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**Analisi funzionale.** — *On surjectivity for  $\epsilon$ -quasibounded multi-valued maps* (\*). Nota (\*\*) di GIUSEPPE CONTI e PAOLO NISTRI, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si danno condizioni di suriettività per applicazioni multivoche in spazi di Banach.

## 1. INTRODUCTION

Let  $E$  be a Banach space and  $T : E \multimap E$  be a multi-valued map. The purpose of this Note is to investigate conditions under which the map  $I \multimap T$  is onto, where  $I$  is the identity on  $E$ .

We obtain results which extend Theorems proved in [4], [9] and [11]. The main tool we have used in this paper is a suitable definition of asymptotic spectrum of a multi-valued map  $T : E \multimap E$  as introduced in [1].

## 2. NOTATIONS AND DEFINITIONS

Let  $E, F$  be two real or complex Banach spaces and  $T : E \multimap F$  be a multi-valued map. We recall that  $T$  is *upper semicontinuous* on  $E$  if, for any  $x \in E, T(x)$  is compact (and not empty) and, for any open set  $U$  containing  $T(x)$ , there exists a neighborhood  $V$  of  $x$  such that  $T(y) \subset U$  for all  $y \in V$ .

If for any  $x \in E$  the set  $T(x)$  is acyclic in the Vietoris homology theory with coefficients in  $\mathbb{Q}$ , then  $T$  is said to be *acyclic-valued*.

Let  $A$  be any bounded set of  $E$ . Denote by  $\alpha(A)$  the Kuratowski measure of non-compactness of  $A$  (see [7]). If there exists  $k \geq 0$  such that  $\alpha(T(A)) \leq k\alpha(A)$  for any bounded set  $A \subset E$ , then  $T$  is called an  $\alpha$ -Lipschitz map with constant  $k$ . According to [2] we shall say that  $T$  is  $\alpha$ -contractive ( $\alpha$ -nonexpansive) if  $k < 1$  ( $k = 1$ ).

If  $\alpha(T(A)) < \alpha(A)$  for any bounded and non precompact set  $A \subset E$ , then  $T$  is said to be *condensing*.

Let  $M \subset E$ . Define  $\Phi(M) = \sup \{\|y\| : y \in M\}$ .

The upper semicontinuous map  $T : E \multimap F$  is *quasibounded* if

$$|T| = \limsup_{\|x\| \rightarrow +\infty} \frac{\Phi(T(x))}{\|x\|} < +\infty.$$

The number  $|T|$  is called the *quasinorm* of  $T$  (see [5]).

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The set  $Q(E)$  of all the multi-valued quasibounded maps  $T: E \multimap E$  cannot be endowed with the structure of a vector space since, for any  $T \in Q(E)$ , we have that  $T - T = 0$  if and only if  $T$  is single-valued.

Let  $T, S \in Q(E)$ . We say that  $T$  satisfies *property (A)* if

$$\lim_{\|x\| \rightarrow +\infty} \frac{\text{diam } T(x)}{\|x\|} = 0$$

where  $\text{diam } T(x)$  is the diameter of  $T(x)$ .

If  $\lim_{\|x\| \rightarrow +\infty} \frac{\delta(T(x), S(x))}{\|x\|} = 0$  then  $T$  and  $S$  are said to be *asymptotically equivalent* ( $T \sim S$ ), where  $\delta$  is the Hausdorff distance between  $T(x)$  and  $S(x)$ . In the case when  $S$  is linear then  $T$  is *asymptotically linear* and  $S = T'_\infty$  is said to be the *asymptotic derivative* of  $T$  (see [1]).

Clearly an asymptotically linear multi-valued map  $T: E \multimap E$  satisfies property (A).

Let  $D_r = \{x \in E : \|x\| \leq r\}$ . A multi-valued map  $T$  satisfies the *Leray-Schauder condition* on  $D_r$  if  $\lambda x \in T(x)$  for some  $x \in S_r$ , the boundary of  $D_r$ , implies that  $\lambda \leq 1$ .

