding quantities.

*Definition.* The viscoelastic body is "stable" if for any  $\delta_1 > 0$  there exists such  $\delta_2 > 0$  that the inequality  $\|\Delta F\| + \|\Delta f\| < \delta_2$  yields the estimate  $|\Delta u(t)| < \delta_1, (t \geq 0)$ .

The aim of this paper is to determine the conditions required for the stability of a viscoelastic body.

2. CONSTITUTIVE LAWS

Denote by  $\mathfrak{X}(\xi) < 0$  the moment of creation of the material element in the neighbourhood of the point  $\xi$ . The function  $\mathfrak{X}(\xi)$  is bounded and piece-wise continuous. The constitutive equations of a nonhomogeneous aging viscoelastic material with elastic volume deformation, may be written in the form [5, 7]

$$\bar{\sigma}(t) = 3K\bar{\varepsilon}(t), \quad s(t) = 2G \left[ e(t) - \int_0^t \frac{\partial Q}{\partial \tau}(t - \mathfrak{X}, \tau - \mathfrak{X}) e(\tau) d\tau \right], \tag{2.1}$$

$$\bar{\sigma} = \text{tr } \sigma, \quad \bar{\varepsilon} = \text{tr } \varepsilon, \quad s = \sigma - \bar{\sigma}I/3, \quad e = \varepsilon - \bar{\varepsilon}I/3.$$

Here  $K$  is volume modulus of elasticity (constant),  $G$  is shift modulus (constant),  $Q(t, \tau)$  is a measure of shift relaxation,  $I$  is unit tensor,  $\text{tr}$  is the first invariant of tensor. For simplicity we omit argument  $\xi$ .

Consider only materials with a regular relaxation measure, where the function  $Q(t, \tau)$  is twice continuously differentiable in its arguments. According to (2.1) we may choose the relaxation measure in such a way that for any  $t \geq 0$

$$Q(t, t) = 0. \tag{2.2}$$

The restrictions on the creep measure of aging viscoelastic material were formulated in [5]. Here we formulate the similar restrictions on the relaxation measure and give its interpretation.

Denote by  $s_{ij}, e_{ij}$  the components of the tensors  $s, e$  in the initial configuration. Consider the deformation processes

$$\bar{\varepsilon}^{(1)}(t, t_0) = 0, \quad (t \geq 0); \quad e_{ij}^{(1)}(t, t_0) = 0, \quad (0 \leq t \leq t_0); \quad e_{ij}^{(1)}(t, t_0) = e_{ij}^0, \quad (t > t_0) \tag{2.3}$$

and

$$\bar{\varepsilon}^{(2)}(t, t_0) = 0, \quad e_{ij}^{(2)}(t, t_0) = e_{ij}^0 \delta(t - t_0), \tag{2.4}$$

where  $t_0$  is a fixed time,  $e_{ij}^0 > 0$  are constant and  $\delta(t)$  is the Dirac function.

From (2.1), (2.2) it follows that the components of the stress tensor deviator and its time derivatives are equal to

$$s_{ij}^{(1)}(t, t_0) = \dot{s}_{ij}^{(1)}(t, t_0) = 0, \quad (0 \leq t < t_0);$$

$$s_{ij}^{(1)}(t, t_0) = 2G[1 + Q(t - \mathfrak{X}, t_0 - \mathfrak{X})]e_{ij}^0, \tag{2.5}$$

$$\dot{s}_{ij}^{(1)}(t, t_0) = 2G \partial Q(t - \mathfrak{X}, t_0 - \mathfrak{X})/\partial t e_{ij}^0, \quad (t > t_0);$$

and

$$s_{ij}^{(2)}(t, t_0) = \dot{s}_{ij}^{(2)}(t, t_0) = 0, \quad (0 \leq t < t_0);$$

$$s_{ij}^{(2)}(t, t_0) = -2G \partial Q(t - \mathfrak{X}, t_0 - \mathfrak{X})/\partial \tau e_{ij}^0, \tag{2.6}$$

$$\dot{s}_{ij}^{(2)}(t, t_0) = -2G \partial^2 Q(t - \mathfrak{X}, t_0 - \mathfrak{X})/\partial t \partial \tau e_{ij}^0, \quad (t > t_0).$$

Suppose that

$$\partial Q(t, \tau)/\partial t \leq 0, \quad (0 \leq \tau \leq t). \tag{2.7}$$

By virtue of (2.5) the condition (2.7) means that after loading according to the program (2.3), the stresses do not increase in time.

Now assume that

$$\frac{\partial Q}{\partial t}(t - \mathfrak{X}, \tau - \mathfrak{X}) \geq \frac{\partial Q}{\partial t}(t, \tau), \quad (0 \leq \tau \leq t). \quad (2.8)$$

Equations (2.5), (2.7), (2.8) imply that the rate of relaxation decreases as the age of the sample increases.

From (2.2) and (2.7) it follows that  $Q(t, \tau) \leq 0$  at  $0 \leq \tau \leq t$ . Assume that the condition of limit creep holds [5]. According to this condition the stresses in the sample tend to the fix limit positive values independent of the loading time  $t_0$  as  $t \rightarrow \infty$ . Hence there exists such a constant  $Q_0 \in (0, 1)$  that

$$\begin{aligned} -Q_0 &\leq Q(t, \tau) \leq 0, & (0 \leq \tau \leq t); \\ -Q_0 &= \lim_t Q(t, \tau), & (t \rightarrow \infty, \tau \geq 0). \end{aligned} \quad (2.9)$$

Suppose that

$$\partial Q(t, \tau) / \partial \tau \geq 0, \quad (0 \leq \tau \leq t). \quad (2.10)$$

By virtue of (2.5), (2.10) at  $t_1 > t_0$  the inequality  $s_{ij}^{(1)}(t, t_1) \geq s_{ij}^{(1)}(t, t_0)$  holds. The condition (2.10) means that the stresses in the sample decrease as the loading time increases because the relaxation process has no time to be observed.

