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AN EXISTENCE RESULT FOR A CLASS OF ASYMPTOTIC BOUNDARY VALUE PROBLEMS*

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Abstract. In this paper we give some results on the solvability of a class of nonlinear boundary value problems, where the boundary condition is of asymptotic type. The results are based on a continuation principle in locally convex spaces. Examples illustrating the meaning of the assumptions are also given.

1. Introduction. In this paper we consider the problem of finding a solution of the following nonlinear boundary value problem (b.v.p.)

$$\begin{cases} \dot{x} = A(t)x + f(t,x), & t \in J = [0, +\infty) \\ x(0) \in V, \ \ x(+\infty) = 0, \end{cases} \tag{1.1}$$

where $t \to A(t)$ is a continuous $n \times n$ matrix function defined on J, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map and V is a given linear subspace of $\mathbb{R}^n$.

An approach for solving boundary value problems of the general form

$$\begin{cases} \dot{x} = \varphi(t,x) \\ x \in S, \end{cases} \tag{1.2}$$

where S is a subset of $C(J, \mathbb{R}^n)$, consists in reducing the b.v.p. to a fixed point problem of the form

$$x = T(x). \tag{1.3}$$

Here T is a (possibly nonlinear) operator defined on a suitable subset X of the Fréchet space $C(J, \mathbb{R}^n)$, equipped with the standard topology of uniform convergence on compact subintervals of J. T is defined in such a way that all its fixed points in X are solutions of the original b.v.p. (see e.g. [3]).

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Obviously this procedure can also be adapted to deal with b.v.p. involving differential inclusions (see [1] and the references therein). Returning to the fixed point equation $x = T(x)$, $x \in X$, the main tool for solving it, is represented by the Schauder-Tychonoff fixed point theorem. This requires that X be closed and convex, $T(X) \subset X$ and that T be continuous and compact with respect to the topology of X (see the references in [2]). However, in general, both the continuity and the compactness of T are very hard to check. In fact even in very simple cases (see [2]), the operator T may be discontinuous with respect to the standard topology of $C(J, \mathbb{R}^n)$. On the other hand, if we consider finer topologies for X then the compactness of T is harder to check.

In [2] a method, based on a variant of Schauder's linearization method, is proposed to prove that, under suitable a priori estimates, the restriction of T to any bounded set $X \subset C(J, \mathbb{R}^n)$ such that $T(X) \subset X$ is compact and continuous with respect to the topology of $C(J, \mathbb{R}^n)$.

Moreover in [5] and [6], by using the results of [2], the authors respectively proposed a continuation method and defined a fixed point index in locally convex spaces.

On the other hand, there are also b.v.p. for which the methods based on the compactness of the operator T are no longer applicable (see e.g. [7], and the references therein).

Our main goal here is to provide sufficient conditions for the solvability of (1.1). We achieve this by means of Theorem 2.1 in Section 2, whose proof is based on the already mentioned continuation principle in locally convex spaces stated in [5] by Furi-Pera. For the reader's convenience we now review this continuation principle.

Consider the boundary value problem

$$\begin{cases} \dot{x} = \varphi(t, x), & t \in I \\ x \in S, \end{cases} \tag{1.4}$$

where $\varphi : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map, I is a (possibly noncompact) interval of $\mathbb{R}$ and S is a non empty subset of $C(I, \mathbb{R}^n)$.

Theorem 1.1([2]). *Let $g : I \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ be continuous and satisfying $g(t, s, s, 1) = \varphi(t, s)$ for all $(t, s) \in I \times \mathbb{R}^n$. Assume that*

(c_1) *there exist a convex closed subset Q_1 of $C(I, \mathbb{R}^n)$ and a bounded closed subset S_1 of S such that the problem*

$$\begin{cases} \dot{x} = g(t, x, q, \lambda), & t \in I,\ \lambda \in [0,1] \\ x \in S_1 \end{cases} \tag{1.5}$$

is uniquely solvable for each $(q, \lambda) \in Q_1 \times [0,1]$;

(c_2) *the solution of (1.5) corresponding to any $(q, 0)$ belongs to Q_1;*

(c_3) *if $\{(x_j, \lambda_j)\}_{j \in \mathbb{N}}$ is a sequence in $S_1 \times [0,1]$ such that $\lambda_j \to \lambda \in [0,1]$ and x_j converges to a solution $x \in Q_1$ of*

$$\dot{x} = g(t, x, x, \lambda), \quad t \in I,\ \lambda \in [0,1],$$

then x_j belongs to Q_1 for j sufficiently large.

