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Some Control Problems Solved via a Sliding Manifold Approach

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Abstract

In this paper by means of singular perturbation methods, we design a dynamical feedback control in order to solve control problems involving sliding manifolds. First, the proposed control strategy will be illustrated for a general control system and the properties of the resulting controller will be pointed out. Among these the elimination of the chattering phenomenon and the robustness with respect to a large class of disturbances. Then two particular control problems are considered. The first one is a tracking problem for a linear control system. The second is the attitude control of a satellite.

1 Introduction

We consider a general nonlinear control system described by the differential equations

$$\dot{x} = f(t, x, u) \tag{1.1}$$

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where f satisfies Carathéodory-type conditions to be specified later. The state variable x belongs to R^n and the control variable u belongs to U, where U is a given subset of R^m and $m \leq n$.

Together with (1.1), consider a sliding manifold

$$S = \{ x \in R^n \ : \ s(x) = 0 \} \tag{1.2}$$

where the function $s : R^n \to R^m$ is a continuously differential function. A classical control problem for (1.1)-(1.2) is the following (see [1] and the references therein).

- **Control of nonlinear variable structure system:** given a time interval $[0, T]$ with $0 < T \leq \infty$, steer and then hold the state vector x of system (1.1) on the sliding manifold S, defined by (1.2) by using feedback control laws $u = u(t, x) \in U$ which are discontinuous along the surfaces

$$s_j(x) = 0 \qquad (j = 1, 2, \ldots, m) .$$

In this paper we consider a different control problem for (1.1)-(1.2), that is we aim at solving (1.1)-(1.2) "approximatively" by using a different class of smooth feedback controls. Specifically, in Section 2 by means of the theory of singularly perturbed ordinary differential equations, we define a dynamical feedback controller as the solution of a differential equation containing a small parameter $\epsilon > 0$. This equation is directly derived from the dynamics f and the function s.

The states corresponding to such controls will not realize, in general, exactly the sliding condition (1.2). However, for any prescribed neighborhood of S, we can determine values of the parameter ϵ for which the corresponding trajectories of equation (1.1) belong to that neighborhood.

Moreover, the controller exhibits robustness with respect to a large class of disturbances. This property follows from the so-called approximability property which will be introduced in Section 3. Another important feature of these controls is that, upon passing to the limit as $\epsilon \to 0$, they converge to the equivalent control and, under suitable assumptions, the corresponding states converge to the (ideal) states determined by the equivalent control.

This approach has been introduced in [7] with the purpose of eliminating the chattering phenomenon which is one of the main drawbacks in the control of variable structure systems. In fact, for any $\epsilon > 0$, the proposed controller turns out to be absolutely continuous and so the chattering phenomenon is practically eliminated.

We like to point out that there are other approaches to solve the problem of the reduction of the chattering phenomenon for system (1.1)-(1.2) and also different notions of approximability property depending on the particular form of the nonlinearity f in (1.1). For a review on this matter we refer to [7] and the references therein.

Finally, in Section 4, we consider two particular cases of (1.1) to which the proposed approach can be successfully applied. Specifically, the first control problem is described by the linear time-invariant dynamical system

$$\dot{x} = Ax + Bu \qquad x \in R^n, \qquad u \in R^m. \tag{1.3}$$

For this system we design a control law in such a way that the state x of (1.3) follows a given reference trajectory during the time interval $[0,\infty)$. The second one deals with the satellite attitude control on a finite time interval whose mathematical model is

$$\dot{x} = A(x)x + B(x)u \qquad x \in R^n, \qquad u \in R^m \tag{1.4}$$

This constitutes an important class of nonlinear control systems for which the proposed control technique applies. Attitude control systems for rigid robotic manipulators also belong to this class, these systems have been considered in [3].

2 Feedback control design

We make the following assumptions on the dynamics f. Recall that in what follows $0 < T \leq \infty$.

(f1) For each $(p,q) \in R^n \times R^m$, the map $t \to f(t,p,q)$ is Lebesgue measurable on $[0,T]$. In addition, for almost all $t \in [0,T]$, the map $(p,q) \to f(t,p,q)$ is continuous in $R^n \times R^m$.

(f2) For each $\rho > 0$, there exists $\gamma_\rho \in L^1([0,T],R_+)$ such that, for almost all $t \in [0,T]$ and every (p,q) with $|p| + |q| \leq \rho$, one has

$$|f(t,p,q)| \leq \gamma_\rho(t) .$$

For our purposes, it is convenient to introduce a sliding manifold S which depends also on the time $t \in [0,T]$. That is, we consider

$$S = \{(t,x) \in [0,T] \times R^n \ : \ s(t,x) = 0\} .$$

We assume that

(H1) for each $t \in [0,T]$ there exists $x \in R^n$ such that $(t,x) \in S$.

For each $\epsilon > 0$, consider the following set of differential equations:

$$\begin{aligned} \dot{x} &= f(t,x,u) \\ \epsilon \dot{u} &= \frac{\partial}{\partial t} s(t,x) + \left(\frac{\partial}{\partial x} s(t,x)\right) f(t,x,u) \end{aligned} \tag{2.1}$$

where $t \in [0,T]$. Define

$$g(t,x,u) = \frac{\partial}{\partial t} s(t,x) + \left(\frac{\partial}{\partial x} s(t,x)\right) f(t,x,u) .$$

We make the following crucial assumption.

