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EXISTENCE AND STABILITY FOR NONPOSITONE ELLIPTIC PROBLEMS†

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1. INTRODUCTION

Let Ω denote a smooth bounded domain in R^m. We consider the problem of establishing the existence of positive solutions to the problem

$$lu = -\Delta u + \sum b_j(x)D_j u = \lambda f(x, u) \qquad x \in \Omega \tag{1}$$

subject to the boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} + k(x)u = 0 \qquad x \in \partial\Omega \tag{2}$$

where $\mathbf{n}$ denotes the outward normal. For simplicity of presentation we assume: $m \geq 3$, $f \in C^1(R^{m+1})$; k, $\mathbf{b} \in C^{1+\alpha_0}(\bar{\Omega})$ for some $\alpha_0 > 0$, where $\mathbf{b} = (b_1, \ldots, b_m)$; $k \geq 0$. Unlike the vast majority of articles on this subject we consider here the nonpositone problem, i.e. we assume $f(x, 0) \leq 0$, $\not\equiv 0$. This will play an important role in what follows. The theory of nonpositone problems arises in studies on symmetry breaking [1, 2] and has been recently developed in a series of significant papers by Brown, Castro and Shivaji [3-6] and in the thesis of Unsurangsie [7] with most of the interest centered on the Dirichlet problem for (1). We are primarily interested here in the superlinear problem, i.e. in the case $f(x, \xi) \sim p(x)\xi^\gamma$ for ξ large, with $\gamma > 1$. This paper can thus be considered a continuation of our earlier work [8] on the superlinear Dirichlet problem. However, apart from the boundary condition (2), we also now no longer require that $p(x)$ be strictly positive in $\bar{\Omega}$. Consequently, our procedures will also lead to new results even for the case of Dirichlet boundary conditions, although we do not pursue this explicitly. The strict positivity of p is replaced by a definiteness condition on an associated linear problem.

This generality in $p(x)$ and the extension of the boundary condition to (2) does, however, restrict the allowed values of γ in our procedures. Specifically, if $k \not\equiv 0$ in (2), we are able to apply degree theory followed by perturbation arguments to show the existence of positive solutions to (1), (2) in the superlinear case for λ small. The general philosophy is in this case the same as employed in [8] for the Dirichlet problem and $p > 0$ in $\bar{\Omega}$ but the details are different. Unfortunately, we are now able to obtain a priori estimates only for $1 < \gamma < m/(m - 2)$ rather

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than for the "full" range $1 < \gamma < (m + 2)/(m - 2)$. These estimates are obtained by adapting the procedures established for the Dirichlet problem and $p > 0$ by Brezis and Turner [9]. Actually in [9] the range for γ is $1 < \gamma < (m + 1)/(m - 1)$ but the boundary condition (2) allows the slightly larger range in γ. As mentioned above, we were unable to obtain existence results for $m/(m - 2) \leq \gamma < (m + 2)$ since we encountered difficulties in extending known a priori bounds such as those of Gidas and Spruck [10] to our conditions.

Related results to the ones obtained for $k \not\equiv 0$ were established by Nussbaum [11] for the positive system case $f(x, 0) = 0$, $\mathbf{b} = \mathbf{0}$, and with conditions which imply that $p(x)$ is positive [11, condition R3, p. 464]. In [11] the bound $\gamma < m/(m - 2)$ was obtained for the above case by arguments similar to the ones presented below. If $p(x)$ is strictly positive, $\mathbf{b} = \mathbf{0}, f \equiv f(u)$ and for Dirichlet boundary conditions, Smoller and Wasserman [12] also obtained existence for $\gamma < m/(m - 2)$ by means of radial procedures followed by upper-lower solution arguments.

We next briefly consider for $k \not\equiv 0$ the sublinear problem $0 < \gamma < 1$. This is inherently simpler than the superlinear case, and we show how the methods mentioned above can be used to obtain the existence of positive solutions. As may be expected, we now obtain the existence of such solutions for λ large and, further, show that the sublinear problem has no positive solutions for λ small.

It is known that for large λ, the superlinear Dirichlet problem does not in general have positive solutions [3, 8]. We conjecture that this is also the case here if $k \not\equiv 0$, but we were unable to prove this fact.

