

Optimal Feedback Control for a Semilinear Evolution Equation¹

M. I. KAMENSKII,² P. NISTRÌ,³ V. V. OBUKHOVSKII,⁴ AND P. ZECCA⁵

Communicated by R. Conti

Abstract. In this paper, we consider a minimization problem of a cost functional associated to a nonlinear evolution feedback control system with a given boundary condition which includes the periodic one as a particular case. Specifically, by using an existence result for a system of inclusions involving noncompact operators (see Ref. 1), we first prove that the solution set of our problem is nonempty. Then, from the topological properties of this set, we derive the existence of a solution of the minimization problem under consideration.

Key Words. Evolution problems in Banach spaces, optimal feedback controls, measure of noncompactness, topological degree.

1. Introduction

In this paper, we will consider the following semilinear feedback control system:

$$y'_i(t) = A_i(t)y_i(t) + f_i(t, y_1(t), \dots, y_n(t), u(t)), \quad (1a)$$

$$y_i(t) \in X_i, \quad i = 1, \dots, n, \quad t \in [0, 1], \quad (1b)$$

$$u(t) \in V(t, y_1(t), \dots, y_n(t)) \subset Z, \quad t \in [0, 1] \quad (1c)$$

¹This research was supported in part by the Research Project MURST (40%) "Teoria del Controllo dei Sistemi Dinamici" and by a CNR Bilateral Project. The authors are grateful to Prof. B. D. Gel'man for helpful discussions.

²Assistant Professor, Department of Mathematics, Voronezh State University, Voronezh, Russia.

³Associate Professor, Dipartimento di Sistemi e Informatica, Università di Firenze, Firenze, Italy.

⁴Professor, Department of Physics and Mathematics, Voronezh State Pedagogical University, Voronezh, Russia.

⁵Associate Professor, Dipartimento di Sistemi e Informatica, Università di Firenze, Firenze, Italy.

Here, X_1, \dots, X_n, Z are Banach spaces; the spaces $X_i, i = 1, 2, \dots, n$, are assumed to be separable; $\{A_i(t)\}_{t \in [0,1]}$, for each $i = 1, 2, \dots, n$, is a family of linear operators in X_i ; $f_i: [0, 1] \times X_1 \times X_2 \cdots \times X_n \times Z \rightarrow X_i, i = 1, 2, \dots, n$, are nonlinear maps; and $V: [0, 1] \times X_1 \times X_2 \cdots \times X_n \multimap Z$ is a feedback multivalued map.

Let $F_i: [0, 1] \times X_1 \times X_2 \cdots \times X_n \multimap X_i, i = 1, 2, \dots, n$, be given by

$$F_i(t, x_1, \dots, x_n) = f_i(t, x_1, \dots, x_n, V(t, x_1, \dots, x_n)).$$

Given $\mathcal{X}: X \times X \multimap X$ upper semicontinuous, acyclic-valued, compact multivalued mapping, $X = X_1 \times X_2 \cdots \times X_n$, under suitable assumptions on A_i, f_i, V , and F_i that will be specified in Section 2, we will show (Theorem 2.4) the existence of a ball $\bar{B}(0, r) \subset X$ and of a mild solution $y(t) = (y_1(t), y_2(t), \dots, y_n(t))$ of the system (1), with corresponding control $u(t) \in V(t, y_1(t), \dots, y_n(t))$, which satisfies the boundary conditions

$$y(0) \in \bar{B}(0, r), \tag{2a}$$

$$y(1) \in y(0) + \mathcal{X}(y(0), y(1)), \tag{2b}$$

and

$$j(y) = \inf_{\bar{y} \in Q} j(\bar{y}), \tag{3}$$

where Q is the set of all mild solutions of the system (1) satisfying the conditions (2) and j is a lower semicontinuous functional.

The paper is organized as follows. In Section 2, we give some definitions and preliminary results which will be used in the sequel. Section 3 contains the main result (Theorem 3.4) as well as several related results. Theorem 3.4 is based on an abstract existence result on the solvability of a system of inclusions in Banach spaces (Theorem 2.2 below), which is an extension of Theorem 2.3 of Ref. 1. Specifically, in order to apply this abstract result, we first reduce the nonlinear boundary-value control problem (1)–(2) to a system of inclusions,

$$y \in \bar{\Gamma}(x, y), \quad x \in \bar{G}(x, y),$$

where $\bar{\Gamma}$ and \bar{G} are suitable multivalued maps associated to (1) and (2), respectively, $y \in Y = \subset([0, 1], X_1)X \cdots X \subset ([0, 1], X_n)$ is the trajectory $y(t), t \in [0, 1]$, and $x \in X$ is its initial condition $y(0)$. Then, we solve the first inclusion with respect to y considering x as a fixed parameter in a suitable ball $\bar{B}(0, r) \subset X$. Replacing the corresponding solution set $S(x)$ to y in the second equation, we obtain a multivalued map $\bar{T}: \bar{B}(0, r) \subset X \multimap X$, given by $\bar{T}(x) = \bar{G}(x, S(x))$, whose fixed points are the solutions of the nonlinear boundary-value control problem (1)–(2).

