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MATHEMATICAL MODELS FOR HYSTERESIS*

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Abstract. The various existing classical models for hysteresis, Preisach, Ishlinskii, Duhem–Madelung, are surveyed, as well a more modern treatments by contemporary workers. The emphasis is on a clear mathematical description of the formulation and properties of each model. In addition the authors try to make the reader aware of the many open questions in the study of hysteresis.

Key words. hysteresis, ordinary differential equations, partial differential equations, differential inclusions, periodic solutions

AMS(MOS) subject classifications. 34A60, 34C25, 94C05

Introduction. During the last twenty years there has been a steady growth in the study of various models for hysteresis. The purpose of this paper is to describe the mathematics of the various models in existence. There is a substantial literature devoted to hysteresis in each of the communities of physicists, engineers and mathematicians, and our aim is to attract mathematicians in particular to this interesting area of research.

1. The phenomenon of hysteresis. We propose to characterize hysteresis as a special type of memory-based relation between an input signal $v(t)$ and an output signal $w(t)$. At the most basic level, the relation between the output and input is determined by a pair of threshold values $\alpha < \beta$ for the input. We will use the names relay (also called passive, or positive) hysteresis and active (also called negative) hysteresis for the two types of relations we will discuss. We remark that there is at present no universally accepted terminology for these phenomena.

Note. Unless explicitly stated otherwise, we assume the input $v(t)$ and the output $w(t)$ are real-valued. Allowing $w(t) \in R^n$ is usually a straightforward extension.

In *relay hysteresis*, the graph (v, w) with output $w(t) = F[v](t)$ moves, for a given continuous piecewise monotone input $v(t)$, on one of two fixed output curves $h_U(v), h_L(v)$ defined, respectively, on $[\alpha, \infty)$, $[-\infty, \beta]$, $\alpha < \beta$ (Fig. 1), depending on which threshold, α or β, was last attained. It is common for h_U (h_L) to be asymptotically constant because of saturation as $v \to +\infty$ $(-\infty)$, and h_U, h_L need not meet (Fig. 1(a)). In relay hysteresis, the memory-based behaviour of the output can be described by the formula

$$(1) \qquad F[v](t) = \begin{cases} h_L(v(t)) & \text{if } v(t) \leq \alpha; \\ h_U(v(t)) & \text{if } v(t) \geq \beta; \\ h_L(v(t)) & \text{if } v(t) \in (\alpha, \beta) \quad \text{and} \quad v(\tau(t)) = \alpha; \\ h_U(v(t)) & \text{if } v(t) \in (\alpha, \beta) \quad \text{and} \quad v(\tau(t)) = \beta; \end{cases}$$

where $\tau(t) = \sup\{s | s \leq t,\ v(s) = \alpha \quad \text{or} \quad v(s) = \beta\}$ (i.e., $\tau(t)$ is the value of time at the last threshold attained). If $\tau(t)$ does not exist (i.e., $v(s) \in (\alpha, \beta)$ for all $s < t$), then we

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need to specifically define $F[v](t)$ depending on the physical problem we are modelling. Thus we define $F[v](t) = \eta$ if no previous threshold exists, where either $\eta = h_L(v(t))$ or $\eta = h_U(v(t))$, as required. Note that $\tau(t)$ is defined for any *continuous* input $v(\cdot)$; thus the domain of F can be taken as $C[0,\infty)$.

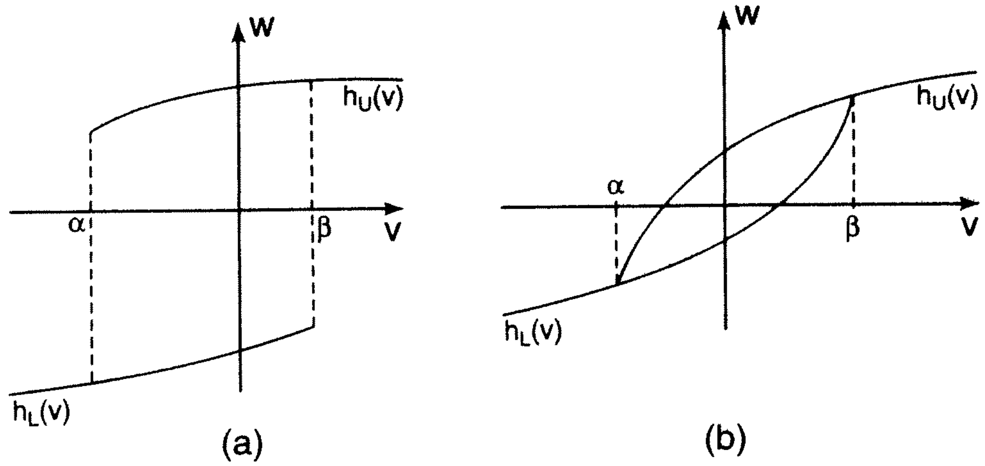

FIG. 1

The hysteresis region $\mathcal{H}$ is defined by

$$\mathcal{H} = \{(v,w) | \alpha < v < \beta,\ h_L(v) < w < h_U(v)\}.$$

Active hysteresis allows trajectories inside the hysteresis region $\mathcal{H}$. Referring to Fig. 2, if the piecewise monotone input $v(t)$ increases to γ, then decreases, the graph (v,w) after $v(t) = \gamma$ moves on a response curve inside $\mathcal{H}$ (dashed); if the input continues to decrease to δ, then increases, the graph moves on another interior path (dotted). The mathematical models for this type of hysteresis require the existence of at least two fixed families of curves filling $\mathcal{H}$, one family for increasing $v(\cdot)$, one family for decreasing $v(\cdot)$. Note that in the case of relay hysteresis, we would have had $f[v](t) = h_L(v(t))$ in the example given. It can also happen that there is a *single* family of interior curves on which two-way motion occurs, i.e., the dashed and dotted curves coincide in Fig. 2.

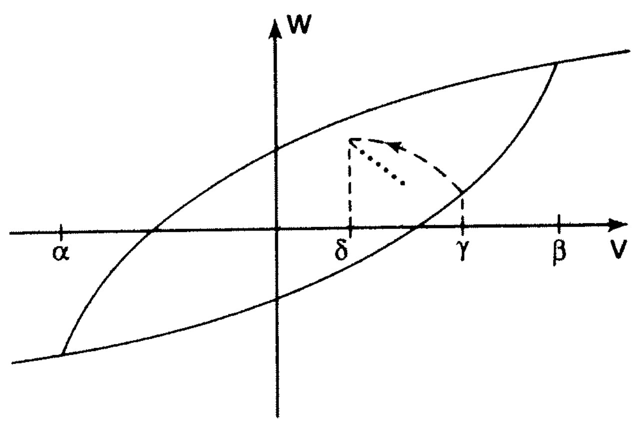

FIG. 2

In all of the above descriptions of basic hysteresis, the relation of response to input is *rate-independent*: the velocity with which the input moves on the v-axis is only reflected in the velocity of the output on the w-axis. The qualitative nature of the response does not change.

The mathematical models for the two types of hysteresis defined above are quite different in character, even though their memory-based behaviour is similar. Both types have been described for $v(t)$ continuous piecewise monotone, but active hysteresis is easily extended to continuous inputs by using approximations and a limiting process [36]. Relay hysteresis $F : v \mapsto F[v]$ is inherently discontinuous as a map between function spaces, since an input function that just reaches a threshold and reverses gives a very different output from a function that reverses just short of the threshold by an arbitrarily small amount. As we shall see below, if the families of interior curves are sufficiently regular, then active hysteresis has excellent continuity properties as a map between appropriate function spaces.

We want to emphasize again that the terminology for, and precise definition of, hysteresis has varied from area to area and paper to paper.

2. The mathematical models.

2.1. The Duhem hysteresis operator. The Duhem model for *active* hysteresis dates from 1897 [19] and focusses on the fact that the output can only change its character when the input changes direction. This model uses a phenomenological approach, postulating an integral operator or differential equation to model the relation described in Fig. 2. Babuška [2] used the differential equation

$$\dot{w}(t) = f_1(w, v)\dot{v}_+(t) + f_2(w, v)\dot{v}_-(t) \tag{2}$$

with $\dot{v}_+(t) = \max[0, \dot{v}(t)]$, $\dot{v}_-(t) = \min\,[0, v(t)]$ to generate the curves of Fig. 2. Bouc [8], [9] used an integral operator of which a particular case is the equation

$$\frac{dw}{dt} + a\left|\frac{dv}{dt}\right| g(v, w) = b\,\frac{dv}{dt}\,. \tag{3}$$

A typical choice for g is $g(v, w) = w - b\phi(v)$, with ϕ chosen (say, piecewise linear) so that $w(\cdot)$ forms a classical hysteresis loop when $v(\cdot)$ is a sinusoid (Fig. 3).