If $k \equiv 0$ in (2), i.e. for the "Neumann problem", then the situation is somewhat different. Clearly in this case the kernel of the relevant operator is not empty, and the perturbation procedure mentioned above is no longer applicable since by the Fredholm alternative, problem (1), (2) cannot have positive solutions in the positive case where $f(x, u) \equiv p(x)u^{\gamma}$. Nevertheless, we can still employ degree theory and Landesman-Lazer arguments [13] to show the existence of positive solutions for small λ. Here we do not require that γ be bounded by $m/(m - 2)$. Indeed observe that the prototype nonpositone problem with $f(x, u) = p(x)[u^{\gamma} - 1]$ has solution $u \equiv 1$ for any γ and for any λ.

Since superlinear nonpositone problems have positive solutions for λ large and sublinear problems have such solutions for λ small, it may be expected that linear nonpositone problems should have positive solutions for λ in some proper interval of R^+. As a final consideration for $k \not\equiv 0$, we adapt the Neumann problem arguments and show that this indeed happens.

We conclude the paper by showing that the associated linearized equations have a negative least eigenvalue. This is connected with lack of stability [14]. Our procedures will employ Barta's inequality [15, 16] which is a consequence of positive operator arguments. We will not require some of the conditions imposed in [14]; in particular we will relax the condition $f_u > 0$. For the variational case, i.e. $\mathbf{b} \equiv \mathbf{0}$, we will briefly indicate how a proof of a more general result can be obtained immediately by the Courant min-max principle.

Finally, we remark that replacement of $-\Delta u$ in (1) by the more general elliptic expression $-\sum D_i(a_{ij}D_j u)$ can be dealt with by means of obvious changes in the proofs. In such a case $\partial u/\partial \mathbf{n}$ in (2) should also be replaced by the relevant conormal derivative. We remark that in this way one can also deal with the boundary condition

$$\nabla u \cdot \mathrm{s} + k(x)u = 0$$

for equation (1), where s denotes a nontangential direction. One need only choose a (non) symmetric matrix (a_{ij}) so that this boundary condition arises naturally. The details of such a choice are given explicitly in [17, p. 181]. Without further mention we shall assume that this procedure is also carried out for (1), (2) so that l with boundary condition (2) is viewed as (a restriction of) a map on the Sobolev space $W^{1,2}$.

2. SUPERLINEAR EXISTENCE RESULTS

We shall first deal with the superlinear problem in the case $k \not\equiv 0$. As mentioned above, we begin by establishing a priori estimates and calculating the degree for a related problem. The final step will then be obtained by a perturbation argument similar to the one used in [8]. We recall that we assume for this part that

$$\lim_{\xi \to \infty} \frac{f(x, \xi)}{\xi^{\gamma}} = p(x) \tag{3}$$

uniformly for x in $\bar{\Omega}$, with $1 < \gamma < m/(m-2)$, and $0 \le p \in C^{1+\alpha}(\bar{\Omega})$.

We require another condition, specifically that for some constant τ the symmetric problem

$$\left.\begin{aligned} -\Delta\omega + \left[-\frac{\nabla \cdot \mathbf{b}}{2} + \tau p\right]\omega &= r_1 \omega \qquad x \in \Omega \\ \frac{\partial \omega}{\partial \mathbf{n}} + \left(k + \frac{\mathbf{b} \cdot n}{2}\right)\omega &= 0 \qquad x \in \partial\Omega \end{aligned}\right\} \tag{4}$$

has the positive least eigenvalue r_1. Note that this will not always be the case, but it is easy to give sufficient conditions to ensure $r_1 > 0$. Possibly the simplest example is to require: (i) $\mathbf{b} \cdot \mathbf{n} + 2k \ge 0$ on $\partial\Omega$; and (ii) $\{x \mid \nabla \cdot \mathbf{b} \ge 0\} \subset \{x \mid p > \varepsilon > 0\}$ for some ε. Note that if $\mathbf{b} = \mathbf{0}$, then we only require that p be nontrivial.

Let l^* denote the formal adjoint of l and let J denote the positive eigenfunction of l^* with weight $p(x)$, i.e. $l^*J = \mu_1 p(x) J$ with μ_1 denoting the "least" real eigenvalue of l^*. The existence of J follows from positive operator arguments as sketched below.

We begin by collecting the following preliminary results. Brief proofs are given for the reader's convenience.