Then system (1.5) *has a solution in* $Q_1 \cap S_1$.

We will apply the previous result with $I = J$, $S = \{x \in C(I, \mathbb{R}^n),\ x(0) \in V,\ x(+\infty) = 0\}$ and $g(t,s,r,\lambda) = A(t)s + \lambda f(t,r)$ for any $(t,s,r) \in I \times \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in [0,1]$.

The paper is organized as follows. In Section 2, under a suitable growth condition on the nonlinearity f, Theorem 2.1 shows that if $A(t)$ has an exponential dichotomy with associated projection P such that $\operatorname{Im} P \oplus V = \mathbb{R}^n$ and if we assume a priori bounds on the solutions of the family

$$\begin{cases} \dot{x} = A(t)x + \lambda f(t,x), & t \in J = [0,+\infty) \\ x(0) \in V,\ x(+\infty) = 0, \end{cases} \tag{1.6}$$

where $\lambda \in [0,1]$, then, by means of Theorem 1.1, the b.v.p. (1.1) is solvable. We give two examples of application of Theorem 2.1 for which the a priori bounds can be found by direct calculation. In Section 3, we assume that A is a $n \times n$ real constant matrix and then we decompose $\mathbb{R}^n$ into the direct sum of the root spaces associated with the eigenvalues of A, respectively with positive or null real part and with negative real part.

Theorem 3.1 gives conditions on the asymptotic behavior of the nonlinearity f with respect to the root spaces, which ensure the required a priori bounds. We end with some examples to illustrate the meaning of the assumptions.

2. Results.

We will make the following assumptions

(H_1) the function $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and, if

$$\alpha_K(t) = \sup_{|x| \le K} |f(t,x)|,$$

where $K > 0$, then $\alpha_K \in L^1(J)$,

(H_2) the $n \times n$ matrix $A(t)$ is continuous on J, and if $X(t)$ is the fundamental matrix such that $X(0) = I$, there exists a projection P for which

$$|X(t)PX^{-1}(s)| \le Le^{-\alpha(t-s)} \quad t \ge s,$$

$$|X(t)(I-P)X^{-1}(s)| \le L \quad s \ge t,$$

with $\alpha, L > 0$;

(H_3) there exists a positive constant M such that if x is a solution of the problem

$$\dot{x} = A(t)x + \lambda f(t,x), \quad x(0) \in V, \quad x(+\infty) = 0, \quad \lambda \in [0,1], \tag{H_λ}$$

we have

$$\sup_{t \in [0,\infty)} |x(t)| \le M;$$

(H_4) $\{0\} \subsetneq V \subsetneq \mathbb{R}^n$ is a linear subspace such that

$$\operatorname{Im} P \oplus V = \mathbb{R}^n.$$

We are now in the position of proving the following:

Theorem 2.1. *If* (H_1), (H_2), (H_3), *and* (H_4) *are satisfied, then problem* (1.1) *has a solution.*

Proof. Let $Q_1 = \{q \in C(J, \mathbb{R}^n) : |q(t)| \le M+1 \text{ for any } t \in J\}$. Clearly Q_1 is a convex, closed subset of $C(J, \mathbb{R}^n)$. By the second inequality in (H_2) we have

$$|(I-P)X^{-1}(s)| \le L \quad \text{for any } \ s \ge 0,$$

so that by definition of α_{M+1} and (H_1) we have for any $q \in Q_1$

$$\left| \int_0^{+\infty} (I-P)X^{-1}(s) f(s, q(s))\, ds \right| \le L \|\alpha_{M+1}\|_{L^1(J)}. \tag{2.1}$$

We let

$$v_q = \int_0^{+\infty} (I-P)X^{-1}(s) f(s, q(s))\, ds.$$

By (H_4), for any $q \in Q_1$, there exists a unique $\xi_q \in \operatorname{Im} P$ such that $\xi_q - v_q \in V$, so that, by (2.1) we deduce that

$$|\xi_q| \le N \|\alpha_{M+1}\|_{L^1(J)} \tag{2.2}$$

for some positive constant N.