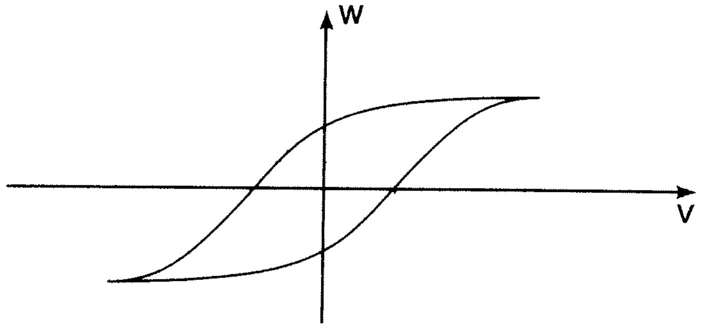

FIG. 3

The work of Hodgdon [26], [27] and Coleman and Hodgdon [13], [14] shows that such a model is useful in applied electromagnetics because the functions and parameters can be fine-tuned to match experimental results in a given situation.

Krasnosel'skii–Pokrovskii [36, §§29–32] call this model the "Madelung model," referring to Madelung's landmark 1905 paper on electromagnetic hysteresis [42]. However, the Madelung paper does not use a differential equation or integral operator. In fact, Madelung allows nonuniqueness of trajectories through a point (for $\dot{v}$ of fixed sign), which would make a differential equation model difficult. Their version of what we call the Duhem model postulates that for piecewise monotone inputs the output is defined by

$$w = F[v] : \frac{dw}{dt} = \Phi(w, v, \dot{v}). \tag{4}$$

Most often, Φ is assumed to have the form

$$\Phi(w, v, \dot{v}) = \begin{cases} f_D(w, v)\dot{v}, & \dot{v}(t) \leq 0; \\ f_I(w, v)\dot{v}, & \dot{v}(t) \geq 0, \end{cases}$$

with f_D, f_I continuous (D is for "decreasing input," I is for "increasing input"); this is clearly equivalent to Babuška's version.

The solutions of (4) give the trajectories interior to the hysteresis region $\mathcal{H}$. To include C^1 boundary curves h_U and h_L, we can modify (4) by adding the constraints

$$\begin{aligned} w'(t) &= h'_U(v(t))v'(t) \quad \text{for } w(t) \geq h_U(v(t)), \\ w'(t) &= h'_L(v(t))v'(t) \quad \text{for } w(t) \leq h_L(v(t)). \end{aligned} \tag{5}$$

This adds the physically irrelevant trajectories

$$w(t) = h_L(v(t)) + c, \qquad w(t) = h_U(v(t)) + c$$

for initial states outside $\mathcal{H}$ and off the curves h_U, h_L.

2.2. The Ishlinskii hysteresis operator. Ishlinskii's model (cf. [36, §35]) dates from 1944. (The referee has pointed out that it appears in a paper of Prandtl in 1928 [50].) It was proposed as a model for plasticity-elasticity. The foundation for this model is a basic active hysteresis map whose behavior is sketched in Fig. 4, called a *stop* by Krasnosel'skii–Pokrovskii [36, §3].

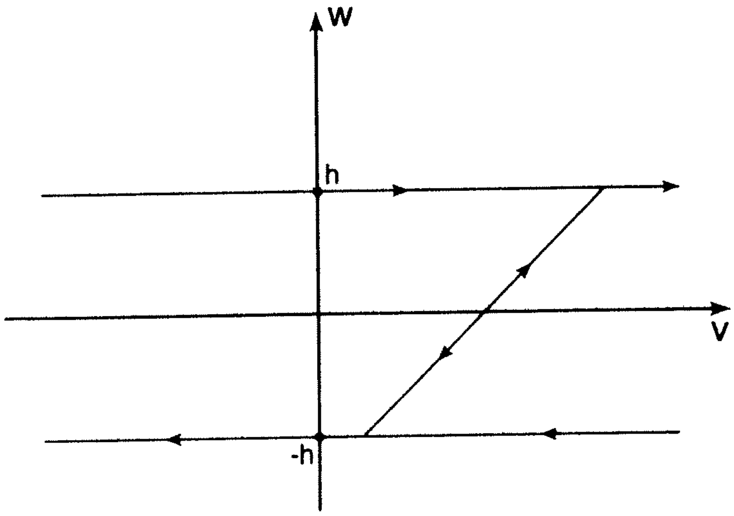

FIG. 4

For the *stop* as sketched in Fig. 4 the bounding curves are $w = \pm h$, the hysteresis region $\mathcal{H}$ is the strip

$$\{(v, w) | -\infty < v < \infty, \quad -h < w < h\},$$

and the corresponding operator $S_h : v \mapsto w$ for v continuous monotone, initial state $(v_0, w_0) \in \mathcal{H}$, according to the following rules:

(i) If $v(\cdot)$ is moving to the right at time t, then $S_h[v](t) = \min\{h, v(t) - v_0 + w_0\}$;

(ii) If $v(\cdot)$ is moving left at time t, then $S_h[v](t) = \max\{-h, v(t) - v_0 + w_0\}$.

When we want to include dependence on (t_0, w_0), we write $S_h(t_0, w_0)[v]$.

The operator S_h can be defined for piecewise monotone inputs by referring to Fig. 4 with its single interior family of two-way curves. We can describe this extension analytically by asking that the *semigroup property* hold:

$$S_h(t_0, w_0)[v](t) = S_h\big(t_1, S_h(t_0, w_0)[v](t_1)\big)[v](t).$$

The simplest form of Ishlinskii hysteresis operator can then be defined as the superposition

$$w(t) = F[v](t) \equiv \int_0^\infty \xi(h) S_h\big(t_0, w_0(h)\big)[v](t)dh. \tag{6}$$

Although defined for piecewise monotone $v(\cdot)$, under reasonable conditions on $\xi(\cdot)$ ($\xi(h) \geq 0$, $\int^\infty h\xi(h)dh < \infty$, say) this operator can be extended to a map from $C(t_0, t_1)$ into $C(t_0, t_1)$ [36, §35.7]; in fact Lipschitz continuous inputs will yield Lipschitz continuous outputs. We can also use a Stieltjes integral to define Ishlinskii operators by replacing $\xi(h)dh$ with $d\xi$.

2.3. The Preisach model. The Preisach model of electromagnetic hysteresis dates from 1935 [51]. It was investigated in the 1950s by Everett and collaborators [21]–[25] and by Biorci and Pescetti [6], [7] and has been studied extensively in recent times. This model uses a superposition of especially simple independent relay hysteresis operators as described in Fig. 1(a). That is,

$$F[v](t) = \int\int \mu(\alpha, \beta)\hat{F}_{\alpha,\beta}[v](t)\, d\alpha\, d\beta, \tag{7}$$

where $\mu(\alpha, \beta) \geq 0$ is a weight function, usually with support on a bounded set in the (α, β)-plane, $\hat{F}_{\alpha\beta}$ is a relay hysteresis operator with thresholds $\alpha < \beta$, and

$$h_U(v) = +1 \quad \text{on } [\alpha, \infty);$$
$$h_L(v) = -1 \quad \text{on } (-\infty, \beta].$$

The arbitrary initial state $\eta = \pm 1$ must be chosen if $v(t_0) \in (\alpha, \beta)$. Whereas the Madelung model is based on a "how can we model this relation" approach, the Preisach model is founded on a physical assumption: hysteresis is the result of the superposition of the behaviour of independent domains within the material, each of which can be modelled by a simple "flip-flop" relay hysteresis operator.

For a thorough discussion of the Preisach model, see Mayergoyz [43] and Brokate–Visintin [12]. The monograph of Krasnosel'skii–Pokrovskii [36] only deals briefly with this model (pp. 367–384).

2.4. The Krasnosel'skii–Pokrovskii hysteron. Krasnosel'skii and Pokrovskii use a geometric approach to define their basic hysteresis operator, called a hysteron.

As a preliminary, they define a hysteresis operator called a "play." The one-dimensional play can be thought of as a piston with plunger, of length $2h$ (Fig. 5).

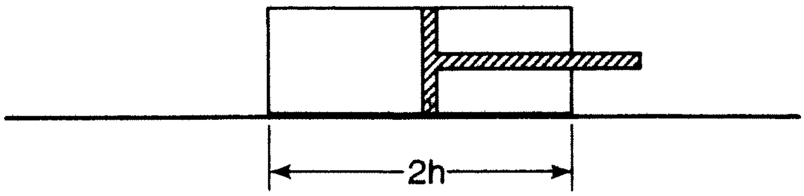

FIG. 5

The output is the position of the center of the piston $w(t)$, the input is the plunger position $v(t)$; Fig. 6 shows the rules of motion. Notice that we always have $|w - v| \leq h$. This defines an active hysteresis operator with $\alpha = -\infty$, $\beta = +\infty$, and a single family of interior curves.