Lemma 1. (a) Let $g \in C^{\delta}$. Then there exists a unique $u \in C^{2+\delta}$ such that u satisfies $lu = g(x)$ in Ω and boundary condition (2). Furthermore

$$\|u\|_{C^{2+\delta}} \le C\|g\|_{\delta}$$

for some C independent of u.

(b) If $g(x) \ge 0$ in part (a), then $u \ge 0$ in $\bar{\Omega}$. If further, $g(x) \not\equiv 0$ then $u > 0$ in $\bar{\Omega}$.

(c) Let $\|u\|_l^2 = \int_{\Omega} |\nabla u|^2 + \int_{\partial\Omega} ku^2$. Then $\|\cdot\|_l \sim \|\cdot\|_{W^{1,2}}$.

(d) J exists and for some positive constants, α, β we have $\alpha < J < \beta$.

(e) $\mu_1 > 0$.

Proof. Parts (a) and (b) are special cases of [17, theorems 3.27–3.29]. Part (c) will follow from part (e) applied to the case $\mathbf{b} \equiv 0$, $p \equiv 1$. Indeed, $\|u\|_l^2 \ge \mu_1 \|u\|_{L^2}^2$ whence $\|\cdot\|_l \ge C_1\|\cdot\|_{W^{1,2}}$ for some C_1. The reverse inequality is obvious by the trace embedding theorem [17, theorem 1.50]. Finally, for parts (d) and (e), we employ simple two parameter arguments [18, 19]: consider the eigenvalue problem $\mathcal{L}\omega = l^*\omega - \lambda p\omega = \zeta(\lambda)\omega$. Theorem 3.17 of [17] implies for τ large that $(\mathcal{L} + \tau)^{-1}$ is a completely continuous operator on C^δ which leaves invariant the cone K of nonnegative functions. Furthermore, if $u \in K$, $u \not\equiv 0$ then there exist positive constants ξ_1, ξ_2 such that $\xi_1 \le (\mathcal{L} + \tau)^{-1}(u) \le \xi_2$ in $\bar{\Omega}$. Indeed, let $\omega = (\mathcal{L} + \tau)^{-1}f$ and observe that since for τ large, $\|\omega\|^2_{W^{1,2}} \sim ((\mathcal{L} + \tau)\omega, \omega)$. Then setting $\omega = \omega_+ - \omega_-$ gives $(f, \omega_-) = ((\mathcal{L} + \tau)\omega, \omega_-) = -((\mathcal{L} + \tau)\omega_-, \omega_-) \sim -\|\omega_-\|^2_{W^{1,2}}$. Consequently $f \ge 0$ implies $\omega_- = 0$, i.e. $\omega \ge 0$. By interior regularity and by the maximum principle, ω cannot have an interior minimum of 0. We conclude that ω is positive in Ω. Finally, by global regularity, $\partial\omega/\partial\mathbf{n} + [k + \mathbf{b} \cdot \mathbf{n}]\omega = 0$ on $\partial\Omega$. If for some $x_0 \in \partial\Omega$ we have $\omega(x_0) = 0$, then $\partial\omega/\partial\mathbf{n}(x_0) = 0$. But by the maximum principle, $\partial\omega/\partial\mathbf{n}(x_0) < 0$ as ω has a minimum at x_0, and we conclude $\omega > 0$ in $\bar{\Omega}$. That $\omega < \xi_2$ in $\bar{\Omega}$ and the compactness of $(\mathcal{L} + \tau)^{-1}$ follows immediately from theorem 3.17 of [17]. We can thus apply the results of [15, Chapter 2] to show for each $\lambda \ge 0$ the existence of a function ω, positive in $\bar{\Omega}$, such that $\mathcal{L}\omega = \zeta(\lambda)\omega$. We observe that $\xi(\lambda)$ is continuous, and $\zeta(0) > 0$ by part (b) since $\zeta(0)$ is also an eigenvalue of l. Next, we recall that Picone's identity, see e.g. [20], gives for any $\phi \in C_0^\infty(\Omega)$ by direct calculation

$$\begin{aligned} 0 &\le \int_\Omega \omega^2 \left|\nabla\left(\frac{\phi}{\omega}\right)\right|^2 + \phi\omega \sum b_j D_j\left(\frac{\phi}{\omega}\right) + \left(\frac{|\mathbf{b}|^2}{4}\right)\phi^2 \\ &= \int_\Omega \left[|\nabla\phi|^2 + \phi \sum b_j D_j \phi + \frac{|\mathbf{b}|^2}{4}\phi^2 - \lambda p \phi^2 \right] - \zeta(\lambda) \int_\Omega \phi^2. \end{aligned}$$

Since p is nontrivial, we see that $\zeta(\lambda) < 0$ for λ large. Hence, there exists μ_1 such that $\zeta(\mu_1) = 0$, and we denote the associated ω by J.