(H2) There exists a neighborhood I of the manifold S such that, for every $(t,x) \in I$, the map

$$u \rightarrow g(t,x,u)$$

is one-to-one on U and its range contains zero.

Definition 2.1 *The unique solution (if it exists) $u \in U$ of the algebraic equation*

$$g(t,x,u) = w$$

for a given $w \in R^n$ will be denoted by $u^*(t,x,w)$. When $w = 0$, the map $(t,x) \rightarrow u^*(t,x,0)$ or simply $u^*(t,x)$ is called the "equivalent control" for the system (1.1)-(1.2).

The notion of equivalent control was introduced in [13] and was recast in the above form in [1].

Observe that, by assumption (H2), for each $(t,x) \in I$ the equilibrium point $u^* = u^*(t,x)$ is isolated. Furthermore, we assume

(H3) there exists $\mu > 0$ such that, if $(t,x) \in I$, $|v - u^*(t,x)| < \mu$, and $v \neq u^*(t,x)$, then $g(t,x,v) \neq 0$;

(H4) for any $(x_0, u_0) \in R^n \times R^m$ and any $\epsilon \geq 0$, system (2.1) has a unique solution $(x(t,\epsilon), u(t,\epsilon))$ defined in the interval $[0,T]$ such that $(x(0,\epsilon), u(0,\epsilon)) = (x_0, u_0)$;

(H5) the equilibrium point $u_0^* = u^*(t_0, x_0)$ of the equation $\epsilon \dot{u} = g(t,x,u)$ is asymptotically stable for all $(t_0, x_0) \in I$. In other words, the solution of the differential equation

$$\dot{z} = g(t_0, x_0, z)$$

corresponding to the initial condition $z(0) = u_0$ converges asymptotically to the point $u_0^* = u^*(t_0, x_0)$ whenever u_0 is sufficiently close to u_0^* and $(t_0, x_0) \in I$. Moreover, we assume that the asymptotic stability is uniform in $(t_0, x_0) \in I$.

In the case when $T = \infty$ we also assume :

(H6) the origin is a uniformly asymptotically stable equilibrium point of (2.1) corresponding to $\epsilon = 0$.

Definition 2.2 *A point $(t_0, x_0, u_0) \in [0,T] \times R^n \times R^m$ such that the solution of the Cauchy problem*

$$\begin{aligned} \dot{z} &= g(t_0, x_0, z) \\ z(0) &= u_0 \end{aligned}$$

satisfies (H5) is said to belong to the domain of influence of $u_0^* = u^*(t_0, x_0)$.

Remark 2.1 Observe that if $(x_0(t), u_0(t))$, $0 \leq t \leq T$, is a solution of system (2.1) corresponding to $\epsilon = 0$ then

$$g(t, x_0(t), u_0(t)) = \frac{d}{dt} s(t, x_0(t)) = 0,$$

where $u_0(t) = u^*(t, x_0(t))$ and $0 \leq t \leq T$. Therefore, if $s(t_0, x_0(t_0)) = 0$ for some $t_0 \in [0, T]$, then $s(t, x_0(t)) = 0$ for all $t \in [0, T]$, and consequently the graph of the function $t \to x_0(t)$ lies on the manifold S.

By using the classical singular perturbation theory, see e.g. [6], [9] and [14], we can state the following theorem.

Theorem 2.1 *Suppose that* $0 < T < \infty$ *and that assumptions (f1)-(f2) and (H1)-(H5) are satisfied. Let* $(0, x_0, u_0) \in [0, T] \times R^n \times R^m$ *be a point in the domain of influence of* $u_0^* = u^*(0, x_0)$*, where* $(0, x_0) \in S$*. Then the solution* $(x(t, \epsilon), u(t, \epsilon))$ *of the Cauchy problem*

$$\begin{aligned} (\dot{x}, \epsilon\dot{u}) &= (f(t, x, u), g(t, x, u)) \\ (x(0), u(0)) &= (x_0, u_0) \end{aligned}$$

has the following properties

$$\lim_{\epsilon \to 0} x(t, \epsilon) = x_0(t) \quad \textit{uniformly in } [0, T], \tag{2.2}$$

$$\lim_{\epsilon \to 0} u(t, \epsilon) = u_0(t) \quad \textit{uniformly in } [t_1, T] \tag{2.3}$$

whenever $0 < t_1 < T$*. Here* $(x_0(t), u_0(t))$ *is the solution of the reduced system*

$$\begin{aligned} \dot{x} &= f(t, x, u) \\ x(0) &= x_0 \\ 0 &= g(t, x, u) \end{aligned}$$

so that in particular $u_0(t) = u^*(t, x_0(t))$.

Furthermore, if $T = \infty$ *and all the previous assumptions are satisfied together with (H6) then (2.2)-(2.3) hold in* $[0, \infty)$.

Proof. It is essentially a straightforward consequence of ([6], Theorem p.523) for the case $T = \infty$ and of ([14], Theorem 39.1 p.258) when T is finite.