### 3. SOME SURJECTIVITY THEOREMS FOR $e$ -QUASIBOUNDED MAPS

Let  $T: E \multimap E$  be a multi-valued upper semicontinuous map. We say that the real number  $\mu$  is an *eigenvalue* of  $T$  if there exists  $x \in E$ ,  $x \neq 0$ , such that  $sx \in T(x)$  for some  $s \in \mathbf{R}$  and

$$\mu = \sup \{s \in \mathbf{R} : sx \in T(x)\}.$$

Then  $x$  is said to be an *eigenvector* of  $T$  belonging to  $\mu$ .

Let  $r > 0$ . Consider the extended real number defined as follows  $b(r, T) = \sup \{\mu \geq 0 : \mu \text{ is an eigenvalue of } T \text{ with eigenvector } x \text{ such that } \|x\| = r\}$  if the above set is not empty, and  $b(r, T) = 0$  otherwise.

For any  $p \in E$  we define

$$1) (T)_0 = \inf \{b(r, T) ; r > 0\}.$$

$$2) (T)_p = (T + p)_0.$$

$$3) (T) = \sup \{(T)_p ; p \in E\}.$$

If  $(T) < +\infty$ , the mapping  $T$  is said to be *essentially-quasibounded* (*e-quasibounded*) and  $(T)$  will be called the *essential quasinorm* (*e-quasinorm*) of the map  $T$ .

We have the following

LEMMA 1

$$(i) \quad (\nu T)_0 = \nu (T)_0 \quad \nu \in \mathbf{R}^+$$

$$(ii) \quad (\nu T) = \nu (T) \quad \nu \in \mathbf{R}^+$$

(iii) *If T is quasibounded then T is  $\epsilon$ -quasibounded and  $(T) \leq |T|$ .*

*Proof.* The first property follows immediately from the fact that  $\rho$  is an eigenvalue of T if and only if  $\nu\rho$  is an eigenvalue of  $\nu T$  ( $\nu > 0$ ).

Definition 2) and (i) imply that

$$(\nu T)_p = (\nu T + p)_0 = (\nu (T + \nu^{-1} p))_0 = \nu (T + \nu^{-1} p)_0 = \nu (T)_{\nu^{-1} p}$$

Hence  $(\nu T) = \nu (T)$ .

In order to prove (iii) observe that

$$b(r, T) \leq \sup_{\|x\| \geq r} \frac{\Phi(T(x))}{\|x\|}.$$

For any  $p \in E$  we have

$$\begin{aligned} (T)_p &= (T + p)_0 \leq \inf_{r > 0} \sup_{\|x\| \geq r} \frac{\Phi(T(x) + p)}{\|x\|} = \\ &= \lim_{\|x\| \rightarrow +\infty} \sup \frac{\Phi(T(x) + p)}{\|x\|} = |T + p| = |T| \end{aligned}$$

hence  $(T) = \sup_{p \in E} (T)_p \leq |T|$

Q.E.D.

*Remark.* There are examples of T such that  $(T) < |T|$  (see [4]).

**THEOREM 1.** *Let  $T : E \rightarrow E$  be an upper semicontinuous acyclic-valued condensing map.*

a) *If  $(T)_0 < 1$  then T has a fixed point.*

b) *If  $(T)_p < 1$  the equation  $p \in x - T(x)$  has a solution.*

c) *If  $(T) < 1$  then  $I - T$  is surjective (i.e.  $(I - T)(E) = E$ ).*

*Proof.* It is easy to see that a)  $\Rightarrow$  b)  $\Rightarrow$  c), therefore it suffices to prove a).

There exists  $r > 0$  such that  $b(r, T) < 1$ , that is all the eigenvalues  $\lambda$  corresponding to eigenvectors  $x$  with  $\|x\| = r$  are smaller than 1.

Therefore the mapping T satisfies the Leray-Schauder condition on the boundary of the ball  $D_r$ . Hence, by a theorem due to Martelli [8], T has at least one fixed point.

Q.E.D.

A surjectivity result proved in [9] for the class of upper semicontinuous acyclic-valued condensing maps with  $|T| < 1$  can be derived from the above Theorem since  $(T) \leq |T|$ .

In the case of single-valued maps we obtain as corollaries some surjectivity results of Granas [5] and Dubrowskij [3] (see also [9]).