It is now a simple matter to follow the procedures in [9, 11] and obtain the following lemma.

LEMMA 2. (a) Let u be a positive solution of

$$lu = p(x)u^\gamma + tp(x)J \tag{5}$$

and boundary condition (2), with $1 < \gamma < m/(m-2)$, $t \ge 0$. Then there exists a δ, K_2 independent of u, such that

$$\|u\|_{C^\delta} + t \le K_2.$$

(b) Set $t = 0$ in (5). Then there exists a $K_1 > 0$ independent of u, such that

$$K_1 \le \|u\|_{C^\delta}.$$

Proof. Owing to the similarity with the arguments in [9, 11] we only give a brief sketch, emphasizing the differences. Since $u^\gamma \ge \xi u - \eta$ for $\xi > \mu_1$ and some constant η, we obtain

$$\mu_1 \int_\Omega puJ = \int_\Omega (lu)J \ge \xi \int_\Omega puJ + \int_\Omega tpJ^2 - N_1.$$

Recalling that J is bounded above and below, we immediately obtain that t, $\int_\Omega puJ$, and thus $\int_\Omega pu^\gamma$, are bounded. Next, integrating by parts, yields

$$\int_\Omega (|\nabla u|^2 + u \sum b_j D_j u) + \int_{\partial\Omega} ku^2 \le \int_\Omega pu^{\gamma+1} + M_1 \int_\Omega Jpu. \tag{6}$$

We observe that

$$\int_\Omega u \sum b_j D_j u = \int_{\partial\Omega} \left(\frac{\mathbf{b} \cdot \mathbf{n}}{2}\right) u^2 - \int_\Omega (\nabla \cdot \mathbf{b}) \frac{u^2}{2}$$

and recall that the symmetric problem (4) has positive least eigenvalue r_1. Hence, there exist $\varepsilon < 1$ and constant $K(\varepsilon)$ such that

$$\int_\Omega u \sum b_j D_j u \geq -\varepsilon \left[\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} k u^2 \right] - K(\varepsilon) \int_\Omega p(x) u^2. \tag{7}$$

Substituting (7) into (6) yields

$$(1 - \varepsilon) \left[\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} k u^2 \right] \leq M_1 \left(1 + \int_\Omega p(x) u^{\gamma+1} \right).$$

We now observe that for $0 < \alpha < 1$ and for any $\varepsilon' > 0$

$$\begin{aligned} \int_\Omega p(x) u^{\gamma+1} &\leq C \left(\int_\Omega p u^\gamma \right)^\alpha \left(\int_\Omega u^{\gamma + 1/(1-\alpha)} \right)^{1-\alpha} \\ &\leq C \left[\varepsilon' \int_\Omega u^{\beta + 1/(1-\alpha)} \right]^{1-\alpha} + C(\varepsilon') \left[\int_\Omega u^{1/(1-\alpha)} \right]^{(1-\alpha)} \end{aligned}$$

where we set $\beta = m/(m - 2)$ and note $u^\gamma \leq \varepsilon' u^\beta + K(\varepsilon')$. That is

$$\int_\Omega p u^{\gamma+1} \leq C \varepsilon'^{(1-\alpha)} \|u\|_{\beta + 1/(1-\alpha)}^{(\beta + 1/(1-\alpha))(1-\alpha)} + C(\varepsilon') \|u\|_{1/(1-\alpha)}.$$

We now choose α so that $1/(1 - \alpha) = m/(m - 2)$ and recall lemma 1(c) to conclude by the Sobolev estimates

$$(1 - \varepsilon) \|u\|_{W^{1,2}}^2 \leq C \varepsilon'^{(1-\alpha)} \varepsilon') \|u\|_{W^{1,2}}^2 + C(\varepsilon') \|u\|_{W^{1,2}}]$$

whence, choosing ε' small enough gives $\|u\|_{W^{1,2}} \leq \tilde{C}$.