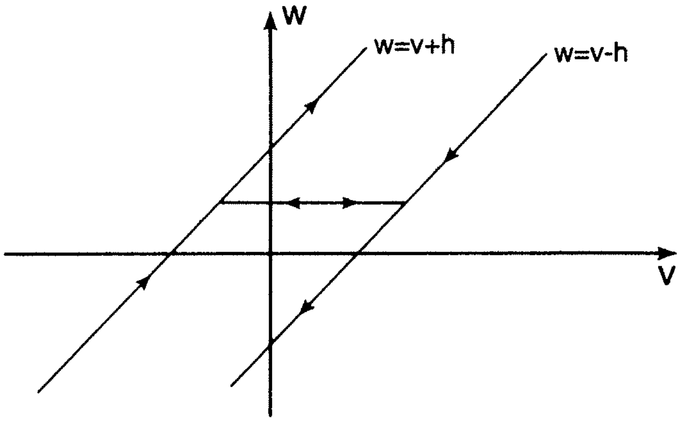

FIG. 6

Note that for a given input $v(\cdot)$, the output $w(\cdot) \equiv P_h[v](t)$ of the play is

$$P_h[v](t) = v(t) - S_h[v](t),$$

where S_h is the stop of Fig. 4. In fact we can give a direct formula for P_h as follows. Define $g_h(v, w) = \min[v + h, \max(v - h, w)]$. If the initial output value $w_0 = w(t_0)$ is given and if the input $v(\cdot)$ is *monotone*, then $P_h[v; w_0](t) = g_h\big(v(t), w_0\big)$. If $v(\cdot)$ is piecewise monotone (alternatively, Brokate [11] uses a step function), with $v(\cdot)$ monotone on $I_i = [t_{i-1}, t_i]$, $i = 1, \ldots, n$, then

$$P_h[v; w_0](t) = g_h\big(v(t), w(t_{i-1})\big) \quad \text{for } t \in I_i.$$

They then define a *hysteron*, F, as a map from piecewise monotone continuous input functions $v(\cdot)$ to output functions $w[v](\cdot)$. The domain $\Omega(F) \subset R^2$ (the interior of which is our hysteresis region $\mathcal{H}$), which defines the input-output relation, is assumed defined by three properties (refer to Fig. 7), which we present in slightly less general form than in [36].

Property 1. The intersection $K(v_0) = \Omega(F) \cap \ell_{v_0}$ of $\Omega(F)$ with the vertical line $\ell_{v_0} = \{(v, w) | v = v_0\}$ is a nonempty interval (perhaps a singleton).

Property 2. The endpoints of $K(v), v \in R$, define two continuous curves $\Phi_L(v)$ (left endpoint) and $\Phi_R(v)$ (right endpoint); when $K(v)$ is a singleton the two curves coincide. The domain of Φ_R is some interval $(-\infty, a_R)$; the domain of Φ_L is some interval (b_L, ∞), $\quad b_L < a_R$.

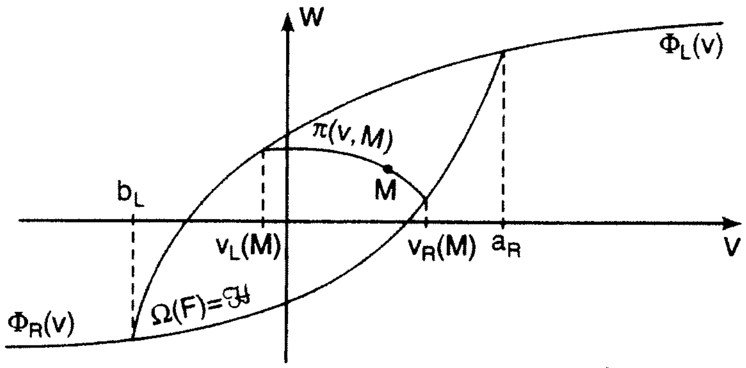

FIG. 7

Property 3. Define $\Omega_0(F) \subset \Omega(F)$ as the finite *open* region bounded by Φ_L and Φ_R; stratify this region by a given family of nonintersecting graphs of continuous functions $\pi(\cdot)$ such that the left endpoint of each graph lies on Φ_L and the right endpoint of each graph lies on Φ_R. The remaining points of each graph do not intersect $\Phi_L \cup \Phi_R$.

The above defines an active hysteresis operator with a single family of interior curves. They then show that any operator defined through the above properties extends uniquely from the class of piecewise monotone inputs to a mapping defined for any continuous input, and in fact admits a canonical representation in terms of *generalized play*. Let $f(v, z)$ be a given real-valued function of two real variables continuous on $\Omega(F)$ and strictly monotone in z. Krasnosel'skii–Pokrovskii show that any hysteron can be represented in the form

$$F[v](t) = f\big(v(t), P(\Gamma_L, \Gamma_R)[v](t)\big), \tag{8}$$

where $P(\Gamma_L, \Gamma_R)$ is a *generalized play* operator defined by the hysteresis diagram in Fig. 8 (for piecewise monotone continuous inputs and then extended to continuous inputs). This operator just replaces the 45° lines in Fig. 6 with functions $\Gamma_L(\cdot), \Gamma_R(\cdot)$.

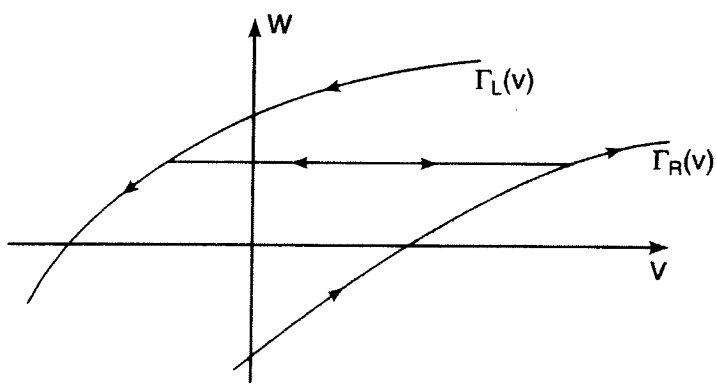

FIG. 8

More precisely, for a *generalized play* we are given two continuous nondecreasing functions $\Gamma_L(v), \Gamma_R(v)$ defined on respective intervals $(-\infty, a_L)$ and (b_R, ∞), $b_R < a_L$, with $\Gamma_L(v) \geq \Gamma_R(v)$ for all $b_R < v < a_L$. The two curves are connected by horizontal lines, and the output $w(\cdot) = P(\Gamma_L, \Gamma_R)[v](\cdot)$ for continuous monotone input $v(t)$, $t \geq t_0$, is defined as

$$(9)\qquad w(t) = \begin{cases} \max\{w(t_0), \Gamma_R[v(t)]\} & \text{when } v(t) \text{ is nondecreasing,} \\ \min\{w(t_0), \Gamma_L[v(t)]\} & \text{when } v(t) \text{ is nonincreasing.} \end{cases}$$

This means that the motions follow the arrows as sketched in Fig. 8.

The shortcoming of the Krasnosel'skii–Pokrovskii hysteron is that it only allows a single family of interior trajectories. To allow two families (as in Fig. 2), Krasnosel'skii and Pokrovskii [36, p. 61] define a *prehysteron* as any *deterministic, static, weakly correct* map ("transducer") which maps continuous piecewise monotone inputs $v(\cdot)$ into continuous outputs $w(t) = F[v](t)$. It is assumed that the initial state (v_0, w_0) at time t_0 is given and is "admissible" for F (intuitively, (v_0, w_0) is in the hysteresis domain $\mathcal{H}$ or on the boundary curves). Axiomatically, it is assumed that a certain domain $\{(v, w)\}$ in R^2 is given—called the feasible states.

Deterministic. $w(t) = F[v](t)$, for $t \geq t_0$, is uniquely determined by $\big(t_0, v(t_0), w(t_0)\big) \equiv (t_0, v_0, w_0)$ and $v(\cdot)$.

Static. (i) $F[v(\cdot)](t)|_{(t_0,v_0,w_0)} = F[v(\cdot)](t - t_0 + t_1)|_{(t_1,v_0,w_0)} \quad (t \geq t_0)$; and

(ii) If we replace the input $v(t)$ by $v\big(\alpha t + (1-\alpha)t_0\big)$ with $\alpha > 0$, then the output $w(t)$ is replaced by

$$w\big(\alpha t + (1-\alpha)t_0\big) \quad (t \geq t_0).$$

Here (t_0, v_0, w_0) specifies the initial conditions $v(t_0) = v_0,\ w(t_0) = w_0$.

Warning. In the monograph [36] of Krasnosel'skii–Pokrovskii, a static transducer is defined on page 4; a completely different meaning is assigned this phrase in Part 5 (p. 212)—the definition in Part 5 is a very special case.

Weakly correct. If $\{v_n(\cdot)\}_{n=1}^{\infty}$ and $v_*(\cdot)$ are continuous monotone inputs, $t_0 \leq t \leq t_1$ with $\big(v_n(t_0), w_n^\circ\big)$ and $\big(v_*(t_0), w_*^\circ\big)$ admissible and if $w_n^\circ \to w_*^\circ$, $\sup_{[t_0,t_1]} |v_n(s) - v_*(s)| \to 0$, then

$$F[v_n(\cdot)](t_1)|_{\left(t_0, v_n(t_0), w_n\right)} \to F[v_*(\cdot)](t_1)|_{\left(t_0, v_*(t_0), w_*\right)}.$$

Under extra conditions ("controllable" and "uniformly correct on harmonic inputs") a given prehysteron will coincide with some hysteron on piecewise monotone continuous inputs. However, since hysterons are based on a single family of curves, for magnetic phenomena this situation will in general not occur.