In the sequel we will refer to $u(t, \epsilon), \epsilon > 0$, as the dynamical feedback control law corresponding to the state $x(t, \epsilon)$.

Some comments on the assumptions (H1)-(H6) are in order. Explicit conditions ensuring that (H2) is satisfied can be found in [12], while conditions ensuring (H5) and (H6) are given in [10]. Specifically, if we wish to employ the first or second Lyapounov method, we can give explicit conditions on g in order to obtain the uniform asymptotic stability of the equilibrium points $u^*(t, x)$ required by hypothesis (H5). Perhaps the simplest such condition is that $g(t, x, z)$ be differentiable with respect to z and that the eigenvalues $\lambda(t, x)$ of the matrix $\frac{\partial g}{\partial z}(t, x, u^*(t, x))$ satisfy $\lambda(t, x) \leq -\lambda_0 < 0$ for all $(t, x) \in I$. Finally, hypotheses (H1)-(H3) guarantee suitable properties of the manifold S and the map $u^*(t, x)$, while (H4) is satisfied under well-known conditions from the classical theory of ordinary differential equations.

3 Approximability property

We now investigate the behaviour of system (1.1) with respect to the presence of a certain class of perturbations. This will be done by means of a concept related to the equivalent control $u = u^*(t,x)$, i.e. the so-called approximability property. It has been illustrated through meaningful examples by Utkin [13] for control systems of the form

$$\begin{aligned} \dot{x} &= A(t,x) + B(t,x)u \\ s(x) &= 0 \\ u &= u(t,x) \in \Omega \subseteq R^n \end{aligned}$$

Using Utkin's discussion as a starting point, a definition of approximability for the fully nonlinear variable structure system

$$\begin{aligned} \dot{x} &= f(t,x,u) \\ s(x) &= 0 \\ u &= u(t,x) \in \Omega \subseteq R^m \end{aligned}$$

was proposed in [1]. There it was proved by means of the notion of G-convergence that the approximability property is satisfied for special cases of the nonlinear equation $\dot{x} = f(t,x,u)$.

Finally, in [11] it was shown that the approximability property is also fulfilled for the fully nonlinear variable structure system, provided that it is satisfied for a subclass of the set of perturbations considered in [1].

We now propose a different definition of approximability which is based on the dynamical feedback control law $u(t,\epsilon)$ and on the equivalent control $u^*(t,x)$.

In order to formulate it, let H_1 be the set of all one-parameter families $\{a_\eta : \eta > 0\}$ of R^n-valued functions $a_\eta \in L^1([0,T], R_+)$ which satisfy

$$|a_\eta(t)| \leq M(t) \qquad (\eta > 0) \tag{3.1}$$

for a.a. $t \in [0,T]$, and that

$$\sup\left\{\left|\int_0^t a_\eta(s)\,ds\right| \ : \ 0 \leq t \leq T\right\} \to 0 \qquad \text{as } \eta \to 0^+ \tag{3.2}$$

for some $M \in L^1([0,T], R_+)$ where I is given by (H2).

Consider the system

$$\begin{aligned} \dot{x} &= f(t,x,u) \\ \epsilon\dot{u} &= g(t,x,u) + a_\eta(t) \ . \end{aligned} \tag{3.3}$$

For each $\epsilon > 0$, $\eta > 0$, let $(x(t,\epsilon,\eta), u(t,\epsilon,\eta))$ be a solution of (3.3). By our assumption on $\{a_\eta\}$, we have for each $\epsilon > 0$

$$\lim_{\eta\to 0}(x(t,\epsilon,\eta), u(t,\epsilon,\eta)) = (x(t,\epsilon,0), u(t,\epsilon,0)) \tag{3.4}$$

uniformly in $[0,T]$ where $(x(t,\epsilon,0),u(t,\epsilon,0))$ is a solution of (2.1). Moreover, under the assumptions of Theorem 2.1, we get

$$\lim_{\epsilon\to 0}(x(t,\epsilon,0),u(t,\epsilon,0)) = (x_0(t),u_0(t)) \tag{3.5}$$

uniformly in $[t_1,T]$ for each $0 < t_1 < T$. We note that, in general, the controls $u(t,0,\eta)$ do not converge to $u_0(t)$ as $\eta \to 0$. Under suitable assumptions and for particular forms of the nonlinear term f this convergence was proved in [1].

The above discussion indicates that the proposed dynamical feedback control exhibits the good behaviour described in (3.4) and (3.5) in the presence of perturbations like $\{a_\eta\} \subset H_1$. Indeed (3.4) implies that the approximability property as given in [1] concerns system (2.1) together with the condition $s(t,x) = 0$ rather than equations (1.1)-(1.2). However, properties (3.4)-(3.5) allow us to formulate an approximability property for equations (1.1)-(1.2), as we will see in the following definition.