Let us assume that

$$\frac{\partial Q}{\partial \tau}(t - \mathfrak{X}, \tau - \mathfrak{X}) \leq \frac{\partial Q}{\partial \tau}(t, \tau), \quad (0 \leq \tau \leq t). \quad (2.11)$$

From (2.6), (2.10), (2.11) it follows that increasing the material age at the loading moment causes a decrease of the absolute stress value.

Suppose that

$$\frac{\partial^2 Q}{\partial t \partial \tau}(t, \tau) \leq 0 \quad (0 \leq \tau \leq t). \quad (2.12)$$

From (2.6), (2.10), (2.12) it follows that under the deformation program (2.4) the absolute value of the stresses decreases in time.

Note that adiabatic loading process conditions (2.7), (2.12) guarantee that the second law of thermodynamics for a viscoelastic body will be valid. These conditions for the difference relaxation kernels were formulated in [8].

Finally, we suppose that there exists a constant  $a > 0$  such that

$$\sup_{t, \tau} [-\partial Q(t, \tau) / \partial t] \leq a, \quad (0 \leq \tau \leq t). \quad (2.13)$$

Relationships (2.5), (2.7), (2.12), (2.13) imply that the rate of stress relaxation in the sample is limited under the deformation program (2.3).

### 3. DETERMINATION OF THE UNPERTURBED STATE

Suppose that the basic load  $(F_0, f_0)$  changes in time slowly and we may neglect the forces of inertia in the analysis of the unperturbed stress state. Furthermore, we assume that under the

action of load  $(F_0, f_0)$  the incremental strains are realized in the body. In this case, tensor  $\varepsilon_0$  is connected with displacement vector  $u_0$  by the relation

$$\varepsilon_0(t) = [\nabla u_0(t) + \nabla u_0'(t)]/2, \tag{3.1}$$

where  $\nabla$  is gradient operator.

Write the equilibrium equation

$$\operatorname{div} \sigma_0(t) + F_0(t) = 0. \tag{3.2}$$

The boundary conditions for the stresses may be written as

$$\sigma_0(t)\mathbf{n} = f_0(t), \quad (\xi \in \Gamma), \tag{3.3}$$

where  $\mathbf{n}$  is the unit outward normal vector to the surface  $\Gamma$ .

Equations (2.1), (3.1)–(3.3) determine the unperturbed stress–strain state in an aging viscoelastic body.

#### 4. THE ENERGY ESTIMATES

Multiply the equality (3.2) by  $u_0(t)$  and integrate over the domain  $\Omega$ . From the received identity, Stoke’s formula and (3.1), (3.3) we find

$$\int_{\Omega} [\frac{1}{3}\bar{\sigma}_0(t)\bar{\varepsilon}_0(t) + s_0(t):e_0(t)] dv = \int_{\Omega} F_0'(t)u_0(t) dv + \int_{\Gamma} f_0'(t)u_0(t) ds, \tag{4.1}$$

where  $s_0:e_0$  is a convolution of tensors.

According to (2.1), (4.1) we get

$$J_0^2(t) = 2G \int_{\Omega} e_0(t): \left[ \int_0^t \frac{\partial Q}{\partial \tau}(t - \mathfrak{X}, \tau - \mathfrak{X})e_0(\tau) d\tau \right] dv + A_0(t) + B_0(t), \tag{4.2}$$

where

$$J_0^2(t) = \int_{\Omega} [K\bar{\varepsilon}_0^2(t) + 2Ge_0(t):e_0(t)] dv,$$

$$A_0(t) = \int_{\Omega} F_0'(t)u_0(t) dv, \quad B_0(t) = \int_{\Gamma} f_0'(t)u_0(t) ds.$$

Estimate the term in the right-hand part of (4.2) with the aid of (2.11) and the Cauchy inequality

$$\left| 2G \int_{\Omega} e_0(t): \left[ \int_0^t \frac{\partial Q}{\partial \tau}(t - \mathfrak{X}, \tau - \mathfrak{X})e_0(\tau) d\tau \right] dv \right| \leq J_0(t) \int_0^t \frac{\partial Q}{\partial \tau}(t, \tau)J_0(\tau) d\tau, \tag{4.3}$$

$$|A_0(t)| \leq |F_0(t)||u_0(t)|, \quad |B_0(t)| \leq |f_0(t)| \left[ \int_{\Gamma} u_0'(t)u_0(t) ds \right]^{1/2}.$$

For any function  $u_0(t, \xi)$ , which satisfies the condition of nondisplacement of the domain  $\Omega$  as a rigid body, there is a constant  $c_1 > 0$  such that [4]

$$|u_0(t)|^2 \leq c_1^2 J_0^2(t), \tag{4.4}$$

and a constant  $C_2 > 0$  such that [4, 9]

$$\int_{\Gamma} u'_0(t)u_0(t) \, ds \leq c_2^2 J_0^2(t). \quad (4.5)$$

From (4.2)–(4.5) we get

$$J_0(t) \leq \int_0^t \frac{\partial Q}{\partial \tau}(t, \tau) J_0(\tau) \, d\tau + c_1 |F_0(t)| + c_2 |f_0(t)|. \quad (4.6)$$