The best example of a prehysteron is a sufficiently regular Duhem model, and Krasnosel'skii and Pokrovskii state (without proof) that prehysterons can be characterized by the operator described in Fig. 2 with continuous curves.

3. Properties of the various models. In this section we present a sampling of the mathematical properties of the hysteresis operators defined by the four models presented in §2.

3.1. Properties of the Duhem model. For the general Duhem model as described by (4), the two families of curves in the (v, w) plane defined, respectively, by

$$(10)\qquad \begin{aligned} &\text{(a)} \quad \frac{dw}{dv} = f_D(w, v), \\ &\text{(b)} \quad \frac{dw}{dv} = f_I(w, v), \end{aligned}$$

represent the dashed and dotted curves in Fig. 2. (However, (4) only makes sense for piecewise absolutely continuous inputs $v(\cdot)$.)

If the boundary curves h_I, h_D are absolutely continuous, if f_D, f_I are continuous, differentiable with respect to w, and satisfy

$$(x-y)[f_I(x,v)-f_I(y,v)] \le \lambda_I(v)(x-y)^2,$$
$$(x-y)[f_D(x,v)-f_D(y,v)] \ge -\lambda_D(v)(x-y)^2,$$

with λ_I, λ_D nonnegative, then the Duhem hysteresis operator maps $AC[t_0,t_1] \to AC[t_0,t_1]$ continuously [36, Thm. 29.1, Lemma 30.1]. Here AC is the space of absolutely continuous functions with norm $\|v\| = v(t_0) + \int_{t_0}^{t_1} |v'(t)|dt$. We can weaken these hypotheses by using differential inclusions involving a convexification of (10) [36, §27, §29.1].

Coleman and Hodgdon [13], [14], [26], [27] have extensively investigated the Duhem model for magnetic hysteresis, using the equation

$$\frac{dB}{dt} = \alpha \left|\frac{dH}{dt}\right| [f(H) - B] + \frac{dH}{dt} g(H), \tag{11}$$

with $\alpha > 0$ constant. They assume

(i) $f(\cdot)$ is piecewise smooth, monotone increasing, odd, with $\lim_{H\to\infty} f'(H)$ finite;

(ii) $g(\cdot)$ is piecewise continuous, even, with

$$\lim_{H\to\infty} g(H) = \lim_{H\to\infty} f'(H);$$

(iii) $f'(H) > g(H) > \alpha e^{\alpha H} \int_H^\infty |f'(\zeta) - g(\zeta)| e^{-\alpha\zeta} d\zeta$ for all $H > 0$.

Here H is the applied magnetic field and B is the level of magnetization of the medium. The solutions of (11) move on the curves defined by

$$\frac{dB}{dH} = \alpha \operatorname{sgn}(\dot H)[f(H) - B] + g(H), \tag{12}$$

and this can be solved explicitly for H piecewise monotone:

$$\begin{aligned} B = {} & f(H) + [B_0 - f(H_0)]e^{-\alpha(H-H_0)\operatorname{sgn}\dot H} \\ & + e^{-\alpha H \operatorname{sgn}\dot H} \int_{H_0}^{H} [g(\zeta) - f'(\zeta)] e^{\alpha\zeta(\operatorname{sgn}\dot H)} d\zeta \end{aligned} \tag{13}$$

for $\dot H$ constant, $B(H_0) = B_0$ (sgn $\dot H = \pm 1$ as $\dot H > 0$ or $\dot H < 0$).

They show [14] that the conditions (i), (ii), (iii) are necessary and sufficient for a model which will give a hysteresis diagram as in Fig. 2. If $B(H; H_0, B_0)$ is the solution of (12) with initial values (H_0, B_0), then they show: if $\dot H > 0$ ($\dot H < 0$) and $H \to +\infty$ ($-\infty$), then

$$\lim_{H\to+\infty} [B(H, H_0, B_0) - f(H)] = 0,$$
$$\left(\lim_{H\to-\infty} [B(H, H_0, B_0) - f(H)] = 0\right),$$

so the curve $B = f(H)$ is asymptotic to the upper and lower bounding curves h_U, h_L of Fig. 2. In fact, they then obtain the upper and lower curves as

$$B_U(H) = \lim_{H_0\to+\infty} B_U(H; H_0, f(H_0)),$$
$$B_L(H) = \lim_{H_0\to-\infty} B_L(H; H_0, f(H_0)),$$

where $B_U(H; H_0, f(H_0))$ is defined for $H < H_0$ as the solution of (12) with $\dot{H} < 0$ and initial state $(H_0, f(H_0))$. ($B_L(H; H_0, f(H_0))$ is defined for $H > H_0$ as the solution of (12) with $\dot{H} > 0$.) In their context $B_U(H; H_0, f(H_0))$ is the *unloading curve* (i.e., H is decreasing from $(H_0, f(H_0))$; $B_L(H; H_0, f(H_0))$ is the *loading* curve; $B_U(t), B_L(H)$ are the unloading, respectively, loading curves when the applied field is brought from saturation ($H = \infty$, respectively, $-\infty$) to value H.

They prove that *any* state in the interior of $\mathcal{H}$ can be reached from $(0, 0)$ by a trajectory with at most one change of sign of $\dot{H}$. They call the curves (pieces of B_U, B_L) bounding $\mathcal{H}$ the *major* loop, and they prove that no trajectory can leave $\overline{\mathcal{H}}$. They define a *primitive* minor hysteresis loop as the trajectory of a periodic solution of (12) which lies in $\mathcal{H}$ and has a single change of sign of $\dot{H}$. They do not explicitly prove that such loops exist (the geometry makes it reasonably clear that they do), but they show that if they exist, then

(a) The values H_m, H_M, of min H, max H on the loop uniquely define the loop;

(b) If $H(t)$ oscillates: $H(t_{2k+1}) = H_m$, $H(t_{2k}) = H_M$, $\dot{H}(t) \neq 0$ for $t \neq t_j$, then $(H(t), B(t))$ approaches the unique primitive loop with extreme values (H_m, H_M).

They demonstrate the usefulness of their model by proving some results on alternating current demagnetization, on hysteresis loss, and on remanence.

Hodgdon [26], [27] has applied the Duhem model, using specific functions $f(\cdot)$ and $g(\cdot)$ first proposed by himself and Coleman. However, he first modifies the model by postulating the governing relation

$$\frac{dH}{dt} = \alpha \left| \frac{dB}{dt} \right| [\tilde{f}(B) - H] + \frac{dB}{dt} \tilde{g}(B), \tag{14}$$

i.e., he interchanges the roles of B and H. If we replace $\tilde{g}(B)$ by $\tilde{g}(B, \dot{B})$, then we can model *rate-dependent* responses $B(t)$: the value of $B(t)$ depends not only on the history of $H(\cdot)$ but also on the history of $\dot{B}(t)$. This corresponds, for example, to the fact that if a pulse $H(\cdot)$ is applied to a material, it takes some time for $B(t)$ to reach an equilibrium.

The simplest model uses $\tilde{f}(B)$ and $\tilde{g}(B)$ piecewise linear (always satisfying (i), (ii), and (iii) above with H replaced by B); a more sophisticated model uses

$$\tilde{f}(B) = \begin{cases} A_1 \tan A_2 B, & |B| < B^*; \\ A_1 \tan(A_2 B^*) + \dfrac{B - B^*}{\mu}, & B > B^*; \\ -A_1 \tan(A_2 B^*) + \dfrac{B + B^*}{\mu}, & B < -B^*; \end{cases}$$

$$\tilde{g}(B) = \begin{cases} \tilde{f}'(B) \left[1 - A_3 \exp\left(\frac{A_4 |B|}{B^* - |B|}\right)\right], & |B| < B^*; \\ \tilde{f}'(B), & |B| > B^*. \end{cases}$$

For rate dependence, he replaces D_3 and/or A_3, by $C(\dot{B})D_3$, $C(\dot{B})A_3$, respectively. He is able to choose parameters so as to accurately reproduce experimental results.

Babuška [2] has proved the existence of periodic input-output curves which are limit cycles. Recall that he describes the input-output relation through (2):

$$\dot{w}(t) = f_1(w, v)\dot{v}_+(t) + f_2(w, v)\dot{v}_-(t).$$

Assume the following.