If we now employ [17, theorem 3.28] then by a simple boot strap procedure we conclude $\|u\|_{C^\delta} \leq \tilde{K}$.

(b) Observe that

$$\mu_1 \int_\Omega p(x) u J = \int_\Omega p(x) u^{\gamma-1} u J$$

whence, $\mu_1 \leq \|u\|_{C^0}^{\gamma-1} \leq [\|u\|_{C^\delta}]^{\gamma-1}$.

Following the procedure in [8, 9, 11] we have the following theorem.

THEOREM 1. There exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, problem (1), (2) has a positive solution.

Proof. Set $T(u) = l^{-1}[p(x) u_+^\gamma]$ for $u \in C^\delta(\bar{\Omega})$. By lemma 2, we conclude that

$$u = T(u) + t l^{-1}[p(x) J]$$

does not have positive solutions for $\|u\|_{C^\delta} = K_2$, and that for $0 \leq s \leq 1$, $u = sT(u)$ does not have positive solutions for $\|u\|_{C^\delta} = K_1$. We conclude, as in [9, 11], that

$$\deg(I - T, B_{K_2} - \bar{B}_{K_1}, 0) = -1,$$

where B_r denotes the ball in C^δ of radius r. Let S denote the set of solutions in $B_{K_2} - \bar{B}_{K_1}$ and let U denote a small neighbourhood of S. Note that since the members of S are positive and S is compact, we may assume that U also consists of positive functions. On ∂U we have $\|(I - T)\omega\|_{C^\delta} > \varepsilon_1$ for some $\varepsilon_1 > 0$, by compactness. Now since for any ε we have $|f(x, \xi) - p(x)\xi^\gamma| \leq \varepsilon\xi^\gamma + K(\varepsilon)$ we conclude

$$\left|f\left(x, \frac{u}{\alpha}\right)\alpha^\gamma - p(x)u^\gamma\right| \leq \varepsilon u^\gamma + K(\varepsilon)\alpha^\gamma, \qquad \alpha > 0.$$

If we choose ε and then α sufficiently small, we conclude, again by [17], that, for $u \in U$, $\|T_1(u)\|_{C^\delta} < \varepsilon_1$ where $T_1(u) = l^{-1}[f(x, u/\alpha)\alpha^\gamma - p(x)u^\gamma]$, i.e.

$$\deg(I - T + T_1, U, 0) \neq 0.$$

Consequently, there exists $v \in U$ such that $v - T(v) + T_1(v) = 0$ and the result follows by setting $u = v/\alpha$, $\lambda = \alpha^{\gamma-1}$.

3. SUBLINEAR RESULTS

Assume the same conditions as in Section 2, except that now $0 < \gamma < 1$. We have the following theorem.

THEOREM 2. There exists $\lambda_1 > 0$ such that for $\lambda_1 < \lambda$ problem (1), (2) has a positive solution. If $f(x, 0) < 0$ in $\bar{\Omega}$ then there exists $\lambda_0 > 0$ such that there are no positive solutions in $0 < \lambda < \lambda_0$.

Proof. We apply the earlier estimates and observe that if $lu = su^\gamma$ with $0 \leq s \leq 1$, then by [17, theorem 3.28] we have

$$\|u\|_{C^\delta} \leq K\|u\|_{C^\delta}^\gamma.$$

We conclude that $\|u\|_{C^\delta} \leq K_2$ for some K_2 independent of u. On the other hand, if $lu = p(x)u^\gamma + tp(x)J$ then

$$\mu_1 \int_\Omega pJu = \int_\Omega l(u)J \geq \int_\Omega p(x)u^{\gamma-1}uJ$$

i.e. $\mu_1 \geq u^{\gamma-1}(x_0)$ for some x_0, and we conclude $\|u\|_{C^\delta} \geq K_1$ for some K_1 independent of u. Following the procedure in the proof of theorem 1 we now obtain

$$\deg(I - T, B_{K_2} - \bar{B}_{K_1}, 0) = 1$$

and, consequently, the existence of positive solutions to

$$lu = \alpha^\gamma f\left(x, \frac{u}{\alpha}\right)$$

for $\alpha > 0$ small. We observe that once again $\lambda = \alpha^{\gamma-1}$ and the result follows since $\gamma < 1$. Finally, for nonexistence, we observe that if u is any positive solution of (1), (2) then, by the same estimates as above,