Let  $J_{01}(t) = \sup_{\tau} J_0(\tau)$ , ( $0 \leq \tau \leq t$ ). By virtue of (2.2), (4.6) we obtain

$$[1 + Q(t, 0)]J_{01}(t) \leq c_1 \|F_0\| + c_2 \|f_0\|.$$

This inequality and (2.9) yield

$$\sup_t J_0(t) \leq (c_1 \|F_0\| + c_2 \|f_0\|)(1 - Q_0)^{-1}. \quad (4.7)$$

According to (2.1) we get

$$\begin{aligned} \alpha_0^2(t) &= \int_{\Omega} \sigma_0(t) : \sigma_0(t) \, dv \\ &= \int_{\Omega} \left\{ 3K^2 \bar{\varepsilon}_0^2(t) + 4G^2 \left[ e_0(t) : e_0(t) - 2e_0(t) : \int_0^t \frac{\partial Q}{\partial \tau}(t - \mathfrak{X}, \tau - \mathfrak{X}) e_0(\tau) \, d\tau \right. \right. \\ &\quad \left. \left. + \int_0^t \int_0^t \frac{\partial Q}{\partial \tau_1}(t - \mathfrak{X}, \tau_1 - \mathfrak{X}) \frac{\partial Q}{\partial \tau_2}(t - \mathfrak{X}, \tau_2 - \mathfrak{X}) e_0(\tau_1) : e_0(\tau_2) \, d\tau_1 \, d\tau_2 \right] \right\} \, dv. \end{aligned}$$

From (2.2), (2.9), (2.11) and the Cauchy inequality it follows that

$$\begin{aligned} \alpha_0^2 &\leq \max(3K, 2G) J_0^2(t) + 4G \int_0^t \frac{\partial Q}{\partial \tau}(t, \tau) J_0(t) J_0(\tau) \, d\tau + 2G \left[ \int_0^t \frac{\partial Q}{\partial \tau}(t, \tau) J_0(\tau) \, d\tau \right]^2 \\ &\leq [\max(3K, 2G) + 4GQ_0 + 2GQ_0^2] \sup_t J_0^2(t). \end{aligned} \quad (4.8)$$

Denote

$$\begin{aligned} u_1(t) &= \dot{u}_0(t), & \varepsilon_1(t) &= \dot{\varepsilon}_0(t), & \sigma_1(t) &= \dot{\sigma}_0(t), \\ \bar{\varepsilon}_1(t) &= \dot{\bar{\varepsilon}}_0(t), & \bar{\sigma}_1(t) &= \dot{\bar{\sigma}}_0(t), & s_1(t) &= \dot{s}_0(t). \end{aligned}$$

According to (2.1), (3.1)–(3.3) we obtain

$$\bar{\sigma}_1(t) = 3K\bar{\varepsilon}_1(t),$$

$$s_1(t) = 2G \left[ e_1(t) + \int_0^t \frac{\partial Q}{\partial t}(t - \mathfrak{X}, \tau - \mathfrak{X}) e_1(\tau) \, d\tau + \frac{\partial Q}{\partial t}(t - \mathfrak{X}, -\mathfrak{X}) e_0(0) \right]; \quad (4.9)$$

$$\varepsilon_1(t) = [\nabla u_1(t) + \nabla u'_1(t)]/2; \quad (4.10)$$

$$\operatorname{div} \sigma_1(t) + \dot{F}_0(t) = 0; \quad (4.11)$$

$$\sigma_1(t)\mathbf{n} = \dot{f}_0(t), \quad (\xi \in \Gamma). \quad (4.12)$$

Multiply (4.11) by  $u_1(t)$  and integrate it at the domain  $\Omega$ . Using Stoke's formula and (4.10), (4.12) we get

$$\int_{\Omega} [\frac{1}{3}\bar{\sigma}_1(t)\bar{\epsilon}_1(t) + s_1(t): e_1(t)] dv = A_1(t) + B_1(t), \tag{4.13}$$

where

$$A_1(t) = \int_{\Omega} \dot{F}_0'(t)u_1(t) dv, \quad B_1(t) = \int_{\Gamma} \dot{f}_0'(t)u_1(t) ds.$$

In accordance with (4.9), (4.13) we find

$$J_1^2(t) = -2G \int_{\Omega} \left[ \frac{\partial Q}{\partial t}(t - \mathfrak{X}, -\mathfrak{X})e_0(0) + \int_0^t \frac{\partial Q}{\partial t}(t - \mathfrak{X}, \tau - \mathfrak{X})e_1(\tau) d\tau \right] : e_1(t) dv + A_1(t) + B_1(t), \tag{4.14}$$

where

$$J_1^2(t) = \int_{\Omega} [K\bar{\epsilon}_1^2(t) + 2Ge_1(t): e_1(t)] dv.$$

Now estimate the expression in the right-hand part of (4.14) using (2.7), (2.8) and the Cauchy inequality

$$\begin{aligned} & \left| 2G \int_{\Omega} \left[ \frac{\partial Q}{\partial t}(t - \mathfrak{X}, -\mathfrak{X})e_0(0) + \int_0^t \frac{\partial Q}{\partial t}(t - \mathfrak{X}, \tau - \mathfrak{X})e_1(\tau) d\tau \right] : e_1(t) dv \right| \\ & \leq - \left[ \frac{\partial Q}{\partial t}(t, 0)J_0(0) + \int_0^t \frac{\partial Q}{\partial t}(t, \tau)J_1(\tau) d\tau \right] J_1(t), \end{aligned} \tag{4.15}$$

$$|A_1(t)| \leq |\dot{F}_0(t)||u_1(t)|, \quad |B_1(t)| \leq |\dot{f}_0(t)| \left[ \int_{\Gamma} u_1'(t)u_1(t) ds \right]^{1/2}.$$