In Section 3, Theorems 3.1, 3.2, 3.3 provide the necessary properties of $\bar{\Gamma}$ and \bar{T} in order to apply Theorem 2.2. Furthermore, from the topological properties of the set of fixed points of the map \bar{T} and the assumptions on the cost functional j , we derive the existence of a solution of (1)–(2) satisfying the optimality condition (3).

Finally, we would like to mention that both boundary-value problems and controllability problems for dynamical systems, modelled by nonlinear ordinary differential equations, have been solved in Refs. 2 and 3, by using an approach similar to the one outlined here.

2. Definitions and Preliminary Results

Let X, Y be topological spaces; a multivalued map M from X into Y will be denoted by the symbol $M: X \multimap Y$.

Definition 2.1. A multivalued map $M: X \multimap Y$ is said to be upper semicontinuous (u.s.c.) if the set

$$M^{-1}(V) = \{x \in X: M(x) \subseteq V\}$$

is open in X for every open $V \subseteq Y$. A multivalued map $M: X \multimap Y$ is said to be closed if its graph

$$\text{Gr } M = \{(x, y) \in X \times Y: y \in M(x)\}$$

is a closed subset of $X \times Y$.

If a multivalued map $M: X \multimap Y$ is closed and compact [i.e., $M(X)$ is relatively compact in Y], then M is upper semicontinuous.

Let us recall the following concepts (see, for example, Refs. 4 and 5).

Definition 2.2. Let X be an infinite-dimensional Banach space. A function $\psi: 2^X \rightarrow [0, \infty)$ is said to be a measure of noncompactness in X if

$$\psi(\overline{\text{co}} \Omega) = \psi(\Omega), \quad \text{for every } \Omega \in 2^X.$$

A measure of noncompactness ψ is called

- (i) monotone, if $\Omega_0, \Omega_1 \in 2^X$ and $\Omega_0 \subseteq \Omega_1$ imply $\psi(\Omega_0) \leq \psi(\Omega_1)$;
- (ii) nonsingular, if $\psi(\{a\} \cup \Omega) = \psi(\Omega)$ for every $a \in X$ and $\Omega \in 2^X$;
- (iii) semiadditive, if $\psi(\Omega_0 \cup \Omega_1) = \max\{\psi(\Omega_0), \psi(\Omega_1)\}$ for every $\Omega_0, \Omega_1 \in 2^X$;
- (iv) algebraically semiadditive if, $\psi(\Omega_0 + \Omega_1) \leq \psi(\Omega_0) + \psi(\Omega_1)$;
- (v) regular, if $\psi(\Omega) = 0$ is equivalent to the relative compactness of Ω .

A well-known example of a measure of noncompactness satisfying the properties (i)–(v) is the Hausdorff measure of noncompactness,

$$\chi(\Omega) = \inf\{\epsilon > 0: \Omega \text{ has a finite } \epsilon\text{-net}\}.$$

Let X be a Banach space, and let $\mathcal{L}(X)$ be the space of all the bounded linear operators in X . The χ -norm of $B \in \mathcal{L}(X)$ is defined as

$$\|B\|^{(\chi)} := \chi(BS),$$

where S is the unit sphere in X (see Ref. 4).

We shall use the following property of the χ -norm.

Lemma 2.1. (See Ref. 4.) If $B \in \mathcal{L}(X)$, then $\chi(B\Omega) \leq \|B\|^{(\chi)}\chi(\Omega)$ for every bounded set $\Omega \subseteq X$.

Definition 2.3. Let X_0 be a closed subset of a Banach space X ; let ψ be a measure of noncompactness in X . A multivalued map $M: X_0 \multimap X$ with compact values is said to be ψ -condensing if there exists k , $0 \leq k < 1$, such that

$$\psi(M(\Omega)) \leq k\psi(\Omega),$$

for every $\Omega \subseteq X_0$.

Let Λ be a topological space; a multivalued map $G: \Lambda \times X_0 \multimap X$ is said to be ψ -condensing in the second variable if there exists k , $0 \leq k < 1$, such that

$$\psi(G(K \times \Omega)) \leq k\psi(\Omega),$$

for every compact $K \subseteq \Lambda$ and $\Omega \subseteq X_0$.

The following fixed-point principle holds (see Ref. 6).

Theorem 2.1. Let X_0 be a bounded, convex closed subset of X , let ψ be a monotone and regular measure of noncompactness in X , and let $M: X_0 \multimap X_0$ be a closed ψ -condensing multivalued map. Then, the set of fixed points of M ,

$$\text{Fix}(M) = \{x \in X_0: x \in M(x)\},$$

is nonempty and compact.

Let X be a separable Banach space.