**COROLLARY 1.** *Let  $H$  be an Hilbert space and let  $T : H \rightarrow H$  be an upper semicontinuous acyclic-valued condensing map.*

*Define  $s(T, x) = \sup \{ \langle z, x \rangle : z \in T(x) \}$ . If*

$$s(T) = \limsup_{\|x\| \rightarrow +\infty} \frac{s(T, x)}{\|x\|^2} < 1$$

*then  $I - T$  is onto.*

*Proof.* By Theorem 1 it suffices to prove that  $(T)_0 \leq s(T)$ . In fact, take  $a > s(T)$ . Then there exists  $\rho > 0$  such that  $\|x\| \geq \rho$  and  $z \in T(x)$  implies  $\langle z, x \rangle \leq a\|x\|^2$ .

The above inequality shows that there are no positive eigenvalues greater than  $a$  corresponding to eigenvectors  $x$  such that  $\|x\| \geq \rho$ . Hence  $(T)_0 \leq a$ . The arbitrariness of  $a > s(T)$  implies  $(T)_0 \leq s(T)$ . Clearly  $s(T + \rho) = s(T)$  for all  $\rho \in E$ . Therefore  $(T + \rho)_0 \leq s(T)$ , and so  $(T) \leq s(T)$ .

Q.E.D.

Notice that in the case of single-valued maps the above result is due to Krasnosel'skij (see [6]).

**COROLLARY 2.** *Let  $T : E \rightarrow E$  be an upper semicontinuous acyclic-valued map. Assume that  $T$  is  $\alpha$ -Lipschitz with constant  $k$ .*

*Then for any  $\nu$  such that  $\nu > \max \{ k, (T) \}$  the map  $\nu I - T$  is onto.*

*Proof.* Clearly  $\nu^{-1}T$  is  $\alpha$ -contractive and  $(\nu^{-1}T) < 1$ .

Q.E.D.

In [1] a notion of asymptotic spectrum for multi-valued quasibounded maps is given. We state now a Theorem which gives a relation between this spectrum and the concept of  $\epsilon$ -quasibounded maps.

Let us recall first some definitions of [1].

Let  $T \in Q(E)$ . Put

$$d(\lambda - T) = \liminf_{\|x\| \rightarrow +\infty} \frac{\delta(T(x), \lambda x)}{\|x\|} \quad \lambda \in \mathbf{C}$$

where  $\lambda - T : E \rightarrow E$  is the map defined by  $(\lambda - T)x = \lambda x - Tx$ . The asymptotic spectrum  $\Sigma(T)$  of  $T$  is the set

$$\Sigma(T) = \{ \lambda \in \mathbf{C} : d(\lambda - T) = 0 \}$$

and the spectral radius  $r(T)$  of  $T$  is the non-negative real number

$$r(T) = \sup \{ |\lambda| : \lambda \in \Sigma(T) \}.$$

Moreover we put  $r^+(T) = \sup \{ \lambda \geq 0 : \lambda \in \Sigma(T) \}$ . ( $r(T) = r^+(T) = 0$  if  $\Sigma(T) = \emptyset$ ).

The following theorem holds

**THEOREM 2.** *Let  $T \in Q(E)$ . If  $T$  satisfies property (A) then  $T$  is  $e$ -quasibounded and  $(T) \leq r^+(T) \leq |T|$ .*

*Proof.* Clearly  $r^+(T) \leq r(T)$ . On the other hand  $r(T) \leq |T|$  (see [1]).

Therefore it is enough to prove that  $(T) \leq r^+(T)$ . Observe that  $d(\lambda - T) = d(\lambda - T + p)$  for all  $p \in E$ .

In fact

$$\begin{aligned} d(\lambda - T + p) &= \liminf_{\|x\| \rightarrow +\infty} \frac{\delta(T(x) - p, \lambda x)}{\|x\|} \leq \\ &\leq \liminf_{\|x\| \rightarrow +\infty} \frac{\delta(T(x), \lambda x)}{\|x\|} + \limsup_{\|x\| \rightarrow +\infty} \frac{\delta(-p, 0)}{\|x\|} = d(\lambda - T). \end{aligned}$$

On the other hand

$$\begin{aligned} d(\lambda - T) &= \liminf_{\|x\| \rightarrow +\infty} \frac{\delta(T(x) - p, \lambda x - p)}{\|x\|} \leq \\ &\leq \liminf_{\|x\| \rightarrow +\infty} \frac{\delta(T(x) - p, \lambda x)}{\|x\|} + \limsup_{\|x\| \rightarrow +\infty} \frac{\delta(0, -p)}{\|x\|} = d(\lambda - T + p). \end{aligned}$$

Therefore  $r^+(T + p) = r^+(T)$  for any  $p \in E$ . Hence it suffices to show that  $(T)_0 \leq r^+(T)$ .