Relationships (4.4), (4.5), (4.14) and (4.15) yield

$$J_1(t) \leq -\frac{\partial Q}{\partial t}(t, 0)J_0(0) - \int_0^t \frac{\partial Q}{\partial t}(t, \tau)J_1(\tau) d\tau + c_1|\dot{F}_0(t)| + c_2|\dot{f}_0(t)|. \tag{4.16}$$

Integrate inequality (4.16) by  $t$  from zero to infinity. With the aid of (2.2), (2.9) we have

$$\begin{aligned} \int_0^{\infty} J_1(t) dt & \leq Q_0J_0(0) - \int_0^{\infty} J_1(\tau) d\tau \int_0^{\infty} \frac{\partial Q}{\partial t}(t, \tau) dt + c_1\|F_0\| + c_2\|f_0\| \\ & \leq Q_0 \left[ J_0(0) + \int_0^{\infty} J_1(t) dt \right] + c_1\|F_0\| + c_2\|f_0\|. \end{aligned}$$

From this inequality it follows that

$$\int_0^{\infty} J_1(t) dt \leq [Q_0J_0(0) + c_1\|F_0\| + c_2\|f_0\|](1 - Q_0)^{-1}. \tag{4.17}$$

Using (2.7), (2.12), (2.13) and (4.16) we get

$$J_1(t) \leq a \left[ J_0(0) + \int_0^\infty J_1(\tau) d\tau \right] + c_1 \|F_0\|_1 + c_2 \|f_0\|_1. \tag{4.18}$$

Using (2.8), (4.9) and the Cauchy inequality, we find

$$\alpha_1^2(t) = \int_\Omega \sigma_1(t) : \sigma_1(t) dv \leq 3KJ_1^2(t) + 2G \left[ J_1(t) - \int_0^t \frac{\partial Q}{\partial t}(t, \tau) J_1(\tau) d\tau - \frac{\partial Q}{\partial t}(t, 0) J_0(0) \right]^2.$$

This relation and (2.7) imply

$$\alpha_1(t) \leq c_3 \left[ J_1(t) - \int_0^t \frac{\partial Q}{\partial t}(t, \tau) J_1(\tau) d\tau - \frac{\partial Q}{\partial t}(t, 0) J_0(0) \right], \tag{4.19}$$

where  $c_3^2 = 3K + 2G$ . Integrating (4.19) from zero to infinity and using (2.2), (2.9), we find

$$\int_0^\infty \alpha_1(t) dt \leq c_3 \left[ (1 + Q_0) \int_0^\infty J_1(t) dt + Q_0 J_0(0) \right]. \tag{4.20}$$

From (2.7), (2.12), (2.13), (4.19) it follows that

$$\alpha_1(t) \leq c_3 \left[ J_1(t) + a \left( J_0(0) + \int_0^\infty J_1(t) dt \right) \right]. \tag{4.21}$$

By virtue of (4.7), (4.8), (4.20), (4.21) it follows that there exists a constant  $c > 0$  such that

$$\sup_t \alpha_0(t) \leq c(\|F_0\|_1 + \|f_0\|_1), \quad \sup_t \alpha_1(t) \leq c(\|F_0\|_1 + \|f_0\|_1), \tag{4.22}$$

$$\int_0^\infty \alpha_1(t) dt \leq c(\|F_0\|_1 + \|f_0\|_1).$$

### 5. STABILITY OF VISCOELASTIC BODY UNDER DYNAMIC PERTURBATIONS

Suppose that the forces  $\Delta F$ ,  $\Delta f$  depend on time. Then at the investigation of perturbed state we need to take inertia forces into consideration. Write equations of the body element movement and boundary conditions in terms of stress, neglecting the second order terms. According to [2] we get

$$\rho \Delta \ddot{u}(t) = \text{div}[\Delta \sigma(t) + \sigma_0(t) \nabla \Delta u(t)] + \Delta F(t); \tag{5.1}$$

$$[\Delta \sigma(t) + (\sigma_0(t) \nabla \Delta u(t))]' \mathbf{n} = \Delta f(t), \tag{5.2}$$

where  $\rho$  is the density of the material.

Perturbation of the stress tensor  $\Delta \sigma(t)$  is connected with perturbation of the strain tensor by the constitutive laws

$$\Delta \bar{\sigma}(t) = 3K \Delta \bar{\varepsilon}(t),$$

$$\Delta s(t) = 2G \left[ \Delta e(t) - \int_0^t \frac{\partial Q}{\partial \tau}(t - \mathfrak{X}, \tau - \mathfrak{X}) \Delta e(\tau) d\tau \right]. \tag{5.3}$$

Perturbation of the strain tensor  $\Delta\epsilon(t)$  may be expressed by the formula

$$\Delta\epsilon(t) = [\nabla\Delta u(t) + \nabla\Delta u'(t)]/2. \tag{5.4}$$

Equations (5.1)–(5.4) with the initial data

$$\Delta u(0) = 0, \quad \Delta \dot{u}(0) = 0 \tag{5.5}$$

determine the perturbed stress–strain state in a nonhomogeneous aging viscoelastic body. For the analysis of stability we use the modified second Liapunov method (e.g. [10]).