Definition 2.4. A multivalued function (multifunction) $G: [a, b] \multimap X$ with closed values is said to be measurable if it satisfies one of the following two equivalent conditions:

- (i) the set $G^{-1}(V)$ is measurable for every open set $V \subseteq X$;
- (ii) there exists a sequence $\{g_n\}_{n=1}^\infty$ of measurable functions $g_n: [a, b] \rightarrow X$ such that

$$G(t) = \overline{\{g_n(t)\}_{n=1}^\infty}, \quad \text{for all } t \in [a, b].$$

By the symbol S_G^1 , we denote the set of all Bochner integrable selectors of the multifunction $G: [a, b] \multimap X$, i.e.,

$$S_G^1 = \{g \in L^1([a, b], X) : g(t) \in G(t), \text{ a.e.}\}.$$

If $S_G^1 \neq \emptyset$, then the multifunction G is called integrable, and by definition,

$$\int_{\mathcal{F}} G(s) \, ds := \left\{ \int_{\mathcal{F}} g(s) \, ds : g \in S_G^1 \right\}, \quad \text{for every measurable set } \mathcal{F} \subset [a, b].$$

Clearly, if G is measurable and integrably bounded [i.e., there exists $\alpha \in L^1_+([a, b])$ such that $\|G(t)\| := \max\{\|y\| : y \in G(t)\} \leq \alpha(t)$, a.e.], then G is integrable.

Lemma 2.2. (See Ref. 7.) Let the multifunction $G: [a, b] \multimap X$ be integrable, integrably bounded, and let $\chi(G(t)) \leq \beta(t)$, a.e., where $\beta \in L^1_+([a, b])$. Then,

$$\chi\left(\int_{\mathcal{F}} G(s) \, ds\right) \leq \int_{\mathcal{F}} \beta(s) \, ds,$$

for any measurable $\mathcal{F} \subseteq [a, b]$. In particular, if G is measurable and integrably bounded, then $\chi(G(\cdot)) \in L^1_+([a, b])$ and

$$\chi\left(\int_{\mathcal{F}} G(s) \, ds\right) \leq \int_{\mathcal{F}} \chi(G(s)) \, ds.$$

Definition 2.5. (See Ref. 8.) Let X, Y be Hausdorff topological spaces. An u.s.c. multivalued map $M: X \multimap Y$ is said to be admissible if there are multivalued maps $Q_i: Z_i \multimap Z_{i+1}$, $i = 0, 1, \dots, m$, with Z_i Hausdorff topological spaces, $Z_0 = X$, $Z_{m+1} = Y$, satisfying

- (a) $M = Q_m \circ \dots \circ Q_0$,
- (b) Q_i is u.s.c. with acyclic, compact values for each $i = 0, 1, \dots, m$.

Let X, Y, Z be Hausdorff topological spaces; a continuous map $p: Z \rightarrow X$ is said to be a Vietoris map provided it is proper, surjective, and the set $p^{-1}(x)$ is acyclic for every $x \in X$.

Definition 2.6. (See Ref. 8.) An u.s.c. multivalued map $M: X \multimap Y$ is said to be generalized acyclic if there exists a pair of continuous maps $p: Z \rightarrow X$, $q: Z \rightarrow Y$ such that p is a Vietoris map and $q(p^{-1}(x)) \subseteq M(x)$ for each $x \in X$.

It is known that every admissible multivalued map is also generalized acyclic (see Refs. 8 and 14).

Let X be a Banach space, let $B = B(0, r)$ be an open ball, let ψ_X be a monotone, nonsingular measure of noncompactness in X , and let $M: \bar{B} \multimap X$ be a generalized acyclic ψ_X -condensing multivalued map. If

$$\text{Fix}(M) \cap \partial B = \emptyset,$$

then the topological degree $\text{Deg}(M, \bar{B})$ is defined (see Refs. 9 and 10), $\text{Deg}(M, \bar{B}) \in \mathbb{Z}$, and it has the usual property of the topological degree. In particular:

- (i) if $H: [0, 1] \times \bar{B} \multimap X$ is a generalized acyclic ψ_X -condensing in the second variable multivalued map such that $\text{Fix}(H(\lambda, \cdot)) \cap \partial B = \emptyset$ for all $\lambda \in [0, 1]$, then $\text{Deg}(H(0, \cdot), \bar{B}) \cap \text{Deg}(H(1, \cdot), \bar{B}) \neq \emptyset$;
- (ii) $\text{Deg}(M, \bar{B}) \neq \{0\}$ implies $\text{Fix}(M) \neq \emptyset$.

To solve our control problem, we need the following existence result which is an extension of Theorem 2.3 of Ref. 1.

Theorem 2.2. Let X, Y be Banach spaces, and let ψ_X, ψ_Y be monotone nonsingular measures of noncompactness in X and Y , respectively. Let $\bar{\Gamma}: X \times Y \multimap Y$ be a closed multivalued map ψ_Y -condensing in the second variable, and let $\bar{G}: X \times Y \multimap X$ be an admissible multivalued map. Let us suppose that, for every $x \in \bar{B}(0, r) \subset X$, the set $S(x) = \{y \in Y: y \in \bar{\Gamma}(x, y)\}$ is nonempty and acyclic. Then:

- (i) the multivalued map $\bar{T}: \bar{B}(0, r) \subset X \multimap X$, defined by $\bar{T}(x) = \bar{G}(x, S(x))$, is admissible;

furthermore, if \bar{T} is ψ_X -condensing and $\text{Deg}(\bar{T}, \bar{B}(0, r)) \neq \{0\}$, then:

- (ii) the system of inclusions

$$y \in \bar{\Gamma}(x, y), \quad x \in \bar{G}(x, y) \tag{4}$$
 has a solution in $\bar{B}(0, r) \times Y$.