Definition 3.1 *We say that the system (1.1)-(1.2) fulfills the approximability property if and only if the following conditions hold*

(i) the hypothesis (H2) is valid;

(ii) there exists $M \in L^1([0,T],R_+)$ such that the set H_1 is not empty and such that

(iii) if $\{a_\eta\} \subset H_1$, if $\epsilon > 0$, $\eta > 0$ are positive numbers, if $(x(t,\epsilon,\eta),\ u(t,\epsilon,\eta))$ is a solution of (3.3) such that $s(0,x(0,\epsilon,\eta)) \to 0$ as $\eta \to 0$, and if $y(t)$ is the solution in $[0,T]$ of the system

$$\begin{aligned} \dot{x} &= f(t,x,u) \\ 0 &= g(t,x,u) \end{aligned}$$

satisfying $s(0,y(0)) = 0$, then the condition $\lim_{\eta\to 0} x(0,\epsilon,\eta) = y(0)$ implies that

$$\lim_{\epsilon\to 0}\lim_{\eta\to 0} x(t,\epsilon,\eta) = y(t)$$

uniformly in $[0,T]$.

Reviewing previous considerations, we can easily see that the system (1.1)-(1.2) fulfills the approximability property. Note that, in general, $\lim_{\eta\to 0}\lim_{\epsilon\to 0} x(t,\epsilon,\eta)$ does not exist.

4 Applications

4.A - A linear tracking problem

Consider the linear, time invariant system

$$\dot{x} = Ax + Bu \qquad x \in \mathbf{R}^n, \qquad u \in \mathbf{R}^m. \tag{4.1}$$

Assume that B is a matrix of full rank m. Therefore, there is a nonsingular $n \times n$ matrix T such that

$$TB = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}; \; B_2 \in \mathbf{R}^{m\times m}. \tag{4.2}$$

Furthermore, if we put

$$z = Tx \tag{4.3}$$

then we can rewrite system (4.1) in the form

$$\dot{z} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} z + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u \tag{4.4}$$

where

$$\begin{gathered} A_{11} \in \mathbf{R}^{(n-m)\times(n-m)}; \; A_{22} \in \mathbf{R}^{m\times m} \\ z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}; \; z_1 \in \mathbf{R}^{n-m}, \; z_2 \in \mathbf{R}^{m}. \end{gathered} \tag{4.5}$$

The following is a well known result of linear system theory.

Lemma 4.1 *If system (4.1) is completely controllable then the pair (A_{11}, A_{12}) is completely controllable.*

We consider now the following tracking problem:

- let $\hat{z}(t)$ be a desired state trajectory differentiable and bounded in $[0, \infty)$. It is required to design a feedback control law such that the corresponding state $z(t)$ of system (4.4), with a given initial condition z_0, follows $\hat{z}(t)$, within a prescribed error, during the time interval $[0, \infty)$.

To solve this tracking problem we use the proposed sliding manifold approach of Section 2. For this, we define a function $s : [0, \infty) \times \mathbf{R}^n \to \mathbf{R}^m$ as follows

$$s(t, z) = H\left(\hat{z}(t) - z - e^{Ct}(\hat{z}_0 - z_0)\right), \tag{4.6}$$

where $\hat{z}_0 = \hat{z}(0)$ and $z_0 = z(0), H = (H_1 \quad H_2)$ with $H_1, H_2,$ $m \times (n-m)$ and $m \times m$, matrices respectively and C is an $n \times n$ symmetric matrix. All these matrices will be chosen in the following Theorem 4.1 in a suitable way. Observe that $s(0, z_0) = 0$.

Define the related sliding manifold S as follows

$$S = \{(t, z) \in \mathbf{R}_+ \times \mathbf{R}^n : s(t, z) = 0\}. \tag{4.7}$$

Thus $(0, z_0) \in S$.

For any $\epsilon > 0$, we form the system of differential equations

$$\dot{z} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} z + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u \tag{4.8}$$

$$\epsilon\dot{u} = (H_1 \quad H_2)\left(\dot{\hat{z}} - \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} z - \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u - Ce^{Ct}(\hat{z}_0 - z_0)\right) \tag{4.9}$$

In the sequel Re$\lambda_{\min}(A)$ (Re$\lambda_{\max}(A)$) denotes the minimum (the maximum) of the real parts of all the eigenvalues of the matrix A. Clearly, we omit Re in the case where A is symmetric.

We have the following

Theorem 4.1 *Let δ, β and γ be given positive numbers. Assume that*

(i) $$Re\lambda_{\min}(H_2B_2) \geq \beta;$$

(ii) $$Re\lambda_{\min}(A_{12}H_2^{-1}H_1 - A_{11}) \geq \beta + \gamma;$$

(iii) $$-\lambda_{\max}(C) \geq \beta.$$

Moreover, assume that the following matching condition is satisfied

(iv) $$\dot{\hat{z}}_1 = A_{11}\hat{z}_1 + A_{12}\hat{z}_2.$$

Then there exists $\epsilon_0 > 0$ such that, for any $\epsilon \in (0, \epsilon_0]$, the solution $(z(t,\epsilon), u(t,\epsilon))$ to (4.8) – (4.9) satisfying $(z(0,\epsilon), u(0,\epsilon)) = (z_0, u_0)$ whenever $u_0 \in \mathbf{R}^m$ is such that

$$\mid \hat{z}(t) - z(t,\epsilon) \mid \leq \delta + a_1 e^{\lambda_{\max}(C)\, t} + a_2 e^{-\beta t} \tag{4.10}$$

$$u(t,\epsilon) = \frac{1}{\epsilon} H\left(\hat{z}(t) - z(t,\epsilon) - e^{Ct}(\hat{z}_0 - z_0)\right) + u_0 \tag{4.11}$$

with $t \in [0,\infty)$ and a_1, a_2 positive constants depending on the data.