Introduce the functional

$$E(t) = \int_{\Omega} \left\{ \frac{1}{2} \rho \Delta \dot{u}'(t) \Delta \dot{u}(t) + \frac{1}{2} K \Delta \bar{\epsilon}^2(t) + G \left[ (1 + Q(t - \mathfrak{X}, -\mathfrak{X})) \Delta e(t) : \Delta e(t) + \int_0^t \frac{\partial Q}{\partial \tau} (t - \mathfrak{X}, \tau - \mathfrak{X}) (\Delta e(t) - \Delta e(\tau)) : (\Delta e(t) - \Delta e(\tau)) d\tau \right] \right\} dv.$$

From this formula and (2.9) it follows that

$$E(t) \geq J(t) = \int_{\Omega} \left[ \frac{1}{2} K \Delta \bar{\epsilon}^2(t) + G(1 - Q_0) \Delta e(t) : \Delta e(t) \right] dv. \tag{5.6}$$

Calculate the derivative of the functional  $E(t)$

$$\begin{aligned} \dot{E}(t) = \int_{\Omega} \left\{ \rho \Delta \dot{u}'(t) \Delta \ddot{u}(t) + K \Delta \bar{\epsilon}(t) \Delta \dot{\bar{\epsilon}}(t) + 2G \left[ \Delta e(t) - \int_0^t \frac{\partial Q}{\partial \tau} (t - \mathfrak{X}, \tau - \mathfrak{X}) \Delta e(\tau) d\tau \right] : \Delta \dot{e}(t) \right\} dv + R(t), \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} R(t) = G \int_{\Omega} \left[ \frac{\partial Q}{\partial t} (t - \mathfrak{X}, -\mathfrak{X}) \Delta e(t) : \Delta e(t) + \int_0^t \frac{\partial^2 Q}{\partial t \partial \tau} (t - \mathfrak{X}, \tau - \mathfrak{X}) (\Delta e(t) - \Delta e(\tau)) : (\Delta e(t) - \Delta e(\tau)) d\tau \right] dv. \end{aligned}$$

From (2.7), (2.12) we get

$$R(t) \leq 0. \tag{5.8}$$

Transform expression (5.7), using (5.1)–(5.4)

$$\dot{E}(t) = \int_{\Omega} \Delta F'(t) \Delta \dot{u}(t) dv + \int_{\Gamma} \Delta f'(t) \Delta \dot{u}(t) ds + R(t) - \int_{\Omega} \text{tr}[(\nabla \Delta \dot{u}(t))' \sigma_0(t) \nabla \Delta u(t)] dv.$$

It is clear that

$$\int_{\Omega} \text{tr}[(\nabla \Delta \dot{u}(t))' \sigma_0(t) \nabla \Delta u(t)] dv = -\dot{D}_0(t) - D_1(t),$$

where

$$D_0(t) = -\frac{1}{2} \int_{\Omega} \text{tr}[\nabla \Delta u'(t) \sigma_0(t) \nabla \Delta u(t)] \, dv,$$

$$D_1(t) = \frac{1}{2} \int_{\Omega} \text{tr}[\nabla \Delta u'(t) \dot{\sigma}_0(t) \nabla \Delta u(t)] \, dv.$$

Hence

$$\dot{E}(t) = \dot{D}_0(t) + D_1(t) + \int_{\Omega} \Delta F'(t) \Delta \dot{u}(t) \, dv + \int_{\Gamma} \Delta f'(t) \Delta \dot{u}(t) \, ds + R(t). \tag{5.9}$$

Integrate this equality from zero to  $t$ . With the aid of (5.5), (5.8) we obtain

$$E(t) \leq D_0(t) + \int_0^t D_1(\tau) \, d\tau + A(t) + B(t), \tag{5.10}$$

where

$$A(t) = \int_{\Omega} \left[ \Delta F'(t) \Delta u(t) - \int_0^t \Delta \dot{F}'(\tau) \Delta u(\tau) \, d\tau \right] \, dv,$$

$$B(t) = \int_{\Gamma} \left[ \Delta f'(t) \Delta u(t) - \int_0^t \Delta \dot{f}'(\tau) \Delta u(\tau) \, d\tau \right] \, ds.$$

Denote by  $\mathcal{U}$  the set of continuously differentiable in  $\Omega$  displacement fields  $w(\zeta)$ , which satisfy the condition of lack of the displacement of the domain  $\Omega$  as a rigid body. For any  $w \in \mathcal{U}$  denote the corresponding incremental strain field by  $\varepsilon_w$ . Let

$$\bar{\varepsilon}_w = \text{tr} \, \varepsilon_w, \quad e_w = \varepsilon_w - \bar{\varepsilon}_w I/3.$$