Consider now the family of systems for $q \in Q_1$,

$$\dot{x} = A(t)x + \lambda f(t, q(t)), \quad x(0) \in V. \tag{$H_{\lambda,q}$}$$

Observe that the only solution of $(H_{\lambda,q})$ such that $x(+\infty) = 0$ is given by

$$\begin{aligned} x_{\lambda,q}(t) = \lambda \Big[X(t)\xi_q &+ \int_0^t X(t)PX^{-1}(s) f(s, q(s))\, ds \\ &- \int_t^{+\infty} X(t)(I-P)X^{-1}(s) f(s, q(s))\, ds \Big], \end{aligned}$$

so that by (H_2), (H_3), (2.1), and (2.2) we have

$$\begin{aligned} |x_{\lambda,q}(t)| &\le L\Big(N\|\alpha_{M+1}\|_{L^1(J)} e^{-\alpha t} + e^{-\alpha t} \int_0^t e^{\alpha s} \alpha_{M+1}(s)\, ds + \int_t^{+\infty} \alpha_{M+1}(s)\, ds \Big) \\ &= \psi(t). \end{aligned}$$

It is easy to verify that we have

$$\lim_{t \to +\infty} \psi(t) = 0.$$

For this aim it will be sufficient to show that

$$\lim_{t \to +\infty} e^{-\alpha t} \int_0^t e^{\alpha s} \alpha_{M+1}(s)\, ds = 0. \tag{2.3}$$

Fix $\epsilon > 0$; by (H_1) there exists $\bar{t} = \bar{t}_\epsilon$ such that

$$\int_{\bar{t}}^{t} \alpha_{M+1}(s)\, ds < \epsilon \quad \forall t \geq \bar{t},$$

and we have for $t \geq \bar{t}$

$$\begin{aligned} e^{-\alpha t} \int_0^t e^{\alpha s} \alpha_{M+1}(s)\, ds &= e^{-\alpha t} \int_0^{\bar{t}} e^{\alpha s} \alpha_{M+1}(s)\, ds + e^{-\alpha t} \int_{\bar{t}}^t e^{\alpha s} \alpha_{M+1}(s)\, ds \\ &\leq C e^{-\alpha t} + \epsilon. \end{aligned}$$

Now if we consider the upper limit for $t \to +\infty$ of each member of the last inequality, with $\epsilon > 0$ arbitrary, we obtain (2.3). Then, if we let

$$S_1 = \{x \in C(J, \mathbb{R}^n) : x(0) \in V,\ |x(t)| \leq \psi(t)\, \forall t \in J\},$$

we have proved that assumption (c_1) of Theorem 1.1 is verified, i.e., S_1 is a closed bounded subset of $C(J, \mathbb{R}^n)$ and for any $q \in Q_1$ there exists a unique solution $x_{\lambda,q} \in S_1$ of $(H_{\lambda,q})$. Moreover, (c_2) holds too because if $\lambda = 0$ then $x(t) \equiv 0$ is the only solution of $(H_{0,q})$ for any $q \in Q_1$, and $0 \in Q_1 \cap S_1$. It remains to check the validity of (c_3), i.e., if $x(t)$ is a solution of

$$\dot{x} = A(t)x + \lambda f(t,x), \quad x(0) \in V,\ x(+\infty) = 0,$$

and if $x_n \to x$ uniformly on compact subsets of J and $\lambda_n \to \lambda \in [0,1]$, then there exists $\bar{n}$ such that $x_n \in Q_1$ for any $n > \bar{n}$. This is easy to prove by virtue of (H_3) and by the definition of S_1.

Remark 2.1. If $V = \{0\}$, $P = I$ then (H_2) means that $A(t)$ is an exponentially stable matrix. If $V = \mathbb{R}^n$, $P = 0$ then (H_2) means that $A(t)$ is uniformly stable.

Remark 2.2. Assumption (H_2) is satisfied if $A(t)$ has an exponential dichotomy. Recall (see [4]) that, if $X(t)$ is a fundamental matrix for the linear differential equation

$$\dot{x} = A(t)x,$$

the $n \times n$ continuous coefficient matrix $A(t)$ is said to possess an exponential dichotomy if there exist a projection P and positive constants L, α such that

$$|X(t)PX^{-1}(s)| \leq Le^{-\alpha(t-s)} \quad \text{for } t \geq s,$$

$$|X(t)(I-P)X^{-1}(s)| \leq Le^{-\alpha(s-t)} \quad \text{for } s \geq t.$$

This is the case for a matrix of the form

$$A(t) = A + B(t),$$

when every eigenvalue of A has real part different from zero and $B(t)$ is bounded on J with sufficiently small $L^\infty(J)$ norm. This follows from [4] Proposition 4.1. Moreover by the same proposition, we have that the projection depends continuously on $B(t)$ in $L^\infty(J)$, so that the complementary condition (H_4) is not destroyed by small $L^\infty(J)$ perturbations.