$$\begin{aligned}\|u\|_{C^\delta} &\le C\lambda\|f(x,u)\|_{C^0}\\ &\le C_1\lambda[\|u\|^{\gamma}_{C^\delta} + C_2]\end{aligned}$$

for some constants C, C_1, C_2 independent of u. It follows that for any $\varepsilon > 0$ there exists a λ_0 such that if $\lambda < \lambda_0$ then $\|u\|_{C^\delta} \le \varepsilon$. On the other hand, by the maximum principle, there exists $x_0 \in \Omega$ such that $f(x_0, u(x_0)) > 0$. This implies $\|u\|_{C^\delta} > \varepsilon_0$ for some $\varepsilon_0 > 0$ as $f(x, 0) < 0$, and the result follows.

4. THE NEUMANN AND LINEAR CASES

We now pass to the consideration of the Neumann problem and assume $k \equiv 0$ in (2). As mentioned earlier, the previous approach based on a perturbation argument on a positone problem no longer works. Instead we use the nonpositone nature of f directly and observe that we shall no longer require estimate (3) or the positivity of r_1 in (4).

Since l, l^* form a Fredholm pair, we note that $\mathrm{Ker}(l^*)$, the kernel of l^*, consists of a single (normalized) function v_1, and l^{-1} is well defined and compact on $\mathrm{Im}(l) = \{g \mid g \perp \mathrm{Ker}(l^*)\}$ to $\mathrm{Im}(l^*) = \{g \mid g \perp \mathrm{Ker}(l)\}$. To see this, we recall that for τ sufficiently large, $((l + \tau)u, u) \sim \|u\|^2_{W^{1,2}}$, and observe first that if $lu = g$ with $u \in \mathrm{Im}(l^*)$, $g \in \mathrm{Im}(l)$ then $C\|u\|_{L^2} \le \|g\|_{L^2}$ for some C, or else we construct a nontrivial function $u' \in \mathrm{Im}(l^*)$ such that $lu' = 0$, a contradiction. Next it follows that

$$\begin{aligned}C_1\|u\|^2_{W^{1,2}} &\le (lu, u) + \tau(u, u)\\ &\le \|g\|_{L^2}\|u\|_{L^2} + (\tau/C)\|u\|_{L^2}\|g\|_{L^2}\end{aligned}$$

and consequently,

$$\|u\|_{W^{1,2}} \le C_2\|g\|_{L^2}.$$

Furthermore, we now apply [17, theorem 3.17 (ii)] and conclude that if also $g \in L^\infty$ and we choose p large then for some α

$$\|u\|_{C^{1+\alpha}} \le C\|u\|_{W^{2,p}} \le C_1[\|g\|_{L^\infty}].$$

Following the classic Landesman–Lazer approach we rewrite (1), (2) in the equivalent form

$$\left.\begin{aligned}u_2 &= \lambda l^{-1}\left[f(x, u_2 + cu_1) - \left(\int_\Omega v_1 f(x, u_2 + cu_1)\right)v_1\right]\\ c &= c - \left[\int_\Omega v_1 f(x, u_2 + cu_1)\right]\end{aligned}\right\} \tag{8}$$

where $u_2 \in \mathrm{Im}(l^*)$, $u_1 \in \mathrm{Ker}(l)$ and λ is viewed as a homotopy parameter.

Theorem 3. Let $J(c) = \int_\Omega v_1 f(x, cu_1)$ and assume $J(c_1) = 0$, $J'(c_1) > 0$, for some $c_1 > 0$. Then system (10), and thus the Neumann problem for (1), has a positive solution for λ small.

We observe that in the classical nonpositone cases we have $J(0) < 0$ and $J(c_1) > 0$ for all large c_1. Thus, the conditions of theorem 2 are satisfied. We also note that the procedures below associate with each different nondegenerate zero of J a different solution of (8) for small λ, thus leading immediately to multiplicity results. Finally, if the problem is variational, i.e. $\mathbf{b} = \mathbf{0}$, then $v_1 \equiv u_1 \equiv [\mu(\Omega)]^{-1/2}$.