Denote by  $\lambda_0(t)$ ,  $\lambda_1(t)$  the values

$$\lambda_0(t) = \sup_w \left| \int_{\Omega} \text{tr}(\nabla w' \sigma_0(t) \nabla w) \, dv \right|$$

$$\times \left\{ \int_{\Omega} [K \bar{\varepsilon}_w^2 + 2G(1 + Q(t - \mathfrak{X}, -\mathfrak{X})) e_w : e_w] \, dv \right\}^{-1}, \tag{5.11}$$

$$\lambda_1(t) = \sup_w \left| \int_{\Omega} \text{tr}(\nabla w' \dot{\sigma}_0(t) \nabla w) \, dv \right|$$

$$\times \left\{ \int_{\Omega} [K \bar{\varepsilon}_w^2 + 2G(1 + Q(t - \mathfrak{X}, -\mathfrak{X})) e_w : e_w] \, dv \right\}^{-1}.$$

From (5.11) it follows that

$$|D_0(t)| \leq \lambda_0(t) E(t), \quad |D_1(t)| \leq \lambda_1(t) E(t). \tag{5.12}$$

According to (4.4), (4.5) and (5.6) we get

$$\int_{\Omega} \Delta u'(t) \Delta u(t) \, dv \leq c_1^2 c_4^2 E(t), \quad \int_{\Gamma} \Delta u'(t) \Delta u(t) \, ds \leq c_2^2 c_4^2 E(t), \tag{5.13}$$

where  $c_4^2 = 2(1 - Q_0)^{-1}$ . Estimate  $A(t)$ ,  $B(t)$  using the Cauchy inequality and (5.13),

$$|A(t)| \leq c_1 c_4 \|\Delta F\| E_1^{1/2}(t), \quad |B(t)| \leq c_2 c_4 \|\Delta f\| E_1^{1/2}(t), \tag{5.14}$$

where  $E_1(t) = \sup_{\tau \in [0, t]} E(\tau)$ ,  $(0 \leq \tau \leq t)$ .

Inequalities (5.10), (5.12), (5.14) imply

$$E(t) \leq \lambda_0(t)E(t) + \int_0^t \lambda_1(\tau)E(\tau) \, d\tau + c_4 E_1^{1/2}(t)(c_1 \|\Delta F\| + c_2 \|\Delta f\|). \tag{5.15}$$

**THEOREM 1.** Suppose that

$$\sup_t \left[ \lambda_0(t) + \int_0^t \lambda_1(\tau) \, d\tau \right] \leq \beta < 1, \quad (t \geq 0). \tag{5.16}$$

Then the viscoelastic body is stable.

*Proof.* According to (5.15), (5.16) we have

$$E(t) \leq \beta E_1(t) + c_4 E_1^{1/2}(t)(c_1 \|\Delta F\| + c_2 \|\Delta f\|).$$

This inequality yields

$$E_1^{1/2}(t) \leq c_4(c_1 \|\Delta F\| + c_2 \|\Delta f\|)(1 - \beta)^{-1}. \tag{5.17}$$

The theorem assertion follows from (5.13), (5.17).

*Remark.* Suppose that the basic load does not depend on time and the corresponding principle [5] holds. Then the stress tensor  $\sigma_0$  is also independent of time and  $\lambda_1(t) = 0$ . By virtue of (2.9), (5.11) the stability condition (5.16) may be written in the following form

$$\sup_w \left| \int_{\Omega} \text{tr}(\nabla w' \sigma_0 \nabla w) \, dv \right| \left\{ \int_{\Omega} [K \bar{\epsilon}_w^2 + 2G(1 - Q_0) e_w : e_w] \, dv \right\}^{-1} \leq \beta < 1. \tag{5.18}$$

Inequality (5.18) coincides with the condition of stability of a nonhomogeneous aging viscoelastic body under quasistatic loading [11].

For nonstationary external forces the stress  $\sigma_0$  depends on the material age  $\mathfrak{X}(\xi)$  and we cannot apply the limit moduli theory for the determination of the critical loads [6]. In this case the condition of the theorem 1 is not sufficient. However, one can prove the stronger assertion, in which the sufficient conditions of stability are close to necessary ones.

**THEOREM 2.** Suppose that

$$\Lambda = \int_0^{\infty} \lambda_1(t) \, dt < \infty \tag{5.19}$$

and

$$\sup_t \lambda_0(t) \leq \beta < 1. \tag{5.20}$$

Then the viscoelastic body is stable.

*Proof.* Fix arbitrary positive value  $\gamma \in (0, (1 - \beta)/2)$ . From (5.19) it follows that there exists such  $t_1(\gamma)$  that

$$\int_{t_1}^{\infty} \lambda_1(t) dt < \gamma. \tag{5.21}$$

Choose arbitrary time  $t > t_1$ . Integrating (5.9) from  $t_1$  to  $t$  and using (5.8) we get

$$E(t) - E(t_1) \leq D_0(t) - D_0(t_1) + \int_{t_1}^t D_1(\tau) d\tau + A_2(t) + B_2(t), \tag{5.22}$$

where

$$A_2(t) = \int_{\Omega} \left[ \Delta F'(t) \Delta u(t) - \Delta F'(t_1) \Delta u(t_1) - \int_{t_1}^t \Delta \dot{F}'(\tau) \Delta u(\tau) d\tau \right] dv,$$

$$B_2(t) = \int_{\Gamma} \left[ \Delta f'(t) \Delta u(t) - \Delta f'(t_1) \Delta u(t_1) - \int_{t_1}^t \Delta \dot{f}'(\tau) \Delta u(\tau) d\tau \right] ds.$$

Reinforce inequality (5.22) by the aid of (5.12)

$$E(t) \leq \lambda_0(t)E(t) + [1 + \lambda_0(t_1)]E(t_1) + \int_{t_1}^t \lambda_1(\tau)E(\tau) d\tau + |A_2(t)| + |B_2(t)|.$$

This relation and (5.20) imply

$$(1 - \beta)E(t) \leq (1 + \beta)E(t_1) + E_2(t) \int_{t_1}^t \lambda_1(\tau) d\tau + |A_2(t)| + |B_2(t)|, \tag{5.23}$$

where  $E_2(t) = \sup_{t_1 \leq \tau \leq t} E(\tau)$ .