Observe that (H_2) is also satisfied if $A(t)$ is of the form $A(t) = A + B(t)$, where A possesses an ordinary dichotomy, that is, the previous inequalities hold with $\alpha = 0$, and $B(t)$ is an $L^1(J)$-small perturbation. Furthermore, the projection P depends continuously on $B(t)$ in $L^1(J)$. This can be shown by adapting the proof of Proposition 4.2 in [4].

Example 2.1. Consider the problem

$$\ddot{x} + [\alpha + a(t)]\dot{x} + [-\beta + b(t)]x = h(t, x, \dot{x}), \quad x(0) = 0,\ x(+\infty) = 0, \tag{2.4}$$

where $a : J \to \mathbb{R}$, $b : J \to \mathbb{R}$ are continuous functions, α and β are positive constants, and $h : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function of (t, x_1, x_2). Assume that there exists a constant M for which

$$x_1 h(t, x_1, 0) > 0 \quad \text{for any } |x_1| > M,\ t \in J \tag{2.5}$$

and that, for any compact set $K \subseteq \mathbb{R}$, there exists $\alpha_K \in L^1(J)$ such that

$$|h(t, x_1, x_2)| \le \alpha_K(t) \quad \text{for any } x_1 \in K,\ x_2 \in \mathbb{R},\ t \in J. \tag{2.6}$$

Then there exists $\delta = \delta(\alpha, \beta)$ such that if

$$\|a\|_{L^\infty(J)} + \|b\|_{L^\infty(J)} < \delta \tag{2.7}$$

then problem (2.4) has a solution. In fact consider the system

$$\dot{z} = [A + B(t)]z + f(t, z), \quad z(0) \in V,\ z(+\infty) = 0, \tag{2.8}$$

where $z = (x_1, x_2)$,

$$A = \begin{pmatrix} 0 & 1 \\ -\beta & \alpha \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ b(t) & a(t) \end{pmatrix}, \quad f(t, z) = \begin{pmatrix} 0 \\ h(t, z) \end{pmatrix},$$

$$V = \{(0, x_2) : x_2 \in \mathbb{R}\}.$$

If $z(t) = (x_1(t), x_2(t))$ is a solution of (2.8), then obviously $x_1(t)$ is a solution of (2.4). We note that A has an exponential dichotomy and that the spectral projection verifies the complementary condition (H_4). Then, by Remark 2.2 there exists a constant $\delta = \delta(\alpha, \beta)$ such that if (2.7) holds then $A + B(t)$ has an exponential dichotomy with the projection Q, associated to the fundamental matrix $Z(t)$ such that $Z(0) = I$, which verifies (H_4).

Since (2.6) holds, hypothesis (H_1) is satisfied. Finally in order to solve (2.8), we have to show that (H_3) is satisfied.

Consider the family of problems

$$\dot{z} = [A + B(t)]z + \lambda f(t, z), \quad z(0) \in V,\ z(+\infty) = 0 \tag{H_λ}$$

with $\lambda \in [0,1]$. Let $z(t) = (x_1(t), x_2(t)) = (x_1(t), \dot{x}_1(t))$ be a solution of (H_λ), for some $\lambda \in [0,1]$, and let $t_0 \in J$ be such that $|x_1(t_0)| = \max\limits_{t\in J} |x_1(t)|$. We have

$$\ddot{x}_1(t_0)x_1(t_0) = [\beta - b(t_0)]x_1^2(t_0) + \lambda x_1(t_0)h(t_0, x_1(t_0), 0),$$

and since $\ddot{x}_1(t_0)x_1(t_0) \leq 0$, if $\delta \leq \beta$, then from (2.5) we obtain

$$|x_1(t)| \leq |x_1(t_0)| \leq M \quad \forall t \in J. \tag{2.9}$$

On the other hand $z(t)$ satisfies the following integral equation

$$z(t) = \lambda\Big[-Z(t)\int_t^{+\infty}(I-Q)Z^{-1}(s)f(s,z(s))\,ds + Z(t)\int_0^t QZ^{-1}(s)f(s,z(s))\,ds\Big],$$

and by (2.6) and (2.9) we obtain

$$|z(t)| \leq C\|\alpha_M\|_{L^1(J)} \quad \forall t \in J,$$

so that hypothesis (H_3) is satisfied.