Proof. Assumption (i) guarantees that the algebraic equation

$$g(t,z,u) := (H_1 \quad H_2)\left(\dot{\hat{z}} - \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} z - \right.$$
$$\left. - \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u - Ce^{Ct}(\hat{z}_0 - z_0)\right) = 0 \tag{4.12}$$

has a unique solution $u^*(t,z)$ for any $(t,z) \in [0,\infty) \times \mathbf{R}^n$ *(the equivalent control)*. Furthermore, for any $(t,z) \in [0,\infty) \times \mathbf{R}^n$ the equilibrium point $u^*(t,z)$ of (4.9) turns out to be globally exponentially stable. In other words, the solution $v = v(\tau)$ of the equation

$$\begin{aligned} \dot{v} &= g(t,z,v) \\ v(0) &= v_0, \end{aligned} \tag{4.13}$$

whenever $v_0 \in \mathbf{R}^m$ tends exponentially to $u^*(t,z)$ as $\tau \to \infty$ and assumption (i) assures that the exponential stability is uniform in $[0,\infty) \times R^n$.

Let $(\tilde{z}(t), \tilde{u}(t))$ be the solution of system (4.8)–(4.9) corresponding to $\epsilon = 0$ *(the reduced system)*. Define

$$\tilde{e}(t) = \hat{z}(t) - \tilde{z}(t). \tag{4.14}$$

We can easily show, by using our assumptions and (4.21), (4.23) below, that $\tilde{e} = 0$ is an equilibrim point of the equation

$$\dot{\tilde{e}} = \phi(t, \tilde{e}) \tag{4.15}$$

and is exponentially stable. Here ϕ is given by

$$\phi(t, \tilde{e}) = \dot{\hat{z}}(t) - A(\hat{z}(t) - \tilde{e}) + Bu^*(t, \hat{z}(t) - \tilde{e}) \tag{4.16}$$

Hence all the assumptions of Theorem 2.1 are satisfied for the pair $(\hat{z}(t) - z(t,\epsilon), u(t,\epsilon))$ and then we have that the solution $(z(t,\epsilon), u(t,\epsilon))$ of (4.8)–(4.9) with $(z(0,\epsilon), u(0,\epsilon)) = (z_0, u_0)$ is such that

$$\begin{aligned} \lim_{\epsilon\to 0} z(t,\epsilon) &= \tilde{z}(t) \quad \text{uniformly in } [0,\infty) \\ \lim_{\epsilon\to 0} u(t,\epsilon) &= \tilde{u}(t) \quad \text{uniformly in } [t_1,\infty) \end{aligned} \tag{4.17}$$

whenever $t_1 > 0$. Here $(\tilde{z}(t), \tilde{u}(t))$ is the solution of the reduced system (4.8) – (4.12), with $\tilde{u}(t) = u^*(t, \tilde{z}(t))$, $t \in [0,\infty)$.
Hence, given $\delta > 0$ there exists $\epsilon_0 > 0$ such that, for any $\epsilon \in (0, \epsilon_0]$ we have that

$$| z(t,\epsilon) - \tilde{z}(t) | \leq \delta \tag{4.18}$$

for any $t \in [0,\infty)$.
Now we want to estimate $| \tilde{z}(t) - \hat{z}(t) |$, $t \in [0,\infty)$.
Equation (4.12) is equivalent to

$$0 = \frac{d}{dt} H\left(\hat{z}(t) - \tilde{z}(t) - e^{Ct}(\tilde{z}_0 - z_0)\right). \tag{4.19}$$

Since for $t = 0$ the term in the bracket vanishes, we have that

$$H\left(\hat{z}(t) - \tilde{z}(t) - e^{Ct}(\tilde{z}_0 - z_0)\right) = 0, \ t \in [0,\infty), \tag{4.20}$$

or equivalently

$$H_1(\hat{z}_1(t) - \tilde{z}_1(t)) + H_2(\hat{z}_2(t) - \tilde{z}_2(t)) = He^{Ct}(\hat{z}_0 - z_0), \tag{4.21}$$

for any $t \in [0,\infty)$.
Using the matching condition (iv) we obtain

$$\begin{aligned} \dot{\tilde{z}}_1 - \dot{\hat{z}}_1 \ = \ & (A_{11} - A_{12}H_2^{-1}H_1)(\tilde{z}_1(t) - \hat{z}_1(t)) - \\ & A_{12}H_2^{-1}He^{Ct}(\hat{z}_0 - z_0). \end{aligned} \tag{4.22}$$

Therefore by assumptions (ii) and (iii) we get

$$| \tilde{z}_1(t) - \hat{z}_1(t) | \leq Le^{-\beta t}\left(| \tilde{z}_1(0) - \hat{z}_1(0) | + \frac{\| A_{12}H_2^{-1}H_1 \| \, | \hat{z}_0 - z_0 |}{\gamma}\right) \tag{4.23}$$