Equations (5.14), (5.23) yield

$$(1 - \beta)E_2(t) \leq (1 + \beta)E(t_1) + E_2(t) \int_{t_1}^{\infty} \lambda_1(\tau) d\tau + c_5 E_2^{1/2}(t)(c_1 \|\Delta F\| + c_2 \|\Delta f\|),$$

where  $c_5 = 2c_4$ .

Transform this inequality using (5.21)

$$(1 - \beta - \gamma)E_2(t) \leq (1 + \beta)E(t_1) + c_5(c_1 \|\Delta F\| + c_2 \|\Delta f\|)E_2^{1/2}(t). \tag{5.24}$$

From (5.24) it follows that

$$E_2^{1/2}(t) \leq c_5(c_1 \|\Delta F\| + c_2 \|\Delta f\|)(2(1 - \beta - \gamma))^{-1} + \{[c_5(c_1 \|\Delta F\| + c_2 \|\Delta f\|)(2(1 - \beta - \gamma))^{-1}]^2 + (1 + \beta)(1 - \beta - \gamma)^{-1}E(t_1)\}^{1/2}. \tag{5.25}$$

The restriction on the value  $\gamma$  implies  $(1 - \beta - \gamma)^{-1} < 2(1 - \beta)^{-1}$ . Rewriting estimate (5.25) we find

$$E_2^{1/2}(t) \leq c_5(c_1 \|\Delta F\| + c_2 \|\Delta f\|)(1 - \beta)^{-1} + \{[c_5(c_1 \|\Delta F\| + c_2 \|\Delta f\|)(1 - \beta)^{-1}]^2 + 2(1 + \beta)(1 - \beta)^{-1}E(t_1)\}^{1/2}.$$

From this relation and the Cauchy inequality we obtain

$$E_2(t) \leq 4\{c_5(c_1\|\Delta F\| + c_2\|\Delta f\|)(1 - \beta)^{-1}\}^2 + (1 + \beta)(1 - \beta)^{-1}E(t_1). \tag{5.26}$$

For the estimation of  $E(t_1)$  we use formula (5.15), which is valid on interval  $[0, t_1]$ . According to (5.20), we get

$$E(t) \leq \beta E(t) + \int_0^t \lambda_1(\tau)E(\tau) d\tau + c_4(c_1\|\Delta F\| + c_2\|\Delta f\|)E_1^{1/2}(t_1).$$

This inequality implies

$$E(t) \leq \left[ \int_0^t \lambda_1(\tau)E(\tau) d\tau + c_4(c_1\|\Delta F\| + c_2\|\Delta f\|)E_1^{1/2}(t_1) \right] (1 - \beta)^{-1}.$$

According to Bellman's inequality

$$\begin{aligned} E(t) &\leq \frac{c_4}{1 - \beta} E_1^{1/2}(t_1)(c_1\|\Delta F\| + c_2\|\Delta f\|) \left\{ 1 + \int_0^t \frac{\lambda_1(\tau)}{1 - \beta} \exp \left[ \int_\tau^t \frac{\lambda_1(\tau_1)}{1 - \beta} d\tau_1 \right] d\tau \right\} \\ &\leq \frac{c_4}{1 - \beta} E_1^{1/2}(t_1)(c_1\|\Delta F\| + c_2\|\Delta f\|) \left[ 1 + \frac{\Lambda}{1 - \beta} \exp \left( \frac{\Lambda}{1 - \beta} \right) \right]. \end{aligned}$$

From this relation and (5.19) we get

$$E_1(t_1) \leq \left[ \frac{c_4}{1 - \beta} (c_1\|\Delta F\| + c_2\|\Delta f\|) \right]^2 \left[ 1 + \frac{\Lambda}{1 - \beta} \exp \left( \frac{\Lambda}{1 - \beta} \right) \right]^2. \tag{5.27}$$

The assertion of theorem 2 follows from (5.13), (5.26) and (5.27).

According to (5.18) and theorem 2, if inequality (5.19) holds, then sufficient conditions of stability of the body at quasistatic and dynamic approach coincide. In the common case we cannot prove that the coefficient  $\Lambda$  is limited. However, for thin-walled elements of constructions one can prove the corresponding estimate. Here we obtain such an estimate for the problem of stability of a viscoelastic plate.

### 6. STABILITY OF A NONHOMOGENEOUS AGING VISCOELASTIC PLATE

Consider viscoelastic plate of constant thickness  $h$ . Introduce Cartesian coordinate system  $x = (x_i)$ , whose axes  $x_1$  and  $x_2$  lie in the middle plane and axis  $x_3$  is perpendicular to the middle plane of the plate in its undeformed state. The plate occupies the domain  $\Omega = \Omega_0 \times [-h/2, h/2]$ , where  $\Omega_0$  is the domain on the plate  $(x_1, x_2)$  with Lipschitz boundary  $\Gamma_0$ .