Assume the contrary. Then there exist  $\varepsilon > 0$  and two sequences  $\{\lambda_n\} \subset \mathbf{R}^+$  and  $\{x_n\} \subset E$ ,  $\|x_n\| \rightarrow +\infty$  as  $n \rightarrow \infty$ , such that  $\lambda_n \geq r^+(T) + \varepsilon$  and  $\lambda_n x_n \in T(x_n)$ . Therefore  $\|\lambda_n x_n\| \leq \delta(T(x_n), 0)$  and so

$$0 \leq \lambda_n \leq \frac{\delta(T(x_n), 0)}{\|x_n\|} \leq |T|.$$

The sequence  $\{\lambda_n\}$  is bounded, hence we may assume without loss of generality that  $\lambda_n \rightarrow \lambda \geq r^+(T) + \varepsilon$ .

We have

$$\frac{\delta(T(x_n), \lambda x_n)}{\|x_n\|} \leq \frac{\delta(T(x_n), \lambda_n x_n)}{\|x_n\|} + |\lambda_n - \lambda| \leq \frac{\text{diam } T(x_n)}{\|x_n\|} + |\lambda_n - \lambda|.$$

Thus

$$\lim_{\|x_n\| \rightarrow +\infty} \frac{\delta(\lambda x_n, T(x_n))}{\|x_n\|} = 0.$$

Then  $d(\lambda - T) = 0$  that is  $\lambda \in \Sigma(T)$ , contradicting the hypothesis.

Q.E.D.

This Theorem extends to multi-valued maps a Theorem of M. Furi and A. Vignoli [4] (see also [11]).

We give now an example which shows that in the above Theorem the assumption  $\lim_{\|x\| \rightarrow +\infty} \frac{\text{diam } T(x)}{\|x\|} = 0$  is essential.

*Example.* Let  $T: \mathbf{R} \rightarrow \mathbf{R}$  be a multi-valued map defined by  $T(x) = \{y \in \mathbf{R} : -|x| \leq y \leq |x|\}$ .  $T$  is compact and quasibounded with  $|T| = 1$ .

We can prove that  $(T) = 1$ , while  $\Sigma(T) = \emptyset$  and so  $r^+(T) = 0$ .

In fact

$$\lim_{\|x\| \rightarrow +\infty} \frac{\text{diam } T(x)}{\|x\|} = 2.$$

**COROLLARY 3.** *Let  $T: E \rightarrow E$  be an upper semicontinuous condensing, acyclic-valued map. Suppose that  $T$  is asymptotically linear.*

*If  $r^+(T'_\infty) < 1$ , then  $I - T$  is onto.*

*Proof.* By definition  $T'_\infty \sim T$  and so  $\Sigma(T) = \Sigma(T'_\infty)$  (see [1]). It follows that  $r^+(T) = r^+(T'_\infty)$ . Hence by Theorem 2 we have  $(T) < 1$ .

Therefore the result follows from Theorem 1.

Q.E.D.

Corollary 3 extends a result obtained in [1].

**COROLLARY 4.** *Let  $T: E \rightarrow E$  be an upper semicontinuous acyclic-valued map. Assume that  $T$  is asymptotically linear and  $\alpha$ -contractive. If*

$$\lambda x \notin T'_\infty(x) \quad \text{for all } x \neq 0 \quad \text{and } \lambda \geq 1,$$

*then  $I - T$  is onto.*

*Proof.*  $T'_\infty$  is an  $\alpha$ -contraction with the same constant (see [10]) and  $r^+(T'_\infty) < 1$  (see [4]). Hence the result is a consequence of Corollary 3.

Q.E.D.

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