Here we use the estimate

$$\| e^{Qt} \| \leq Le^{(-\alpha+\gamma)t} \tag{4.24}$$

where $Re\lambda_{\max}(Q) \leq -a$, $L = L(\gamma)$ and $\gamma > 0$ is sufficiently small ([5], Proposition 3, p.4).
From (4.21) and (4.23) and assumptions (ii) and (iii) we obtain

$$
\begin{aligned}
| z(t,\epsilon) - \hat{z}(t) | \leq \delta + L \left(1 + \| H_2^{-1} H_1 \|\right) \\
\left(| \tilde{z}_1(0) - \hat{z}(0) | + \frac{\| A_{12} H_2^{-1} H_1 \| \, | \hat{z}_0 - z_0 |}{\gamma} \right) e^{-\beta t} \\
+ \| H_2^{-1} H \| \, | \hat{z}_0 - z_0 | \, e^{\lambda_{\max}(C)\, t}
\end{aligned}
\tag{4.25}
$$

$t \in [0, \infty)$, which is the assertion.

Remark 4.1 Observe that, in virtue of Lemma 4.1 it is possible to choose H_1 in such a way that assumption (ii) is satisfied. Furthermore, since $\det(B_2) \neq 0$ we can also choose H_2 to satisfy assumption (i).

Remark 4.2 Since the eigenvalues of the matrices $H_2 B_2$ and $B_2 H_2$ are the same, our assumptions $(i)-(ii)$ are equivalent to the assumptions of [10] and [17]. In these papers system (4.4) was controlled by the high gain feedback control

$$u = \frac{1}{\epsilon} H x, \quad \epsilon > 0. \tag{4.26}$$

The substitution of this control in (4.4) produces a two time scale system for which, under assumptions $(i) - (ii)$ the singular perturbation theory applies. This allows to solve several control problems for system (4.4). We want to point out that, even if the assumptions are the same, our approach is different from that based on the high-gain feedback controls (4.26). In fact, for instance, in our case the state and the control do not present, in general, the peaking phenomenon. Indeed, for $\epsilon > 0$ sufficiently small, the state is confined in a prescribed neighbourhood of the manifold S for any time $t > 0$ and the control, except for a very fast transient depending only on its initial value u_0, remains within a neighbourhood of the equivalent control. In particular, if we choose $u_0 = u^*(0, z_0)$ then the transient disappears.

Remark 4.3 If the reference trajectory is $\hat{z}(t) \equiv 0$, then the tracking problem reduces to the regulation of system (4.1) up to a prescribed error $\delta > 0$. Observe that in this case the matching condition (iv) is satisfied. In [2] an application of the above result to the problem of bringing the state x of (4.4) to zero minimizing an LQ performance index is provided together with simulation results.

4.B - Attitude control of a satellite

The second control problem concerns the attitude control of a satellite. This problem was treated in full detail in [4]. Here we emphasize the use of the proposed sliding manifold approach for solving this control problem. We first introduce in the sequel the dynamical equations.
The rigid body dynamical equations with respect to the principal axes of inertia can be written as

$$
\begin{aligned}
M_x &= I_{xx}\dot{\omega}_x + \omega_y \omega_z (I_{zz} - I_{yy}) \\
M_y &= I_{yy}\dot{\omega}_y + \omega_x \omega_z (I_{xx} - I_{zz}) \\
M_z &= I_{zz}\dot{\omega}_z + \omega_x \omega_y (I_{yy} - I_{xx})
\end{aligned}
\tag{4.27}
$$

where ω_x; ω_y and ω_z are the angular velocities, M_x, M_y and M_z the applied torques and I_{xx}, I_{yy} and I_{zz} the momenta of inertia.

Introducing the attitude error angles ϕ, θ and ψ, which are known as roll, pitch and yaw errors respectively, the angular velocities can be expressed as in [8] and [15].

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\theta & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} - \\ - \omega_0 \begin{pmatrix} \cos\theta\sin\psi \\ \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi \\ -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi \end{pmatrix}$$

where ω_0 is the orbital rate. It is known that such a description is valid only when $\theta \in (-\pi/2, \pi/2)$ and $\phi \in (-\pi/2, \pi/2)$.

Moreover the vector of applied torques can be split into two components

$$\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} = M_a + T_g$$

where M_a is the vector of torques due to the actuators and

$$T_g = 3\omega_0^2 \begin{pmatrix} (I_{zz} - I_{yy}) \sin\phi\cos\phi\cos^2\theta \\ (I_{zz} - I_{xx}) \sin\theta\cos\theta\cos\phi \\ (I_{xx} - I_{yy}) \sin\theta\cos\theta\sin\phi \end{pmatrix}$$

is the vector of gravity gradient torques.