3. Results

We will consider problem (1)–(3) under the following assumptions:

- (A1) For each $i = 1, \dots, n$, the family $\{A_i(t)\}_{t \in [0,1]}$ of closed, linear operators in the separable Banach space X_i generates a strongly continuous evolution operator $U_i: \Delta \rightarrow \mathcal{L}(X_i)$, where

$$\Delta = \{(t, s) \in [0, 1] \times [0, 1]: 0 \leq s \leq t \leq 1\}.$$

We also assume that the operators $U_i, i = 1, \dots, n$, are continuous with respect to the norm of $\mathcal{L}(X_i)$ when $s < t$. Moreover, they satisfy the following estimates:

$$\|U_i(t, s)\|^{(x)} \leq \exp(-\delta_i(t - s)), \text{ where } \delta_i > 0, i = 1, \dots, n; \quad (5a)$$

$$1 \notin \text{spectrum of } U_i(1, 0) \text{ for any } i = 1, 2, \dots, n. \quad (5b)$$

Let Z be the Banach space in which the control u takes values. We will assume that:

- (A2) for any $i = 1, \dots, n$, the map $f_i: [0, 1] \times X_1 \times \dots \times X_n \times Z \rightarrow X_i$ is continuous;
- (A3) the feedback multivalued map $V: [0, 1] \times X_1 \times \dots \times X_n \rightarrow Z$ has compact values, is u.s.c., and sends bounded sets into bounded sets;
- (A4) for any $i = 1, \dots, n$, the set $F_i(t, x_1, \dots, x_n) = f_i(t, x_1, \dots, x_n, V(t, x_1, \dots, x_n))$ is convex for every $(t, x_1, \dots, x_n) \in [0, 1] \times X_1 \times \dots \times X_n$;
- (A5) for any $i = 1, \dots, n$, $F_i(t, x_1, \dots, x_n) \subset W_i$ for every $(t, x_1, \dots, x_n) \in [0, 1] \times X_1 \times \dots \times X_n$, where W_i is a weakly compact, convex subset of X_i ;
- (A6) there exists an $n \times n$ matrix \mathcal{M} with nonnegative components m_{ij} such that

$$\chi_i(F_i(t, D_1, \dots, D_n)) \leq \sum_{j=1}^n m_{ij} \chi_j(D_j),$$

for every $t \in [0, 1]$ and any bounded $D_i \subset X_i$, where χ_i denotes the Hausdorff measure of noncompactness in the space $X_i, i = 1, \dots, n$.

Let us observe that, from the above conditions, it follows (see Ref. 6) that the multivalued maps $F_i, i = 1, \dots, n$, are u.s.c. and that, for any choice of the functions $y_i \in C([0, 1], X_i), i = 1, \dots, n$, the sets of integrable selections $S_{F_i(y_1(\cdot), \dots, y_n(\cdot))}^1$ are nonempty.

We will consider mild solutions of problem (1), i.e., the collection

$$(y_1, \dots, y_n, u) \in C([0, 1], X_1) \times \dots \times C([0, 1], X_n) \times L^\infty([0, 1], Z),$$

such that

$$y_i(t) = U_i(t, 0)y_i(0) + \int_0^t U_i(t, s)f_i(s, y_1(s), \dots, y_n(s), u(s)) ds,$$

$$i = 1, 2, \dots, n,$$

where $u(t) \in V(t, y_1(t), \dots, y_n(t))$.

We assume that the Banach space $X = X_1 \times \dots \times X_n$ is equipped with the norm

$$\|x\| = \|(x_1, \dots, x_n)\| = \max_{1 \leq i \leq n} \|x_i\|_i,$$

where $\|\cdot\|_i$ denotes the norm of the space X_i . Moreover, let

$$Y = C([0, 1], X_1) \times \dots \times C([0, 1], X_n)$$

be the Banach space equipped with the norm

$$\|y\| = \|(y_1, \dots, y_n)\| = \max_{1 \leq i \leq n} \|y_i\|_i,$$

where $\|\cdot\|_i$ denotes the norm of the space $Y_i = C([0, 1], X_i)$.

Let us consider the multioperator $\bar{\Gamma}: X \times Y \rightarrow Y$ defined as

$$\bar{\Gamma}(x, y) = \bar{\Gamma}(x_1, \dots, x_n, y_1, \dots, y_n)$$

$$= \left\{ z \in Y: z_i(t) = U_i(t, 0)x_i + \int_0^t U_i(t, s)g_i(s) ds: g_i(\cdot) \in S_{F_i(\cdot, y_1(\cdot), \dots, y_n(\cdot))}^1 \right\}.$$

Following the methods in Ref. 7 one can show that the multioperator $\bar{\Gamma}$ is closed.