Example 2.2. Consider the problem

$$\ddot{x} + [\alpha + a(t)]\dot{x} + b(t)x = h(t, x, \dot{x}), \quad x(0) = 0,\ x(+\infty) = 0, \tag{2.10}$$

where $a, b : J \to \mathbb{R}$ are continuous functions which are absolutely integrable in J, $\alpha > 0$, $h(t, x_1, x_2) \in C(J \times \mathbb{R} \times \mathbb{R})$ satisfies (2.6) and the following assumption:

$$\liminf_{|x_1|\to+\infty} \frac{h(t,x_1,0)}{x_1} = \beta(t) > b(t) \quad \text{exists uniformly for } t \in J. \tag{2.11}$$

Then there exists $\delta > 0$ such that if

$$\|a\|_{L^1(J)} + \|b\|_{L^1(J)} < \delta$$

problem (2.10) has a solution. In fact consider the problem

$$\dot{z} = [A + B(t)]z + f(t,z), \quad z(0) \in V,\ z(+\infty) = 0,$$

where z, $f(t,z)$, V are defined as in Example 2.1 and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ -b(t) & -a(t) \end{pmatrix},$$

A possesses an ordinary dichotomy with projection P, where

$$P = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix},$$

and P is such that

$$\left|e^{tA}Pe^{-sA}\right| \leq Le^{-\alpha(t-s)}, \quad t \geq s,$$

$$\left|e^{tA}(I-P)e^{-sA}\right| \leq L, \quad s \geq t.$$

As $\operatorname{Im} P = \{c(1,-\alpha) : c \in \mathbb{R}\}$, the complementarity condition holds. By Remark 2.2 it follows that if δ is sufficiently small (H_2) and (H_4) hold for the fundamental matrix $Z(t)$ of $\dot{z} = [A + B(t)]z$ such that $Z(0) = I$ and for a projection Q. By using (2.11) and the same arguments as employed in Example 2.1 we obtain the bounds (H_3) on the solutions of (H_λ), and so, by Theorem 2.1, the existence of a solution of our problem.

3. The autonomous case. In this section we consider the case when the matrix A is constant in (1.1). Thus, we consider the b.v.p.

$$\dot{x} = Ax + f(t,x), \quad x(0) \in V, \ x(+\infty) = 0. \tag{3.1}$$

In this case we can provide conditions which guarantee that (H_2) and (H_3) of Theorem 2.1 are verified. For this, assume that

(H_2') If an eigenvalue of A has null real part, then its algebraic multiplicity is equal to its geometric multiplicity.

Let V_1, V_2 be the root subspaces associated with the eigenvalues of A respectively with positive or null real part and with negative real part. We denote by P_1 the projection of $\mathbb{R}^n$ onto V_1 in the direction of V_2, and let $P_2 = I - P_1$.

By (H_2') we can choose a base $q_1, \dots, q_p$ for V_1, and a base for V_2, $q_{p+1}, \dots, q_n$, such that the $n \times n$ matrix

$$Q = (q_1 \cdots q_n)$$

has the property that: the change of coordinates

$$x = Q\xi$$

transforms the solutions $x : t \to x(t)$ of (3.1) into the solutions $\xi : t \to \xi(t)$ of

$$\frac{d\xi}{dt} = Q^{-1}AQ\xi + Q^{-1}f(t, Q\xi) = A_Q\xi + g(t,\xi),$$

where

$$A_Q = Q^{-1}AQ = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Moreover, if J_1 and J_2 are the canonical isomorphisms between V_1 and $\mathbb{R}^p$ and between V_2 and $\mathbb{R}^{n-p}$, and if $\xi_1 = J_1P_1x$, $\xi_2 = J_2P_2x$, we have

$$(\xi_1, A_1\xi_1) \geq 0, \quad \xi_1 \in J_1V_1, \tag{3.2}$$

$$(\xi_2, A_2\xi_2) < 0, \quad \xi_2 \in J_2V_2, \ \xi_2 \neq 0. \tag{3.3}$$

We now need some more assumptions on the nonlinear part g of the vector field in order to obtain *a priori* bounds on the solutions. In the sequel, $\xi = (\xi_1, \xi_2)$.