Denote by $\xi = (\phi, \theta, \psi)^T$ the vector of error angles. Then eqn (4.27) can be rewritten as

$$J(\xi)\ddot{\xi} + \big(C(\xi)Z(\dot{\xi}) + B(\xi)\big)\,\dot{\xi} + G(\xi)\xi = M_G(\xi)\xi + M_a \tag{4.28}$$

where

$$Z^T(\dot{\xi}) = \begin{pmatrix} \dot{\phi} & 0 & 0 & 0 & 0 & 0 \\ 0 & \dot{\phi} & 0 & \dot{\theta} & 0 & 0 \\ 0 & 0 & \dot{\phi} & 0 & \dot{\theta} & \dot{\psi} \end{pmatrix}$$

and the *total inertia matrix* is

$$J(\xi) = \begin{pmatrix} I_{xx} & 0 & -I_{xx}\sin\theta \\ 0 & I_{yy}\cos\phi & I_{yy}\cos\theta\sin\phi \\ 0 & -I_{zz}\sin\phi & I_{zz}\cos\theta\cos\phi \end{pmatrix}$$

Note that this matrix, when $\theta \in (-\pi/2, \pi/2)$ and $\phi \in (-\pi/2, \pi/2)$, has all its eigenvalues in the right half complex plane.

The remaining matrices can be expressed via standard algebra. In particular the matrix $C(\xi)$ takes into account Coriolis and centrifugal effects, $B(\xi)$ torques depending on gyroscopic effects, $G(\xi)$ torques depending on the attitude and $M_G(\xi)$ gravity–gradient torques.

Rearranging eqn (4.28) in terms of the state vector $x = (\xi^T, \dot{\xi}^T)^T$ we obtain

$$\dot{x} = \begin{pmatrix} 0 & I \\ J^{-1}(\xi)\,(M_G(\xi) - G(\xi)) & -J^{-1}(\xi)\left(C(\xi)Z(\dot{\xi}) - B(\xi)\right) \end{pmatrix} x + \\ + \begin{pmatrix} 0 \\ J^{-1}(\xi) \end{pmatrix} M_a \tag{4.29}$$

The expression of M_a depends on the choice of the actuators.

By using two magnetic coils, the first along the x axis, the second along the z axis and a reaction wheel whose spin axis is aligned with the y axis, M_a can be written as in [16]

$$M_a = \begin{pmatrix} 0 & -B_y & 0 \\ -B_z & B_x & 1 \\ B_y & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_x \\ \mu_z \\ \dot{h} \end{pmatrix} + \begin{pmatrix} 0 & 0 & -h \\ 0 & 0 & 0 \\ h & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

where $(B_x, B_y, B_z)^T$ is the Earth magnetic field expressed in body axes, h is the angular momentum of the wheel and μ_x, μ_z are the components of the vector of dipole momenta in body axes on the x and z–axes respectively.

Under the usual assumption to neglect the wheel dynamics, the input–output behaviour of the wheel is described by $\dot{h} = K_v V$ where V is the control voltage of the wheel and K_v is the wheel gain. Then eqn. (4.29) can be rewritten as

$$\dot{z} = \begin{pmatrix} 0 & I & 0 \\ J^{-1}(\xi)\,(M_G(\xi) - G(\xi)) & -J^{-1}(\xi)\left(C(\xi)Z(\dot{\xi}) - B(\xi)\right) & D(\xi, \dot{\xi}) \\ 0 & 0 & 0 \end{pmatrix} + \\ + \begin{pmatrix} 0 \\ J^{-1}(\xi)T_{WB}(\xi) \\ 0 \quad 0 \quad K_v \end{pmatrix} u \tag{4.30}$$

where $z = (x^T, h)^T$ and $u = (\mu_x, \mu_z, V)^T$,

$$T_{WB}(\xi) = \begin{pmatrix} 0 & -B_y & 0 \\ -B_z & B_x & K_v \\ B_y & 0 & 0 \end{pmatrix}$$

and the vector

$$D(\xi, \dot{\xi}) = \begin{pmatrix} \dfrac{\omega_z}{I_{xx}} + \dfrac{\omega_x \tan\theta \cos\phi}{I_{zz}} \\ -\dfrac{\omega_x \sin\phi}{I_{zz}} \\ \dfrac{\omega_x \cos\phi}{I_{zz}\cos\theta} \end{pmatrix}$$

expresses the interaction between the angular momenta of the wheel and the capsule. Note that the matrix T_{WB} depends only on the vector ξ. Moreover, the entries of this matrix can be measured by means of magnetometers on the capsule.

We are now in a position to formulate our control problem. Let $\mathcal{R}$ be the open connected subset of R^6 in which the state variables x of system (4.30) take the values. Observe that $0 \in \mathcal{R}$.

We address the problem of regulating x to zero. This regulation problem can be stated as follows:

- given $\beta > 0$, $\delta > 0$ and $x_0 \in \mathcal{R}$. It is required to design a feedback control law such that the solution $z(t) = (x(t)^T, h(t))^T$ of system (4.30) with $z(0) = (x_0^T, h_0)^T$ satisfies
$$| x(t) | \leq \delta + Ae^{-\beta t} \tag{4.31}$$
for any $t \in [0, T]$, where A is a constant depending on the data and $T = 5$ days (the nominal duration of the operative phase).