Proof. Set $\mathbf{u} = (u_2, c)^T$ and define T, Q by

$$T\begin{pmatrix} u_2 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ c - \left[\int_\Omega v_1 f(x, u_2 + (c + c_1)u_1)\right] \end{pmatrix}$$

$$Q\begin{pmatrix} u_2 \\ c \end{pmatrix} = \begin{pmatrix} l^{-1}\left[f(x, u_2 + (c + c_1)u_1) - v_1 \int_\Omega v_1 f(x, u_2 + (c + c_1)u_1)\right] \\ 0 \end{pmatrix}.$$

Note that (8) may be expressed as

$$\mathbf{u} = T(\mathbf{u}) + \lambda Q(\mathbf{u}).$$

We observe that T is continuous, compact and, furthermore, a one-dimensional argument shows that there exists $\varepsilon > 0$ such that $I - \mu T$ has no zero on the boundary of $B_\varepsilon(0) \times (-\varepsilon, +\varepsilon)$ for $0 \leq \mu \leq 1$. Once again, $B_\varepsilon(0)$ denotes the ball of radius ε centered at the origin in $C^\alpha(\bar{\Omega})$. Without loss of generality, we may also assume $c_1 \inf(u_1) - \varepsilon(\sup(u_1) + 1) > 0$. We conclude that

$$\deg(I - T, B_\varepsilon(0) \times (-\varepsilon, \varepsilon), 0) = 1.$$

By our earlier arguments we observe that Q is well-defined continuous and compact on $B_\varepsilon(0) \times (c_1 - \varepsilon, c_1 + \varepsilon)$ since

$$\left\| l^{-1}\left(\tau - v_1 \int_\Omega \tau \cdot v_1\right)\right\|_{C^{1+\alpha}} \leq C\left\|\tau - v_1 \int_\Omega \tau \cdot v_1\right\|_{L^\infty} \leq K\|\tau\|_{L^\infty}.$$

Choosing λ small enough gives

$$\deg(I - T - \lambda Q, B_\varepsilon(0) \times (-\varepsilon + \varepsilon), 0) = 1$$

and thus for λ small, system (8) has a solution. Observe that the solution of the original equation (1) is given by $u = u_2 + (c_1 + c)u_1$ with $u_2 \in B_\varepsilon(0)$, i.e.

$$u_2 + (c_1 + c)u_1 \geq c_1 \inf(u_1) - \varepsilon(\sup u_1 + 1) > 0,$$

and u is strictly positive. Finally, note that T, Q are defined on $B_\varepsilon(0) \times (c_1 - \varepsilon, c_1 + \varepsilon)$ and we recover $u_2 \in \mathrm{Im}(l^*)$ from the properties of l^{-1}.

To conclude this section we briefly consider the linear problem $f(x, u) = p(x)(u - q(x))$ with $k \not\equiv 0$, $p(x) \geq 0$, $q(x) \geq 0$, $p \cdot q$ nontrivial. Let λ_0 denote the least eigenvalue of $lu = \lambda_0 p(x)u$. Maximum principle arguments indicate that problem (1), (2) cannot have a positive solution for $\lambda < \lambda_0$. We thus consider the following equation

$$lu - \lambda_0 p(x)u = \varepsilon p(x)(u - q(x)) \tag{9}$$

subject to condition (2). The arguments used in the proof of theorem 1 are immediately applicable with l replaced by $l - \lambda_0 p$ and we obtain the existence of positive solutions to (9), (2) for small ε. Finally, setting $v = (\lambda_0 + \varepsilon)u/\varepsilon$ yields the following theorem.

THEOREM 4. Let $f(x, u) = p(x)(u - q(x))$. Then there exists $\varepsilon > 0$ such that problem (1), (2) has a positive solution for $\lambda_0 < \lambda < \lambda_0 + \varepsilon$.

We remark that theorem 4 is also valid for the Dirichlet problem: one need only work with C^1 norms and observe that $\partial u_1/\partial \mathbf{n}$ is strictly negative. Consequently $u_2 + cu_1$ will be positive if $\varepsilon > 0$ is small enough and $u_2 \in B_\varepsilon(0)$, $u_2 = 0$ on $\partial\Omega$.

We observe that the problem $-y'' = \lambda^2(y - 1)$; $y(0) = y(2\pi) = 0$ has the explicit solution $y = 1 - \cos(\lambda(\pi - x))/\cos(\pi\lambda)$. This indicates that in general one cannot conclude the existence of positive solutions outside of (λ_0, λ_1), where λ_1 denotes the second eigenvalue of $lu = \lambda pu$.