(1) f_1 and f_2 are continuous and nonnegative on $R \times R$.
(2) For w fixed, f_1 (respectively, f_2) is strictly increasing (decreasing) in v except for the possibility that $f_1(w,v) \equiv 0 \quad (f_2(w,v) \equiv 0)$ on one interval.
(3) For v fixed, f_1 (respectively, f_2) is strictly decreasing (increasing) in w except perhaps $f_1 \equiv 0 \quad (f_2 \equiv 0)$ on one interval.
(4) There exist $D_1 > 0,\ D_2 > 0$ such that

$$f_1(w,v) \equiv 0 \quad \text{for} \quad w \geq 0; \qquad f_2(w,v) \equiv D_1 \quad \text{for } w < -D_2.$$

(5) There are no points (w,v) such that

$$f_1(w,v) = f_2(w,v) = 0.$$

He shows that for a given continuously differentiable input $v(\cdot),\ 0 \leq t < T$, and a given initial output value $w(0) = w_0$, there is a unique solution of (2), $w(t; v(\cdot), w_0)$ defined on $[0,T)$. If $w_0 \in [-D_2, D_1]$, then $w(t; v(\cdot), w_0) \in [-D_2, D_1]$ for all $t \in [0,T)$ and $w(t, v(\cdot), w_0)$ is a continuous function of w_0.

Now suppose we are given a nonconstant periodic continuously differentiable input $v_p(\cdot)$, but do not specify w_0. Then there exists a *unique* nonconstant periodic solution of (2), $w_p(\cdot)$, corresponding to $v_p(\cdot)$. If we choose w_0 so that the point $(0, w_0)$ does not lie on the corresponding closed curve in the (v,w)-plane, then $\lim_{t\to\infty} [w(t; v_p(\cdot), w_0) - w_p(t)] = 0$, i.e., it is a "limit cycle," which may in fact coincide with the closed curve $\{(v_p(t), w_p(t)) | 0 \leq t \leq T\}$ after some time.

Bouc [8], [9] used the operator

$$w(t) = \widehat{\mathcal{F}}[v](t) = \mu^2 v(t) + \int_{t_0}^{t} F(V_s^t v(\cdot)) d\Phi(v(s)) + f(v(t))$$

with $f(0) = \Phi(0) = 0$, f and Φ locally Lipschitzian; $F(u)$ is continuous, positive-valued, and decreasing for $u \geq 0$; $V_s^t v(\cdot)$ is the total variation of $v(\cdot)$ on the interval (s,t), i.e., $\int_s^t |\dot{v}(r)| dr$ for $v(\cdot)$ absolutely continuous. If $F(\cdot)$ is globally Lipschitzian, i.e., $|F(u_1) - F(u_2)| \leq M|u_1 - u_2|$ for some $M > 0$ and all $u_1 \geq 0,\ u_2 \geq 0$, then to each periodic input $v_p(\cdot)$ corresponds a unique periodic output $w_p(\cdot)$. As with the results of Babuška, the curve $(v_p(\cdot), w_p(\cdot))$ is a "limit cycle" for any initial condition not lying on it (keeping the input $v_p(\cdot)$ fixed).

Krasnosel'skii and Pokrovskii devote a major effort to extending the Duhem model from piecewise monotone inputs to continuous inputs. They define the differential equation

$$\frac{dw}{dt} = \Phi(w, v, \dot{v}), \qquad t_0 \leq t \leq t_1, \tag{4}$$

to be *vibrocorrect* if

(i) The equation has a unique solution $w(t; t_0, x_0, v(\cdot))$ defined on $[t_0, t_1]$ for any x_0 in an appropriate domain and any $v(\cdot) \in C^1(t_0, t_1)$;
(ii) For any $v^*(\cdot) \in C(t_0, t_1)$ and any sequence $\{v_n(\cdot)\} \subset C^1(t_0, t_1)$ with $\lim_{n\to\infty} \|v_n(\cdot) - v^*(\cdot)\|_\infty = 0$, the solutions $w_n(t) \equiv w(t; t_0, x_0, v_n(\cdot))$ converge uniformly to $v^*(\cdot)$.

This is a particular case of extending an operator by continuity from a dense subset of a Banach space B to all of B. They prove that the Duhem hysteresis operator $w = F[v]$ defined by the above equation is in general *not* vibrocorrect (it will be vibrocorrect under reasonable smoothness hypotheses if Φ is linear in $\dot{v}$, which is the same as saying that interior motions occur on a *single* family of curves).

Since the Duhem hysteresis operator is in general not vibrocorrect, Krasnosel'skii and Pokrovskii suggest working with a modification of the limit process which defines vibrocorrectness. It is also more convenient to work with the spaces of absolutely continuous functions $(AC(t_0, t_1))$ and functions of bounded variation $(BV[t_0, t_1])$ rather than $C[t_0, t_1]$. Let $\mathrm{Var}_{\tau_1}^{\tau_2}[f(\cdot)]$ denote the variation of $f(\cdot)$ on $[\tau_1, \tau_2]$. They prove the following interesting theorem on approximations [36, p. 301].

THEOREM 3.1. *Let* $v(\cdot) \in AC(t_0, t_1)$, $\{v_n(\cdot)\} \subset BV(t_0, t_1)$ *with* $\lim_{n\to\infty} \sup_{[t_0,t_1]} |v(t) - v_n(t)| = 0$ *and*

$$\lim_{n\to\infty} \mathrm{Var}_{t_0}^{t}[v_n(\cdot)] = \mathrm{Var}_{t_0}^{t}[u(\cdot)] + \psi(t), \qquad t_0 \le t \le t_1$$

with $\psi(\cdot) \in AC(t_0, t_1)$. *Then the outputs* $w_n = F[v_n]$ *defined by the Duhem operator* (4) *converge uniformly to the unique solution of*

$$w'(t) = \Phi(w, v, v') + \psi(t)[f_I(w, v) - f_D(w, v)], \qquad w(t_0) = \lim w_n(t_0),$$

subject to the constraints $h_U(\cdot), h_L(\cdot)$.

3.2. Properties of the Ishlinskii model. The basic continuity results for the Ishlinskii operator F defined by (6) are given in Krasnosel'skii–Pokrovskii [36, §35–36]. As mentioned earlier this operator maps $C[t_0, \infty)$ into $C[t_0, \infty)$ if $\xi(h) \ge 0$ with $(*)$ $\int^{\infty} h\xi(h)dh < \infty$, which we assume throughout this section. In fact, under these assumptions the operator maps the space of Lipschitz continuous functions into itself.

Krejči [37], [39], Krejči and Lovicar [38] have established continuity results under special alternative hypotheses, proved the existence of an inverse operator under certain conditions, and studied partial differential equations with hysteresis term.

For example, Krejči assumes a weight function of the form $\xi(h) = -\varphi''(h)$, where $\varphi(\cdot) \in C^2[0, \infty)$ is such that $\varphi''(h) \le 0$ for all h with $\varphi''(0) < 0$. He then shows that F is a Lipschitz continuous map from $C[0, \tau]$ into $C[0, \tau]$, and in fact maps absolutely continuous (AC) functions into AC functions. He also shows that the hysteresis loops of his operator consist of parts of the graph of $\pm\varphi(\cdot)$ when $v(\cdot)$ is increasing and decreasing, respectively.

Krejči and Lovicar have shown that the Ishlinskii operator (6) is a continuous map from $W^{1,p}(0, \tau)$ into itself for $1 \le p < \infty$ (in fact, Lipschitz continuous when $p = 1$ under the minimal assumption $\xi(h) \ge 0$, $\xi(\cdot) \in L^1(0, \infty)$).

Krejči has used his continuity results for F to prove existence theorems for partial differential equations with periodic boundary conditions.

3.2.1. A connection between Ishlinskii and Preisach operators. Krejči [37] and Krejči and Lovicar [38] have recently shown how to connect a large class of Preisach operators with Ishlinskii operators. Let a Preisach operator be defined by

$$Z[v](t) = \lim_{K\to\infty} \frac{1}{2} \int_0^K \int_{-K}^K z_{\rho,h}[v](t) \frac{\partial m(\rho, h)}{\partial \rho} \, d\rho \, dh, \tag{15}$$

where the following hold.

(1) $z_{\rho,h}$ is a relay as sketched in Fig. 1(a) with thresholds $\alpha = \rho - h$, $\beta = \rho + h$:

$$z_{\rho,h}[v](t) = \begin{cases} +1 & \text{if } v(t_m) = \rho + h; \\ -1 & \text{if } v(t_m) = \rho - h, \end{cases}$$

where $t_m = \max\{\tau \in [0,t];\ v(\tau) = \rho \pm h\}$ (i.e., last threshold attained). If the set of previous thresholds is empty, then $z_{\rho,h}[v](t) = z_{\rho,h}[v](0)$. The initial state is chosen as $J\big(v(0) - \rho - h\big)$ for $\rho \geq 0$ and as $-J\big(-v(0) + \rho - h\big)$ for $\rho < 0$, where $J(r) = \operatorname{sgn} r \quad \big(J(0) = +1\big)$.