Now, let us define in the spaces $Y_i = C([0, 1], X_i)$, $i = 1, \dots, n$, the following measure of noncompactness:

$$\phi_i(D) = \sup_{t \in [0, 1]} (\chi_i(D(t) \exp(-rmt))),$$

where

$$D(t) = \{y(t): y \in D\},$$

$$m = \max\{m_{ij}, 1 \leq i \leq n, 1 \leq j \leq n\},$$

$r > n$ is a given number, and χ_i is the Hausdorff measure of noncompactness in X_i .

The measure of noncompactness in the space Y can be defined as

$$\psi_Y(\Omega) = \max_{1 \leq i \leq n} \phi_i(\Omega_i), \quad \text{where } \Omega_i = \text{Proj}_{Y_i}(\Omega).$$

It is not difficult to verify that ψ_i is monotone and nonsingular.

Theorem 3.1. The multioperator $\bar{\Gamma}$ is ψ_Y -condensing in the second variable.

Proof. Let $K \subset X$ be an arbitrary compact set, and let $\Omega \subset Y$ be an arbitrary bounded set. Let us estimate $\psi_Y(\bar{\Gamma}(K, \Omega))$.

We can easily see that

$$\bar{\Gamma}(K, \Omega) \subset \bar{\Gamma}_1(K, \Omega) \times \cdots \times \bar{\Gamma}_n(K, \Omega),$$

where

$$\bar{\Gamma}_i(K, \Omega) = \left\{ z_i \in Y_i : y_i(t) = U_i(t, 0)x_i + \int_0^t U_i(t, s)F(s, \Omega_1(s), \dots, \Omega_n(s)) ds : x_i \in \text{Proj}_{X_i}(K) \right\},$$

with

$$\Omega_j(s) = \text{Proj}_{X_j}(\Omega(s)), \quad j = 1, \dots, n.$$

Then,

$$\chi_i(\bar{\Gamma}_i(K, \Omega)(t)) = \chi_i\left(\int_0^t U_i(t, s)F_i(s, \Omega_1(s), \dots, \Omega_n(s)) ds\right).$$

By Lemma 2.1 and Conditions (A1) and (A6), we get

$$\begin{aligned} & \chi_i(U_i(t, s)F_i(s, \Omega_1(s), \dots, \Omega_n(s))) \\ & \leq \|U_i(t, s)\|^{(x)} \chi_i(F_i(s, \Omega_1(s), \dots, \Omega_n(s))) \\ & \leq \chi_i(F_i(s, \Omega_1(s), \dots, \Omega_n(s))) \\ & \leq \sum_{j=1}^n m_{ij} \chi_j(\Omega_j(s)) \leq m \sum_{j=1}^n \chi_j(\Omega_j(s)) \\ & = m \sum_{j=1}^n \chi_j(\Omega_j(s)) \exp(rms) \exp(-rms) \\ & \leq m \sum_{j=1}^n \phi_j(\Omega_j) \exp(rms) \\ & \leq mn\psi_Y(\Omega) \exp(rms). \end{aligned}$$

Hence by Lemma 2.2, it follows that

$$\chi_i(\bar{\Gamma}_i(K, \Omega)(t)) \leq mn\psi_Y(\Omega) \int_0^t \exp(rms) ds < (n/r)\psi_Y(\Omega) \exp(rmt).$$

Therefore,

$$\chi_i(\bar{\Gamma}_i(K, \Omega)(t)) \exp(-rmt) \leq (n/r)\psi_Y(\Omega),$$

$$\phi_i(\bar{\Gamma}_i(K, \Omega)(t)) \leq (n/r)\psi_Y(\Omega),$$

and finally

$$\psi_Y(\bar{\Gamma}(K, \Omega)) \leq (n/r)\psi_Y(\Omega),$$

concluding the proof. □

Now consider the multioperator $\bar{\Gamma}(x, \cdot)$ for $x \in X$. From Condition (A4), it follows that it is convex-valued. By using the methods in Ref. 7, it can be shown that the set $\bar{\Gamma}(\Lambda, Y)$ is bounded and equicontinuous for any compact $\Lambda \subset X$. By this fact and the regularity of the measure of noncompactness ψ_Y on the equicontinuous subsets, we may deduce from Theorem 2.1 that $\bar{\Gamma}(x, \cdot)$ restricted to $\text{co } \bar{\Gamma}(x, Y)$ has a fixed point. Moreover from Theorem 3.1, it follows that, for each $x \in X$, the nonempty set

$$S(x) = \{y \in Y: y \in \bar{\Gamma}(x, y)\}$$

is compact.

Since the u.s.c. multivalued maps $F_i(t, x_1, \dots, x_n)$, $i = 1, \dots, n$, can be approximated by continuous multivalued maps (see Ref. 11), by using Condition (A5) and some methods of Refs. 12 and 13, one can show that the sets $S(x)$ are acyclic.