5. STABILITY

We now consider the question of the stability of the semilinear elliptic equation (1), subject to any of the standard conditions

$$Bv = \alpha \frac{\partial v}{\partial \mathbf{n}} + k(x)(1 - \alpha)v = 0 \tag{2'}$$

with $0 \le \alpha \le 1$, so that Dirichlet boundary conditions are included. This part of the paper was motivated by the recent results of Brown and Shivaji [14]. As in [14], we show instability of u by showing that the equation linearized about u has a negative principal eigenvalue. Specifically, let $f \in C^2$ and define $l'(u)$ by

$$l'(u)v = -\Delta v + \sum b_j D_j v - \lambda f_u(x, u)v. \tag{10}$$

We then have the following theorem.

THEOREM 5. Assume that the nonlinear problem (1), (2') has a positive solution u in Ω with $\lambda > 0$, and that $f(x, 0) < 0$, and $f_{uu}(x, \xi) \ge 0$ for $\xi > 0$. If μ denotes the principal eigenvalue of (10), (2'), i.e. $l'(u)v = \mu v$, then $\mu < 0$.

Proof. We apply Barta's inequality to (10), (2') with the trial function u, i.e.

$$\mu \le \sup_{x \in \Omega} \left[\frac{l'(u)u}{u}\right].$$

This inequality is a consequence of [15, theorem 2.19]. We thus have

$$\mu \le \lambda \sup_{x \in \Omega} \frac{[f(x, u) - f_u(x, u)u]}{u}.$$

Now $f(x, 0) < 0$, and

$$\frac{d}{d\xi}(f(x, \xi) - f_u(x, \xi)\xi)\bigg|_{\xi = \xi_1} = -f_{uu}(x, \xi_1)\xi \le 0 \qquad \text{if } \xi, \xi_1 > 0.$$

We conclude that $\mu < 0$.

In some cases a modest improvement can be obtained. In particular, if $\alpha \neq 0$ in (2′) and $u > 0$ in $\bar{\Omega}$ then we can replace the assumptions that

$$f(x, 0) < 0, \qquad f_{uu}(x, \xi) \geq 0$$

by $f(x, 0) \leq 0, f_{uu}(x, \xi) > 0$ with no change in the proof. On the other hand, if $\alpha = 0$ and $k > 0$ in (2′) then $u = 0$ on $\partial\Omega$. The result still holds if we assume $f_{uu}(x, \xi) \leq 0, f(x, 0) \leq 0$ in Ω but also that $f(x_0, 0) < 0$ for some $x_0 \in \partial\Omega$. To see this, note that if $\mu = 0$ then

$$0 = \mu = \sup_{x \in \Omega} \left[\frac{l'(u)u}{u} \right].$$

Again by [15, theorem 2.19], we conclude that $cu \equiv v$ for some $c \in \mathbb{R}$, i.e. $f(x, u) \equiv f_u(x, u)u$ in Ω. But then $f(x, 0) \equiv 0$ on $\partial\Omega$ by continuity, contradicting our assumptions.

Similar improvements can be obtained if $\mathbf{b} = \mathbf{0}$. Specifically we have the following corollary.

COROLLARY 1. Assume now that $\mathbf{b} = \mathbf{0}$ and that $f(x, 0) \leq 0$, $f_{uu}(x, \xi) \geq 0$ for $\xi > 0$. Assume that for some $x_0 \in \Omega$ one of the inequalities is strict. Then $\mu < 0$.

Proof. Suppose $\alpha > 0$. The case $\alpha = 0$ is identical. Since $\mathbf{b} = \mathbf{0}$ we apply the Courant min–max principle and conclude

$$\mu = \inf_{\substack{\varphi \in W^{1,2} \\ \varphi \not\equiv 0}} \left[\frac{\int_\Omega (|\nabla\varphi|^2 - \lambda f_u(x, u)\varphi^2) + \int_{\partial\Omega} k(x)((1 - \alpha)/\alpha)\varphi^2}{\int_\Omega \varphi^2} \right].$$

If we again employ u as a test function, we obtain

$$\mu \leq \lambda \left[\frac{\int_\Omega u[f(x, u) - f_u(x, u)u]}{\int_\Omega u^2} \right].$$

Exactly as before we observe that $f(x, u) - f_u(x, u)u \leq 0$, with strict inequality holding at x_0, whence $\mu < 0$.

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