(2) If $m(\rho, h) : R \times [0,\infty) \to R$ is C^2 in ρ, odd in ρ with $\frac{\partial m}{\partial \rho} > 0$, then

$$Z[v](t) = \int_0^\infty m\big(v(t) - S_h[v](t), h\big)dh, \tag{16}$$

where the simple stop S_h is defined below Fig. 4, just before (6). The operator $\ell_h[v] = v - S_h[v]$ (which is just the play of Fig. 6) figures prominently in the analysis of the Ishlinskii model, and (16) gives a representation of the Preisach operators described by (15), which allows the application of techniques normally used to analyze Ishlinskii operators. On the other hand, replacing $\partial m/\partial \rho$ in (15) by

$$\mu(\rho, h) = \rho(\varphi^{-1})''(h) \quad \text{with } \varphi \in C^2[0,\infty),$$

concave, $\varphi(0) = 0,\ 0 < \varphi'(0+) < \infty$, Krejči shows that in this case

$$Z = \frac{-1}{\varphi'(0+)} I + F^{-1},$$

where I is the identity and F is the Ishlinskii operator generated by φ.

3.3. Properties of the Preisach model. Recall that the Preisach model defined a hysteresis operator as a superposition $F[v](t) = \int\int \mu(\alpha,\beta)\hat{F}_{\alpha,\beta}[v](t)d\alpha\, d\beta$ with $\mu(\alpha,\beta) \geq 0$ and (usually) has bounded support. This formulation is quite appealing mathematically, since if $\mu(\alpha,\beta)$ is bounded and measurable with bounded support, then $F[v]$ is a well-defined map on $C(t_0, t_1)$. A disadvantage of the Preisach model is the lack of direct information about the actual hysteresis curves. There is, however, an interesting geometric picture associated with the model, based on the (α,β) plane.

Assume $\mu(\alpha,\beta)$ has support in a triangle T (Fig. 9) defined by the lines $\beta = \alpha$, $\beta = \beta_m$, $\alpha = \alpha_m$, and define, for a given continuous piecewise monotone input $v(\cdot)$ and time t,

$$\begin{aligned} S^+(t) &= \{(\alpha,\beta) \in T \mid \hat{F}_{\alpha\beta}[v](t) = 1\}, \\ S^-(t) &= \{(\alpha,\beta) \in T \mid \hat{F}_{\alpha\beta}[v](t) = -1\}. \end{aligned} \tag{17}$$

Then the boundary $L(t) = \bar{S}^+(t) \cap \bar{S}^-(t)$ is a descending "staircase" with vertices at values of α or β corresponding to a subset of previous local maxima or minima of the input $v(\cdot)$ (Fig. 10).

The final link of $L(t)$ is attached to $\alpha = \beta$ at $\big(v(t), v(t)\big)$ by a vertical segment when $v(\cdot)$ is decreasing at t, by a horizontal segment when $v(\cdot)$ is increasing at t. Thus we can write

$$w(t) \equiv F[v](t) = \int\int_{S^+(t)} \mu(\alpha,\beta)d\alpha\, d\beta - \int\int_{S^-(t)} \mu(\alpha,\beta)d\alpha\, d\beta, \tag{18}$$

and $F[\,\cdot\,]$ is characterized to a large extent by the evolution in time of $S^+(t), S^-(t)$ and the measure $\mu(\alpha,\beta)d\alpha\, d\beta$.

In fact, Brokate in [11] describes a remarkable connection between the curve $L(t)$ in the (α,β) plane (t to be thought of as a parameter, fixed for each curve) and the play

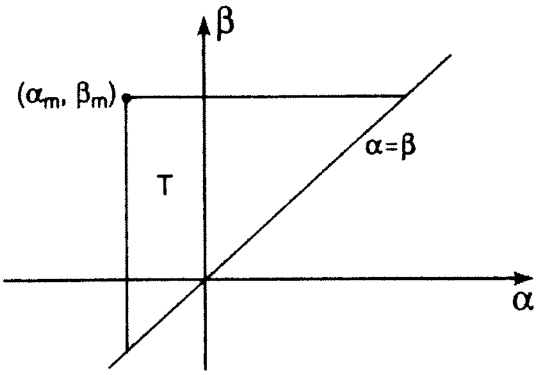

FIG. 9

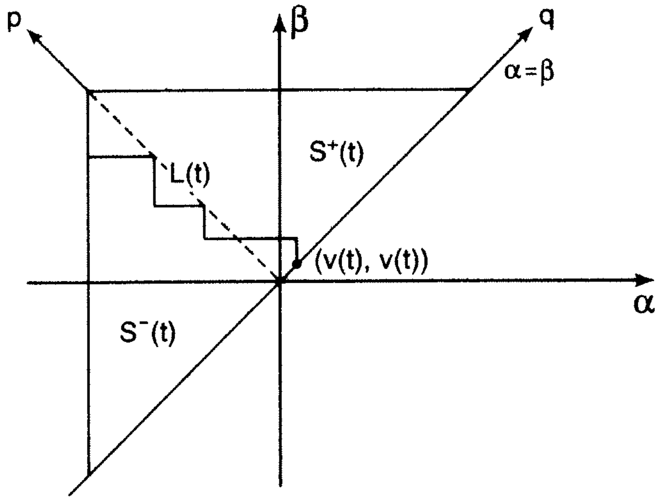

FIG. 10

operator $P_h[v; w_0]$ described in Fig. 6. If the curve $L(t)$ is given by the parametrized functional relation $q = \varphi(t,p)$ in rotated coordinates $p = (\beta - \alpha)/2, \quad q = (\alpha + \beta)/2$, then

$$\varphi(t,p) = P_p\big(v; \varphi(0,p)\big)(t).$$

We could also replace μ in (18) by a Borel measure $d\mu(\alpha, \beta)$. Integrals of the above type are often called Everett integrals, since he was the first to systematically exploit the representation (18). For more details on the history and use of these diagrams see the excellent discussions in Atherton, Szpunar, and Szpunar [1], Brokate and Visintin [12], and Brokate [11].

If the support of μ is a singleton, then $F[\,\cdot\,]$ is a relay hysteresis operator. If the support of μ is sufficiently "broad," then $F[\,\cdot\,]$ can be an active hysteresis operator; the continuity properties in the two cases are radically different. We will survey continuity results below.

It is straightforward to show (Mayergoyz [43]) that the hysteresis operator (18) has the following two properties as an operator from continuous piecewise monotone functions into piecewise smooth functions: Assume that $\mu(\cdot,\cdot)$ is piecewise continuous with at most jump discontinuities on a set of measure zero.

Property A (*wiping out property*). Whenever $v(t)$ attains a local maximum, all vertices on $L(t)$ with α coordinate below this maximum are wiped out; dually, each local minimum of $v(\cdot)$ wipes out all vertices on $L(t)$ with β-coordinates above this minimum.

Property B (*congruence property*) (Fig. 11). All primitive hysteresis loops corresponding to the same extreme values of input are congruent. More explicitly, if it is the case that $v(t)$ oscillates between v_m and v_M, then for all initial states the resulting loops are congruent.

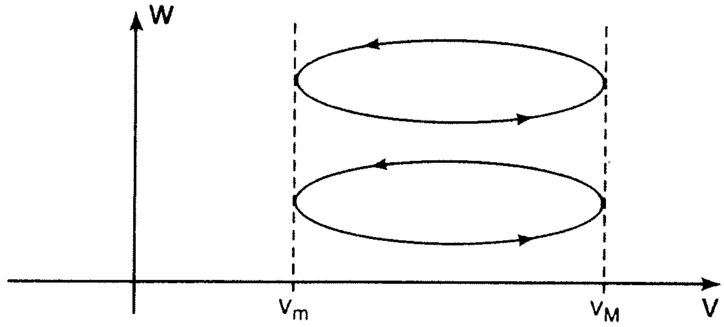

FIG. 11

Unfortunately, Property B is not valid in general for electromagnetic hysteresis. Usually, the higher the initial value of output $w(t_0) = f[v(t_0)]$, the smaller the hysteresis loop (the upper loop in Fig. 11). In practice it can take many cycles for a trajectory to stabilize so as to be indistinguishable from a fixed periodic loop (see Kadar and Della Torre [31]). Several authors have proposed models for circumventing this problem (see [62], [3], [52], [43]). One idea is to replace $\mu(\alpha,\beta)$ by either $\mu(\alpha,\beta,v(t))$ or by $\mu(\alpha,\beta,\dot{w}(t))$ ($v(\cdot)$ the input, $w(\cdot)$ the assumed differentiable output) and also to add to the integral a term related to the output $w(\cdot)$ at the points where $\dot{v}(t)$ last changed sign. As we shall discuss below, it is quite easy to see how to experimentally determine $\mu(\alpha,\beta)$ in the basic Preisach model (7); it is not so clear in the more complex models.

To circumvent this particular problem, Mayergoyz has proposed using $\mu_0(\alpha,\beta) + \mu_1(\alpha,\beta)\dot{w}(t)$ as an approximation to $\mu(\alpha,\beta,\dot{w}(t))$; this permits experimental determination of μ_0, μ_1. Salling and Schulz [52] propose shifting the (finite) domain of the Everett integrals (18) as the input v increases and decreases.

To model a given physical situation, the determination of $\mu(\alpha,\beta)$ from experiment is of critical importance and is carried out as follows. For various a, b $(a < b)$ one uses an input with $v(0) = -\infty$, $v(\cdot)$ increasing to the value b, then reversing and decreasing to the value a.