Consider now the multioperator $\bar{G}(x, y): X \times Y \multimap X$ defined as

$$\bar{G}(x, y) = y(1) - \mathcal{K}(x, y(1)),$$

where \mathcal{K} is the multivalued map of the boundary condition (2).

The multivalued map \bar{G} is admissible, since it may be represented as the composition of the following maps:

$$W_1: X \times Y \rightarrow X \times X, \text{ defined as } W_1(x, y) = (x, y(1));$$

$$W_2: X \times X \multimap X \times X, \text{ defined as } W_2(x_1, x_2) = (x_2, \mathcal{K}(x_1, x_2));$$

$$W_3: X \times X \rightarrow X, \text{ defined as } W_3(x_1, x_2) = x_1 - x_2.$$

From Conditions (A1) and (A5), it follows that the set $\mathcal{L} \subset X$, given by

$$\mathcal{L} = \left\{ (x_1, \dots, x_n): x_i = \int_0^1 U_i(1, s)g_i(s) ds: g_i(\cdot) \in S_{F_i(\cdot, y_1(\cdot), \dots, y_n(\cdot))}^1, \right. \\ \left. y_i \in Y_i, i = 1, \dots, n \right\},$$

is bounded. Let $N > 0$ be such that

$$\|x\| \leq N, \quad \text{for all } x \in \mathcal{L} - \mathcal{X}.$$

Let us consider the linear operator $U(1, 0): X \rightarrow X$, defined by

$$U(1, 0)x = (U_1(1, 0)x_1, \dots, U_n(1, 0)x_n).$$

From the condition (5b), it follows that we may choose $r > 0$ in such a way that

$$\|x - U(1, 0)x\| > N, \tag{6}$$

if $\|x\| = r$.

Let us define, on the ball $\bar{B}(0, r) \subset X$, the multivalued map $\bar{T}: \bar{B}(0, r) \multimap X$ as follows:

$$\bar{T}(x) = \bar{G}(x, S(x)).$$

Let

$$\mathcal{N} = (\eta_{ij})_{i=1, j=1}^n$$

be the matrix given by

$$\eta_{ij} = \begin{cases} m_{ij} - \delta_i, & \text{if } i = j, \\ m_{ij}, & \text{if } i \neq j, \end{cases}$$

where the coefficients m_{ij} are those given by Condition (A6) and δ_i are the coefficients from estimation (5a).

Let us suppose now that:

(N) the zero solution of the linear system in \mathbb{R}^n

$$z'(t) = \mathcal{N}z(t)$$

is exponentially stable.

Under this condition, the linear operator $B = e^{\cdot \mathcal{N}}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ has the spectral radius less than 1. Let l be such that $\|B^l\| < q^l$, where $q < 1$.

Let the map $\kappa: 2^X \rightarrow \mathbb{R}_+^n$ be the vector-valued measure of noncompactness in X defined as

$$\kappa(\Omega) = \{\chi_1(\Omega_1), \dots, \chi_n(\Omega_n)\},$$

where χ_i is the Hausdorff measure of noncompactness in X_i . Then, $\psi_X: 2^X \rightarrow \mathbb{R}_+$ defined as

$$\psi_X(\Omega) = q^{l-1} \|\kappa(\Omega)\| + q^{l-2} \|B_\kappa(\Omega)\| + \dots + \|B^{l-1}\kappa(\Omega)\|, \quad \Omega \subset X,$$

is a monotone, nonsingular measure of noncompactness in X .

Let us prove the following statement.

Theorem 3.2. The multivalued map $\bar{T}: \bar{B}(0, r) \rightarrow X$ is ψ_X -condensing.

Proof. For $\Omega \subset \bar{B}(0, r)$, let us consider the translation operator along the trajectories of the system (1), $P: [0, 1] \times \Omega \rightarrow X$, defined by

$$P(t, x) = \left\{ p(t, x) \in X: p_i(t, x) = U_i(t, 0)x_i + \int_0^t U_i(t, s)g_i(s) ds: g_i(\cdot) \in S_{F_i(\cdot, p_1(\cdot, x), \dots, p_n(\cdot, x))}^1 \right\}.$$

Let us denote

$$P_i(t, \Omega) = \text{Proj}_{X_i} P(t, \Omega).$$

We have the following estimate:

$$\begin{aligned} & \chi_i(\{U_i(t, s)g_i(s): g_i(\cdot) \in S_{F_i(\cdot, p_1(\cdot, x), \dots, p_n(\cdot, x))}^1: x \in \Omega\}) \\ & \leq \chi_i(\{U_i(t, s)g_i(s): g_i(\cdot) \in S_{F_i(\cdot, P_1(\cdot, \Omega), \dots, P_n(\cdot, \Omega))}^1\}) \\ & \leq \chi_i(\{U_i(t, s)F_i(s, P_1(s, \Omega), \dots, P_n(s, \Omega))\}) \\ & \leq \|U_i(t, s)\|^{(\omega)} \chi_i(F_i(s, P_1(s, \Omega), \dots, P_n(s, \Omega))) \\ & \leq \exp(-\delta_i(t-s)) \sum_{j=1}^n m_{ij} \chi_j(P_j(s, \Omega)). \end{aligned}$$