Suppose that the moment of creation of material elements  $\mathfrak{X}$  does not depend on  $x_3$  and under the action of the load  $(F_0, f_0)$  the generated plane stress state is realized in the plate

$$\begin{aligned} \sigma_{0ij} &= \sigma_{0ij}(t, x_1, x_2), \quad (i, j = 1, 2), \\ \sigma_{013} &= \sigma_{023} = \sigma_{033} = 0. \end{aligned}$$

For a thin plate we have  $h/r \ll 1$  where  $r$  is a characteristic dimension of the domain  $\Omega_0$ .

According to [13] the admissible displacement fields  $w(x)$  may be written in the following form

$$w_1 = -x_3 y_{,1}, \quad w_2 = -x_3 y_{,2}, \quad w_3 = y, \quad (6.1)$$

where  $y(x_1, x_2)$  is the deflection of the plate,  $y_{,i} = \partial y / \partial x_i$ .

For simplicity we assume that the boundary of the plate is stringently fixed

$$y = 0, \quad y_{,1} = y_{,2} = 0, \quad (x_1, x_2) \in \Gamma_0. \quad (6.2)$$

Neglecting the terms of the higher order by  $h/r$  we find with the help of (5.11), (6.1), (6.2)

$$\lambda_0(t) = \sup_y M_0(t)/N(t), \quad \lambda_1(t) = \sup_y M_1(t)/N(t), \quad (6.3)$$

where

$$\begin{aligned} M_0(t) &= h \int_{\Omega_0} (\sigma_{011} y_{,1}^2 + 2\sigma_{012} y_{,1} y_{,2} + \sigma_{022} y_{,2}^2) dx_1 dx_2, \\ M_1(t) &= h \int_{\Omega_0} (\dot{\sigma}_{011} y_{,1}^2 + 2\dot{\sigma}_{012} y_{,1} y_{,2} + \dot{\sigma}_{022} y_{,2}^2) dx_1 dx_2, \\ N(t) &= \frac{h^3}{12} \int_{\Omega_0} \left[ K(y_{,11} + y_{,22})^2 + \frac{4G}{3}(1 + Q(t - \mathfrak{X}, -\mathfrak{X})) \right. \\ &\quad \left. \times (y_{,11}^2 + y_{,22}^2 - y_{,11} y_{,22} + 3y_{,12}^2) \right] dx_1 dx_2. \end{aligned} \quad (6.4)$$

From (2.9) and (6.4) it follows that

$$N(t) \geq \frac{h^3}{12} \min[3K, 2G(1 - Q_0)] \int_{\Omega_0} (y_{,11}^2 + 2y_{,12}^2 + y_{,22}^2) dx_1 dx_2. \quad (6.5)$$

Estimate  $M_0(t)$ ,  $M_1(t)$  using the Cauchy inequality

$$\begin{aligned} M_0(t) &\leq h \left\{ \int_{\Omega_0} [\sigma_{011}^2(t) + 2\sigma_{012}^2(t) + \sigma_{022}^2(t)] dx_1 dx_2 \right\}^{1/2} \left[ \int_{\Omega_0} (y_{,1}^4 + 2y_{,1}^2 y_{,2}^2 + y_{,2}^4) dx_1 dx_2 \right]^{1/2} \\ &= h^{1/2} \left[ \int_{\Omega} \sigma_0(t) : \sigma_0(t) dv \right]^{1/2} Y = h^{1/2} \alpha_0(t) Y, \\ M_1(t) &\leq h \left\{ \int_{\Omega_0} [(\dot{\sigma}_{011}(t))^2 + 2(\dot{\sigma}_{012}(t))^2 + (\dot{\sigma}_{022}(t))^2] dx_1 dx_2 \right\}^{1/2} \\ &\quad \times \left[ \int_{\Omega_0} (y_{,1}^4 + 2y_{,1}^2 y_{,2}^2 + y_{,2}^4) dx_1 dx_2 \right]^{1/2} \\ &= h^{1/2} \left[ \int_{\Omega} \dot{\sigma}_0(t) : \dot{\sigma}_0(t) dv \right]^{1/2} Y = h^{1/2} \alpha_1(t) Y, \end{aligned} \quad (6.6)$$

where

$$Y^2 = \int_{\Omega_0} (y_{,1}^4 + 2y_{,1}^2 y_{,2}^2 + y_{,2}^4) dx_1 dx_2.$$

By virtue of [9] there exists a constant  $c_6 > 0$  such that for any function  $y(x_1, x_2)$ , which satisfies to the boundary conditions (6.2) we get

$$Y \leq c_6 \int_{\Omega_0} (y_{,11}^2 + 2y_{,12}^2 + y_{,22}^2) dx_1 dx_2.$$

According to this inequality and (4.22), (6.3), (6.5), (6.6) there exists such a constant  $c_7 > 0$  that

$$\lambda_0(t) \leq c_7(\|F_0\|_1 + \|f_0\|_1), \quad \Lambda = \int_0^\infty \lambda_1(t) dt \leq c_7(\|F_0\|_1 + \|f_0\|_1). \quad (6.7)$$

From (6.7) and theorem 2 it follows that inequality (5.20) is a sufficient condition of the dynamic stability of the viscoelastic plate. This condition coincides with the stability condition of viscoelastic plate under quasistatic loading, which has been obtained in [12].

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