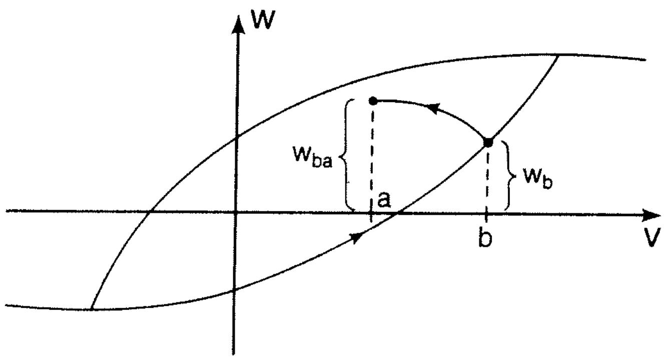

FIG. 12

The output when $v(t_1) = b$ is labelled w_b, and the subsequent output when $v(t_2) = a$ is labelled w_{ba}. It is easy to see that

$$\frac{w_b - w_{ba}}{2} = \int\int_{T(a,b)} \mu = \int_a^b \int_a^y \mu(x,y)dx\,dy, \tag{19}$$

where $T(a,b) = \{(x,y) | a \le x \le y,\ a \le y \le b\}$ (Fig. 9). Thus $2\mu(a,b) = -\partial^2(w_b - w_{ba})/\partial a \partial b$. Unfortunately, numerical differentiation is highly inaccurate, so a better method is needed.

Barker et al. [3] and Doong and Mayergoyz [18] have shown that we can reduce the numerical difficulties by noting that the function $p(a,b) \equiv (w_b - w_{ba})/2$ generates the output for *any input* (with successively decreasing (respectively, increasing) maxima (respectively, minima)) by the following formula:

$$w(t) = -p(a_0, b_1) + 2\sum_{k=1}^{n} [p(a_k, b_k) - p(a_k, b_{k+1})], \tag{20}$$

where $\{a_k\}$, $\{b_k\}$ are, respectively, increasing and decreasing sequences of the a (respectively, b) coordinates of the corners on $L(t)$, and n is the number of horizontal links of $L(t)$. Thus one can avoid numerical differentiation.

To further refine the Preisach model, Mayergoyz [43] suggests the more complex model

$$\begin{aligned} w(t) = \hat{\Gamma}[v](t) &= \int\int_{\alpha\le\beta} \mu(\alpha,\beta,v(t))\hat{F}_{\alpha\beta}[v](t)d\alpha\,d\beta \\ &+ \int_{-\infty}^{\infty} \lambda(\alpha)\hat{\lambda}_\alpha[v](t)d\alpha, \end{aligned} \tag{21}$$

where $\hat{\lambda}_\alpha[v](t) = -1$ if $v(t) < \alpha$, $+1$ otherwise, and $\lambda(\cdot)$ is an unknown weight function representing the reversible part of the process. This reversible part is related to physical effects like "domain wall bowing" in which the elementary hysteresis domains change shape.

Szpunar, Atherton, and Schonbachler [54] and Atherton et al. [1] have outlined methods for overcoming various deficiencies in the Preisach model. If a material is magnetized with $w(0) = B(0) = 0,\ v(0) \equiv H(0) = 0$ (B = magnetization, H = applied field), then the initial susceptibility

$$\frac{dB}{dH}\Big|_{(0,0)} = 0,$$

whereas the Preisach model in general predicts a nonzero value. Also, the loops in Fig. 11 are in practice not always closed, contrary to the model. These two papers provide an excellent overview of efforts by the engineering and physics community to develop more accurate models, and raise a number of interesting mathematical questions.

Kadar and Della Torre [31], [32] propose a "product" model based on the representation

$$\frac{dw^*}{dv} = R(w^*)\int_{v_1}^{v} 2Q(\alpha,\beta)d\alpha, \tag{22}$$

where w^* is the reduced magnetisation w/w_{sat}, w_{sat} is the saturation value (the assumed constant value of $h_U(v)$ in Fig. 1 for v large), v_1 is the input value at the last extremum of $v(\cdot)$, Q is a Preisach-type weight function, $Q(\alpha,\beta) = Q(\beta,\alpha)$, $R(w^*) = R(-w^*)$. As an example they choose $R(m) = 1 - m^2$, which gives noncongruent loops. This type of model seems ripe for deeper mathematical analysis.

Wiesen and Charap [61], [62] propose a practical improvement of the Preisach model which allows noncongruent loops and zero initial susceptibility. In practice one is interested in determining $w(t)$ after $v(\cdot)$ has gone through a finite number of maxima and minima (turning points). Let (v_0, w_0) be an initial state on either of $h_L(\cdot), h_U(\cdot)$ and $\mathcal{L} = \{v_0, v_1, \ldots, v_n\}$ be the list of successive turning points. Then the change Δw as $v(\cdot)$ goes from v_k to $v(t) \in (v_k, v_{k+1}]$ would ideally be a function $D(\mathcal{L}_k, v_k, v(t))$, which depends on $v_k, v(t)$ and the previous history $\mathcal{L}_k = \{v_0, v_1, \ldots, v_{k-1}\}$. In contrast, the standard Preisach model only uses the history since the last turning point v_k. In practice Wiesen and Charap propose to approximate their ideal model by the following.

(1) Initially treating only *first-order* curves—those extending from one turning point to the next, using the standard Preisach model to compute $\Delta w(v_k, v_{k+1})$ (Fig. 13).

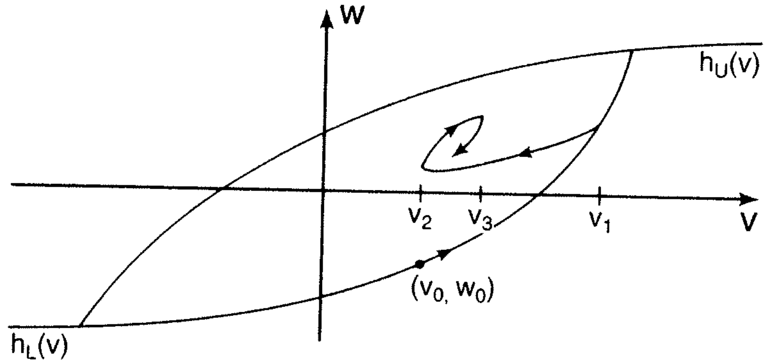

FIG. 13

For any list of turning points, these first-order approximations $\Delta w(v_k, v_{k+1})$ can be determined experimentally. They propose two axioms to carry the model further:

(i) $D(\mathcal{L}, v, v^*) = -D(-\mathcal{L}, -v, -v^*)$;

(ii) $D(\mathcal{L}, v, v^*) = -D(\mathcal{L} \cup \{v\}, v^*, v)$.

The first axiom guarantees symmetry with respect to field reversal, the second guarantees closed loops when the applied field cycles between v and v^*.

(2) To determine the change in magnetization as v moves from v_1 to v_2 in Fig. 13, the standard Preisach model would ignore the history previous to v_1.

They instead define

$$D(\mathcal{L}_1, v_1, v_2) = -\left[\frac{\Delta w(v_1, v_2)}{\Delta w(-v_2, v_1)}\right] \Delta w(-v_1, -v_2),$$

and for the next phase,

$$D(\mathcal{L}_2, v_2, v_3) = -\left[\frac{D(\mathcal{L}_1, v_1, v_2)}{\Delta w(v_2, v_1)}\right] \cdot \Delta w(v_2, v_3).$$

The idea being that they *scale* the first-order (Preisach) computation by the factor in brackets to account for magnetization history.

There presently does not exist any embedding of this scaling technique into the modified Preisach models described earlier.

Wiesen and Charap also develop a model based on a factorization of the measure in the Everett integral. For the operator

$$W[v](t) = E\big(v_1, v(t)\big) = \int_{v_1}^{v(t)} \int_{v_1}^{a} \mu(a,b)\,db\,da,$$

they assume $\mu(a,b) = \rho(a)\rho(-b)$. (Biorci and Pescetti [6], [7] assume a more general factorization.) Here v_1 is the input value at the last change in monotonicity of the input $v(\cdot)$. We can then write $\rho(v) = \sqrt{2}\,\frac{d}{dv}\,q(v)$, $q = 0$ for $v = \alpha$, the left intersection of h_L and h_U, while $q(\beta) = \sqrt{2h_U(\beta)}$ at the right intersection point β (Fig. 2).

This leads to the exact formula

$$E[v_1, v(t)] = [q\big(v(t)\big) - q(v_1)][q(-v_1) - q\big(-v(t)\big)],$$

which yields a quadratic equation for $q(v)$ and the requirement that for $q(v)$ to be real-valued,

$$-h_L(\alpha)\big[h_U(\beta) + W[v](t) + W[-v](t)\big] + \tfrac{1}{4}\big[W[v](t) - W[-v](t)\big]^2 \geq 0.$$

(They assume $\alpha = -\beta$ and $w_{\text{sat.}} \equiv h_L(v) = h_U(-v)$ for $v \notin [-\alpha, \alpha]$.)