(H_3') There exists $M_1 > 0$ such that

$$(\xi_1, J_1P_1g(t,\xi)) \geq 0, \tag{3.4}$$

for any $|\xi_1| \geq M_1$, $t \in J$ and any $\xi_2 \in J_2V_2$, and

$$|J_2P_2g(t,\xi)| \leq a_K(t) + b_K(t)|\xi_2|, \tag{3.5}$$

for any $\xi_2 \in J_2V_2$, $t \in J$ and for any $\xi_1 \in K$, where K is an arbitrary compact set of J_1V_1 and $a_K, b_K \in L^1[0,+\infty)$ are functions depending on K.

We can prove the following:

Theorem 3.1. *If* (H_1), (H_2') *and* (H_3') *are satisfied, and* V *is a subspace of* $\mathbb{R}^n$ *such that*

$$V \oplus V_2 = \mathbb{R}^n, \tag{3.6}$$

then problem (3.1) *is solvable.*

Proof. Let us consider the family of problems

$$\dot{x} = Ax + \lambda f(t,x), \quad x(0) \in V,\ x(+\infty) = 0, \tag*{(H_λ)}$$

where $\lambda \in [0,1]$. By (3.6), if $\lambda = 0$ then the only solution of (H_0) is $x(t) = 0$ for any $t \in J$. Let $\lambda \in (0,1]$ and $\xi(t) = Q^{-1}x(t)$, where $x(t)$ is any possible solution of (H_λ). Then

$$\frac{1}{2}\frac{d}{dt}|\xi_1(t)|^2 = (\dot{\xi}_1(t), \xi_1(t)) = (\xi_1(t), A_1\xi_1(t) + \lambda J_1 P_1 g(t, \xi(t))),$$

and by assumptions (3.2) and (3.4) it follows that

$$|\xi_1(t)| \le M_1, \quad \text{for any } t \in J. \tag{3.7}$$

In particular

$$|\xi_1(0)| \le M_1,$$

and since P_1 is a linear homeomorphism between V and V_1, we have that there exists $D > 0$ such that

$$|\xi_2(0)| \le D. \tag{3.8}$$

On the other hand

$$\xi_2(t) = e^{tA_2}\xi_2(0) + \int_0^t e^{(t-s)A_2} P_2 g(s, \xi(s))\, ds.$$

It is well known that there exists $\alpha > 0$, $L \ge 1$ such that

$$|e^{tA_2}| \le Le^{-\alpha t} \quad t \ge 0, \tag{3.9}$$

and by (3.5), (3.8) and (3.9) we have that

$$|\xi_2(t)| \le De^{-\alpha t} + L\int_0^t [a_{M_1}(s) + b_{M_1}(s)|\xi_2(s)|]\, ds.$$

Thus by Gronwall's lemma it follows that there exists $M_2 > 0$ with the property that

$$|\xi_2(t)| \le M_2, \quad \text{for any } t \in J. \tag{3.10}$$

By (3.7) and (3.10), we have that there exists $M > 0$ such that if $x(t)$ is a solution of (H_λ), then

$$|x(t)| \le M, \quad \text{for any } t \in J. \tag{3.11}$$

We point out that the constant M depends on the subspace V. Thus we proved that (H_3) of Theorem 2.1 is satisfied. Since (H_2') implies (H_2), we have the solvability of the considered b.v.p. (3.1) by means of Theorem 2.1.

Example 3.1. In this example we show that, in general, assumption (H_4) cannot be dropped if the problem (1.1) has to have a solution. Furthermore, the following example shows that the function $V \to M(v)$ is not necessarily bounded,

$$\begin{cases} \dot{\xi}_1 = \xi_1 + v(t,\xi_1)h(\xi_2) \\ \dot{\xi}_2 = -\xi_2[1 + w(t,\xi_1)] + a(t) \\ \alpha\xi_1(0) + \beta\xi_2(0) = 0 \\ \xi_1(+\infty) = \xi_2(+\infty) = 0, \quad \alpha,\beta \in \mathbb{R},\, \beta \neq 0, \end{cases} \tag{3.12}$$

$v : J \times \mathbb{R} \to \mathbb{R}$, $w : J \times \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}^+$, $a : \mathbb{R} \to \mathbb{R}$ are all continuous maps. If we fix $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, and we assume further that

$$v(t,\xi_1)\xi_1 \geq 0 \ \text{ for any } |\xi_1| \geq M_1 \text{ and } t \in J,$$

and that, if

$$\alpha_K(t) = \sup_{|\xi_1| \leq K} [v^2(t,\xi_1) + w^2(t,\xi_1)] + a^2(t)$$

then

$$\alpha_K \in L^1(J),$$

the problem (3.12) has a solution.