Hence,

$$\begin{aligned} & \chi_i(P_i(t, \Omega)) \\ & \leq \|U_i(t, 0)\|^{(\omega)} \chi_i(\Omega_i) + \int_0^t \exp(-\delta_i(t-s)) \sum_{j=1}^n m_{ij} \chi_j(P_j(s, \Omega)) ds \\ & \leq \exp(-\delta_i t) \chi_i(\Omega_i) + \int_0^t \exp(-\delta_i(t-s)) \sum_{j=1}^n m_{ij} \chi_j(P_j(s, \Omega)) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} & \chi_i(P_i(t, \Omega)) = \kappa_i(P(t, \Omega)) \\ & \leq \exp(-\delta_i t) \kappa_i(\Omega) + \int_0^t \exp(-\delta_i(t-s)) (\mathcal{M} \kappa(P(s, \Omega)))_i ds, \end{aligned}$$

and from the Gronwall lemma it follows that

$$\kappa(P(t, \Omega)) \leq B \kappa(\Omega),$$

in the sense of component-wise comparison of vectors. But then

$$\begin{aligned} &\psi_X(P(1, \Omega)) \\ &= q^{l-1} \|\kappa(P(1, \Omega))\| + q^{l-2} \|B\kappa(P(1, \Omega))\| + \dots + \|B^{l-1}\kappa(P(1, \Omega))\| \\ &\leq q^{l-1} \|B\kappa(\Omega)\| + q^{l-2} \|B^2\kappa(\Omega)\| + \dots + \|B^l\kappa(\Omega)\| \\ &\leq q^{l-1} \|B\kappa(\Omega)\| + q^{l-2} \|B^2\kappa(\Omega)\| + \dots + q \|B^{l-1}\kappa(\Omega)\| + q^l \|\kappa(\Omega)\| \\ &\leq q(q^{l-1} \|\kappa(\Omega)\| + q^{l-2} \|B\kappa(\Omega)\| + \dots + q \|B^{l-2}\kappa(\Omega)\| + \|B^{l-1}\kappa(\Omega)\|) \\ &\leq q\psi_X(\Omega). \end{aligned}$$

Note now that the following relation holds:

$$\bar{T}(x) \subseteq P(1, x) - \mathcal{K}(x, P(1, x)), \quad \text{for all } x \in \bar{B}(0, r).$$

Since \mathcal{K} is compact and the measure of noncompactness ψ_X is monotone, regular, and algebraically semiadditive, it follows that \bar{T} is ψ_X -condensing. □

Now let us evaluate the topological degree of the map \bar{T} .

Theorem 3.3. The following property holds: $\text{Deg}(\bar{T}, \bar{B}(0, r)) \neq \{0\}$.

Proof. From the condition (5a), it follows that the linear operator $U(1, 0): X \rightarrow X$ is ψ_X -condensing. Since the measure of noncompactness ψ_X is algebraically semiadditive, we have the multivalued map $H: [0, 1] \times X \rightarrow X$, defined as

$$H(\lambda, x) = \lambda\bar{T}(x) + (1 - \lambda)U(1, 0)x,$$

is ψ_X -condensing in the second variable. It is known that a multivalued map of such a form is generalized acyclic (Ref. 14).

It remains only to show that

$$\text{Fix}(H(\lambda, \cdot)) \cap \partial B(0, r) = \emptyset.$$

Assuming the contrary, we will have the existence of $x \in X$, $\|x\| = r$, and $\lambda \in [0, 1]$ such that

$$x \in \lambda\bar{T}(x) + (1 - \lambda)U(1, 0)x.$$

But then

$$x - U(1, 0)x \in \lambda\bar{T}(x) - \lambda U(1, 0)x \subseteq \lambda(\mathcal{L} - \mathcal{K}),$$

and hence,

$$\|x - U(1, 0)x\| \leq \lambda N,$$

in contradiction with our choice of r in (5). Therefore,

$$\text{Deg}(U(1, 0), \bar{B}(0, r)) \in \text{Deg}(\bar{T}, \bar{B}(0, r));$$

but from Ref. 4, it follows that

$$|\text{Deg}(U(1, 0), \bar{B}(0, r))| = 1,$$

and the theorem is proved. \square

We are now in a position to prove our main result.

Theorem 3.4. Under Assumptions (A1)–(A6), (5a), (5b), and (N), there exists a mild solution $y(\cdot)$ of the system (1) satisfying conditions (2) (3).