To solve this problem first we rewrite system (4.30) in the following form.

$$\dot{z} = A(z)z + B(z)u$$

where $z \in \mathcal{R}_1 = \mathcal{R} \times \mathbf{R}$, $u \in \mathbf{R}^3$ and the applications $z \to A(z)$ and $z \to B(z)$ satisfy a local Lipschitz condition in $\mathcal{R}_1$. Then we define a function $s : [0, T] \times \mathcal{R}_1 \to R^3$ as follows

$$s(t, z) = H\left(-z + e^{Ct} z_0\right)$$

where $H = (H_1, H_2, 0)$ and C are matrices of suitable dimensions to be chosen later in Theorem 4.2. Finally, define the corresponding sliding manifold as

$$S = \{(t, z) \in [0, T] \times \mathcal{R}_1 : s(t, z) = 0\}.$$

Consider the function $g(t, z, u) : [0, T] \times \mathcal{R}_1 \times \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$g(t, z, u) = \frac{\partial}{\partial t} s(t, z) + \left(\frac{\partial}{\partial z} s(t, z)\right)(A(z)z + B(z)u)$$

and for any $\epsilon > 0$ consider the system of ordinary differential equations

$$\dot{z} = A(z)z + B(z)u' \tag{4.32}$$
$$\epsilon \dot{u}' = g(t, z, u') \tag{4.33}$$

where $u' = T_{WB} u$.

Now we can prove the following.

Theorem 4.2 *Let $\beta > 0$, $\delta > 0$, $\gamma > 0$. Let x_0 be given in such a way that $\tilde{x}(t) \in \mathcal{R}$ for any $t \in [0, T]$. Assume that*

$$\mathrm{Re}\lambda_{\min}(H_2^{-1} H_1) \geq \beta + \gamma$$

and

$$\mu(C) < -\mathrm{Re}\lambda_{\min}(H_2^{-1} H_1) + \gamma$$

then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ the solution $(z(t, \epsilon), u'(t, \epsilon))$, with $z(t) = (x^T(t, \epsilon), h(t, \epsilon))^T$, to (4.32)-(4.33) satisfying $(z(0, \epsilon), u'(0, \epsilon)) = (z_0, u'_0)$ is such that

$$| x(t, \epsilon) | < \delta + a_1 e^{\mu(C)t} + a_2 e^{-\beta t} \tag{4.34}$$

with

$$u'(t, \epsilon) = \frac{1}{\epsilon}(H_1\ H_2\ 0)\left(-z(t, \epsilon) + e^{Ct} z_0\right) + u'_0 \tag{4.35}$$

where $t \in [0, T]$ and a_1 and a_2 are positive constants depending on H_1, H_2, C and x_0.

Proof. Denote by $(\tilde{z}(t), \tilde{u}'(t))$ the solution of the reduced system associated to (4.32)–(4.33), with $\tilde{z}(0) = z_0$. Since $s(t, \tilde{z}(t)) = 0$ for all $t \in [0, T]$, we have

$$(H_1\ H_2)\tilde{x}(t) = He^{Ct}z_0.$$

This implies

$$\dot{\tilde{\xi}}(t) = -H_2^{-1}H_1\tilde{\xi}(t) + H_2^{-1}He^{Ct}z_0 \tag{4.36}$$

so by assumptions on the matrices $H_2^{-1}H_1$ and C we have

$$\lim_{t\to\infty} \tilde{x}(t) = 0$$

exponentially. Consider now a neighbourhood I of S such that the set $\{z \in R^7 : (t, z) \in I$ for some $t \in [0, T]\}$ is bounded with respect to the x–variable and $I \subset [0, T] \times \mathcal{R}_1$. For $\epsilon > 0$ sufficiently small, under our assumptions it is not hard to show that conditions (H1)–(H5) are verified in I. In fact, (H1) is a direct consequence of the definition of s and (H2), (H3), (H5) can be derived from the fact that for any compact set $K \subset \mathcal{R}$ there exists a constant $\alpha_K > 0$ such that $\mathrm{Re}\lambda_{\min}(H_2J^{-1}(\xi)) \geq \alpha_K$ for each $\xi \in K$. Finally, (H4) holds since $\tilde{z}(t) \in \mathcal{R}_1$, it is defined on all $[0, T]$ and the maps $A(\cdot)$, $B(\cdot)$ are locally Lipschitz. Hence by Theorem 2.1 we get the following inequalities

$$| x(t,\epsilon) | \leq | x(t,\epsilon) - \tilde{x}(t) | + | \tilde{x}(t) | < \delta + | \dot{\tilde{\xi}}(t) | + | \tilde{\xi}(t) | \tag{4.37}$$

Finally, using our assumptions, eqn. (4.36) and ([5], Proposition 3, p. 4), for sufficiently small $\gamma > 0$ from (4.37) we obtain

$$| x(t,\epsilon) | < \delta + a_1e^{\mu(C)t} + a_2e^{(-\mathrm{Re}\lambda_{\min}(H_2^{-1}H_1)+\gamma)t}$$

where, since $h_0 = 0$ implies $|x_0| = |z_0|$, we obtain

$$a_1 = \| H_2^{-1}H \| \, | x_0 |$$

and

$$a_2 = L(1 + \| H_2^{-1}H_1 \|)\left(| \xi(0) | + \frac{\| H_2^{-1}H \| \, | x_0 |}{| \mu(C) + \mathrm{Re}\lambda_{\min}(H_2^{-1}H_1) - \gamma |}\right)$$

where $L = L(\gamma)$ is the constant of Proposition 3 in [5] . Then (4.34) can be easily derived.

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