But if we suppose that

$$v(t,\xi_1) > 0 \quad \text{for any } \xi_1 \geq 0 \text{ and } t \in J, \tag{3.13}$$

then any solution starting from the ξ_2-axis has the property that $\xi_1(t)$ increases as $t \to +\infty$, so that the problem (3.12) has no solution for $\beta = 0$. We note further that if (3.13) holds and the vector field is sufficiently smooth, say C^1, then if we let

$$E = \{\xi(0) \in \mathbb{R}^2 : \xi(t) \text{ is a solution of (3.12) for some } \alpha, \beta \in \mathbb{R},\, \beta \neq 0,\},$$

it results that E is not bounded. In fact suppose by contradiction that there exists $C > 0$ such that

$$|\xi| \leq C \quad \text{for any } \xi \in E, \tag{3.14}$$

consider the subspaces

$$V_n = \{\xi_1 + \beta_n\xi_2 = 0 : \beta_n \to 0\},$$

and let $\xi_n(t)$ be a solution of (3.12) with $\xi_n(0) \in V_n$. By (3.14) we can assume $\xi_n(0) \to \xi \in V_2 = \{(0,\xi_2) : \xi_2 \in \mathbb{R}\}$. Let $\xi(t)$ be the solution of the differential equation with $\xi(0) = \xi$. By the continuous dependence of the solution on the initial value we have

$$\liminf_{t \to +\infty} |\xi(t)| = 0,$$

while for (3.13), $\xi_1(t)$ increases with t. So (3.14) is false, and E is unbounded.

Example 3.2. As in (H_3') we use a coordinate system involving V_1 and V_2. It is natural to consider vector fields with some "symmetry property" with respect to the root subspaces of A and which is represented by conditions similar to (3.4) and (3.5). For example,

$$g(t,\xi) = \phi(t,\xi_1) + \eta(t,\xi), \tag{3.15}$$

$$\liminf_{|\xi_1|\to+\infty} (\xi_1, J_1P_1\phi(t,\xi_1)) \geq \varepsilon > 0 \tag{3.16}$$

uniformly for $t \in J$,

$$\lim_{|\xi_1|\to+\infty} |\xi_1|\,|J_1P_1\eta(t,\xi)| = 0 \tag{3.17}$$

uniformly for $t \in J$ and $\xi_2 \in J_2V_2$,

$$|J_2P_2\eta(t,\xi)| \leq a_K(t) + b_K(t)|\xi_2|, \tag{3.18}$$

where $\xi_1 \in K \subseteq J_1V_1$, K is an arbitrary compact set, $\xi_2 \in J_2V_2$ and a_K, b_K are functions depending on K and belonging to $L^1(J)$.

We note that (3.15), (3.16) and (3.17) imply (3.4), while (3.18) is implied by (3.5). All together they ensure the required a-priori bounds on the solutions of (H_λ).

REFERENCES

[1] G. Anichini and P. Zecca, *Problemi ai limiti per equazioni differenziali multivoche su intervalli non compatti*, Riv. Mat. Univ. Parma, 4 (1975), 199–212.

[2] M. Cecchi, M. Furi and M. Marini, *On continuity and compactness of some nonlinear operators associated with differential equations in noncompact intervals*, Nonl. Anal., TMA, 9 (1985), 171–180.

[3] R. Conti, *Recent trends in the theory of boundary value problems for ordinary differential equations*, Boll. Un. Mat. Ital., 22 (1967), 135–178.

[4] W.A. Coppel "Dichotomies in Stability Theory," Lectures Notes in Mathematics, 629, Springer Verlag, 1978.

[5] M. Furi and M.P. Pera, *A continuation method on locally convex spaces and applications to ordinary differential equations on noncompact intervals*, Ann. Pol. Math., 47 (1987), 331–346.

[6] M. Furi and M.P. Pera, *On the fixed point index in locally convex spaces*, Proc. Roy. Soc. of Edinburgh, 106A (1987), 161–168.

[7] W. Kryszewski, *An application of A-mapping theory to boundary value problems for ordinary differential equations*, Nonl. Anal., TMA, 15 (1990), 697–717.