Proof. We see that all conditions of Theorem 2.2 are fulfilled, and hence the system (4) has a solution in $\bar{B}(0, r) \times Y$. Moreover, from Theorems 3.1 and 3.2, it follows that the set of all the solutions of system (4) in $\bar{B}(0, r) \times Y$ is compact. Hence, the set of all functions $y \in Y$ of the form

$$y_i(t) = \left\{ U_i(t, 0)y_i(0) + \int_0^t U_i(t, s)g_i(s) ds: g_i(\cdot) \in S_{F_i(\cdot, y_1(\cdot), \dots, y_n(\cdot))}^1, \right. \\ \left. y_i \in Y_i, i = 1, \dots, n \right\},$$

satisfying the boundary conditions (2), is compact. On the other hand, by the Filippov implicit function lemma (see, for example, Ref. 6), it follows that this set coincides with the set of all mild solutions of the control system (1) satisfying the conditions (2). Therefore, there exists a mild solution $y(\cdot)$ with corresponding admissible control $u(\cdot)$ satisfying the optimality property (3). This ends the proof. \square

Remark 3.1. Let us note that, if $\mathcal{X} \equiv \{0\}$, then we have the existence of a solution of the periodic problem $y(0) = y(1)$ which minimizes the given cost functional.

Concerning the periodic problem for differential inclusions and control systems in Banach spaces, see also Refs. 1, 15 and 16.

References

1. NISTRI, P., OBUKHOVSKII, V. V., and ZECCA, P., *On the Solvability of Systems of Inclusions Involving Noncompact Operators*, Transactions of the American Mathematical Society, Vol. 342, pp. 543–562, 1994.

2. CONTI, G., NISTRI, P., and ZECCA, P., *Systems of Set-Valued Equations in Banach Spaces*, Delay Differential Equations and Dynamical Systems, Edited by S. Busemberg and M. Martelli, Springer Verlag, Berlin, Germany, pp. 98–109, 1991.
3. CONTI, G., NISTRI, P., and ZECCA, P., *Systems of Multivalued Equations for Solving Controllability Problems*, Pure Mathematics and Applications, Series B, Vol. 2, pp. 215–225, 1991.
4. AKHMEROV, R. R., KAMENSKII, M. I., POTAPOV, A. S., RODKINA, A. E., and SADOVSKII, B. N., *Measure of Noncompactness and Condensing Operators*, Birkhauser Verlag, Berlin, Germany, 1992.
5. BORISOVICH, YU. G., GEL'MAN, B. D., MYSHKIS, A. D., and OBUKHOVSKII, V. V., *Topological Methods in the Fixed-Point Theory of Multivalued Maps*, Russian Mathematical Surveys, Vol. 35, pp. 65–143, 1980.
6. BORISOVICH, YU. G., GEL'MAN, B. D., MYSHKIS, A. D., and OBUKHOVSKII, V. V., *Multivalued Mappings*, Journal of Soviet Mathematics, Vol. 24, pp. 719–791, 1984.
7. OBUKHOVSKII, V. V., *Semilinear Functional-Differential Inclusions in a Banach Space and Controlled Parabolic Systems*, Soviet Journal Automat. Inform. Sci., Vol. 24, pp. 71–79, 1992.
8. GÓRNIIEWICZ, L., *Homological Methods in Fixed-Point Theory of Multivalued Mappings*, Dissertations Mathématiques, Vol. 129, pp. 1–71, 1976.
9. OBUKHOVSKII, V. V., *On the Topological Degree for a Class of Noncompact Multivalued Mappings*, Funktsional'nyi Analiz (Ul'yanovsk), Vol. 23, pp. 82–93, 1984 (in Russian).
10. BORISOVICH, YU. G., GEL'MAN, B. D., MYSHKIS, A. D., and OBUKHOVSKII, V. V., *Multivalued Analysis and Operator Inclusions*, Journal of Soviet Mathematics, Vol. 39, pp. 2772–2811, 1987.
11. DE BLASI, F., *Characteristics of Certain Classes of Semicontinuous Multifunctions by Continuous Approximations*, Journal of Mathematical Analysis and Applications, Vol. 106, pp. 1–18, 1985.
12. KAMENSKII, M. I., and OBUKHOVSKII, V. V., *On the Translation Operator along Trajectories of Solutions of Semilinear Differential Inclusions and Control Systems in Banach Spaces*, Proceedings of the Conference on Qualitative Methods in Boundary-Value Problems, Voronezh, Russia, p. 68, 1992.
13. PAPAGEORGIOU, N. S., *On Multivalued Evolution Equations and Differential Inclusions in Banach Spaces*, Commentarii Mathematici Universitatis Sancti Pauli, Vol. 36, pp. 21–39, 1987.
14. BRYCZEWSKI, J., and GÓRNIIEWICZ, L., *A Poincaré-Type Coincidence Theorem for Multivalued Maps*, Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques, Vol. 24, pp. 593–598, 1976.
15. KAMENSKII, M. I., and OBUKHOVSKII, V. V., *On Periodic Solutions of Differential Inclusions with Unbounded Operators in Banach Spaces*, Zbornik Radova Prirodno-Matematichko Fakulteta, Serija za Matematiku, Vol. 21, pp. 173–191, 1991.
16. KAMENSKII, M. I., and OBUKHOVSKII, V. V., *Condensing Multioperators and Periodic Solutions of Parabolic Functional-Differential Inclusions in Banach Spaces*, Nonlinear Analysis: Theory, Methods and Applications, Vol. 20, pp. 781–792, 1993.

L40]