Finally, we mention the work of Beardsley, Ortenburger, Potter, and Schmulian in various combinations [47]–[49]. They propose a self-consistent numerical approach to determining $\mu(\alpha,\beta)$ which avoids numerical differentiation.

Continuity properties of the Preisach model. If we are dealing with a problem involving an ordinary or partial differential equation with a hysteresis term, then the continuity properties (as a map between function spaces) of the hysteresis operator are important for proofs of existence, uniqueness, and qualitative properties. The principle work on continuity properties has been carried out by Krasnosel'skii and collaborators [36], Visintin, and Brokate and Visintin [57]–[60], [11], [12]. The recent definitive papers of Brokate and Visintin [12] and Brokate [11] provide a good summary. We present a sampling of their results. We use a Borel measure $d\mu$ in place of $\mu(\alpha,\beta)d\alpha\,d\beta$.

Let $P = \{(\alpha,\beta) | \alpha \leq \beta\}$, and let μ be a finite Borel measure on P. We define

$$w(t) = \int_P\!\!\int \hat{F}_{\alpha,\beta}[v](t)d\mu \equiv F[v](t), \tag{23}$$

where $\hat{F}_{\alpha\beta}$ is a simple hysteresis relay, defined by (1) with $h_L(v) \equiv -1$, $h_U(v) \equiv +1$. As before, we must choose $\hat{F}_{\alpha\beta}[v](t)$ when $v(s) \in (\alpha,\beta)$ for $0 \leq s \leq t$. Thus $\hat{F}_{\alpha\beta}$ is really a mapping from continuous functions to a pair $\big(w(\cdot), \pm 1\big)$, the latter entry reflecting our choice of $w(0)$ if $v(0) \in (\alpha,\beta)$. To keep this clearly in mind we can write $\hat{F}_{\alpha\beta}\big(v(\cdot), \eta(\alpha,\beta)\big)$, where η is a measurable function from P into $\{-1,+1\}$, reflecting our choice of output as $+1$ or -1 when $v(0) \in (\alpha,\beta)$.

Here are some of the basic continuity results.

(1) If $|\mu|(\ell) = 0$ for every horizontal or vertical straight line ℓ, then F maps $C([0,T])$ into $C([0,T])$ and is continuous in the uniform topology.

(2) Let $p = (\beta - \alpha)/2$, $\quad q = (\alpha + \beta)/2$ be rotated coordinates in the (α,β)-plane, so P becomes $\{(p,q) | p \geq 0\}$. For a curve $q = \psi(p)$ from $\Psi_1 = \{\psi | \psi \in C[0,\infty)$, has bounded support, and is Lipschitz continuous with constant $\leq 1\}$, define

$$N(\psi,\varepsilon) = \{(p,q) | p \geq 0, \quad \psi(p) - \varepsilon < q < \psi(p) + \varepsilon\}.$$

Then $F : C([0,T]) \to C([0,T])$ is Lipschitz continuous if and only if there exists a K such that for all $\varepsilon > 0$,

$$\sup_{\psi\in\Psi_1} |\mu|(N(\psi,\varepsilon)) \le K\varepsilon.$$

(3) Let $\mu \ge 0$ satisfy $\mu(P) < \infty$ with $\mu(\ell) = 0$ for any vertical or horizontal line (in (α,β) coordinates). Let $\lambda \in R$, $\xi > 0$, and define $R_\alpha(\lambda, \lambda+\xi)$, $R_\beta(\lambda, \lambda+\xi)$ as the horizontal, respectively, vertical strip defined by $\lambda \le \alpha \le \lambda+\xi$, respectively, $\lambda \le \beta \le \lambda+\xi$. Define $k(\xi) = \sup\{\mu(R_\nu)|\nu = \alpha,\beta;\ \lambda \in R\}$. Assume $k(\xi) \le C\xi$ for all $\beta \ge 0$. Then H maps the Sobolev space $W^{1,s}(0,T)$ into itself for $1 \le s \le \infty$, and for $s > 1$ it is sequentially weak-star continuous.

The above results clearly give powerful tools for existence theorems for problems involving ordinary or partial differential equations—for examples, see [57], [37], [34].

3.4. Properties of the Krasnosel'skii–Pokrovskii hysteron. Continuity properties of the Krasnosel'skii–Pokrovskii hysteron described in Fig. 7 can be summarized as follows:

(a) F maps $C([0,T])$ into $C([0,T])$;

(b) For a point M in the hysteresis region $\mathcal{H}$ let $\pi(v,M)$ denote the curve from Property 3 which passes through M.

Define the graph of possible outputs for monotone input $v(\cdot)$ with $v(0) = M$:

$$T(v;M) = \begin{cases} \Phi_L(v) & \text{for } v \le v_L(M), \\ \pi(v;M) & \text{for } v_L(M) \le v \le v_R(M), \\ \Phi_R(v) & \text{for } v > v_R(M), \end{cases} \tag{24}$$

where (Fig. 7) $v_L(M)$ and $v_R(M)$ are the v-coordinates of the endpoints of $\pi(v;M)$. If $M = (v_0, w_0)$, then we ask that $T(v;(v_0,w_0))$ satisfy a Lipschitz condition with respect to (v_0, w_0, v) and assume that the functions Φ_L and Φ_R satisfy a Lipschitz condition. Then $F[v]$ satisfies a Lipschitz condition as a map from $C([0,T])$ into $C([0,T])$.

(c) Assume that Φ_L and Φ_R intersect only at a_R and b_L (Fig. 7), that they are locally absolutely continuous, and for any sequence $\{M_n\} \subset \mathcal{H}$, $M_n \to M \in \overline{\mathcal{H}}$, the function $\pi(v, M_n)$ satisfies a *uniform* Lipschitz condition.

We define a truncated version of $T(v;M)$ (Fig. 14)

$$\chi(v,M) = \begin{cases} \Phi_L[v_L(M)] & \text{for } v \le v_L(M), \\ \pi(v;M) & \text{for } v_L(M) \le v \le v_R(M), \\ \Phi_R[v_R(M)] & \text{for } v_R(M) \le v. \end{cases}$$

Then F is continuous from $AC([0,T])$ to $AC([0,T])$ if and only if for any sequence $\{M_n\} \subset \mathcal{H}$, with $M_n \to M \in \overline{\mathcal{H}}$, the functions $\chi(v;M_n)$ form a precompact set in $AC([t_1,t_2])$ for any $(t_1,t_2) \subset [0,T]$.

(d) Assume that $\pi(v;M)$, $\Phi_L(v)$, $\Phi_R(v)$ are all Hölder continuous with exponent γ. Then F is a continuous map from H_α into H_β for $0 < \beta < \alpha\gamma$, $0 < \alpha \le 1$, with respect to the Hölder norm defined for H_δ by

$$\|v\| = |v(t_0)| + \sup_{\tau_i\in[0,T]} \frac{|v(\tau_1) - v(\tau_2)|}{|\tau_1 - \tau_2|^\delta}.$$

Krasnosel'skii and Pokrovskii prove that their hysteron is vibrocorrect under reasonable conditions on Φ_L, Φ_R and the family $\pi(\cdot)$.

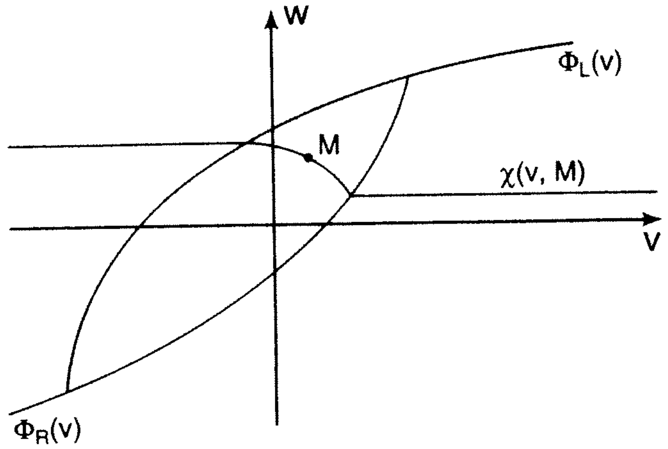

FIG. 14

4. Vector models of hysteresis. There have been several suggestions for extending hysteresis models beyond the scalar case (any or all of $t, v(\cdot)$ and $w(\cdot)$ vector-valued). One clear and easy extension is the case when $t \in R$, $v(t) \in R^p$, $w[v(\cdot)](t) \in R^q$ with each component $w_j[\,\cdot\,]$ exhibiting hysteresis in its dependence on a single component $v_{i(j)}(\cdot)$.

Since Krasnosel'skii and Pokrovskii characterize their scalar active hysteresis model as $F[v](t) = f\big(v(t), P(\Gamma_L, \Gamma_R)[v](t)\big)$ with P a generalized play (§2.4), they extend this characterization to higher dimensions.

For $v(t)$ and $w(t)$ both in R^2, for example, their idea is to replace the one-dimensional piston by a two-dimensional convex body, in the simplest case a $2h_1 \times 2h_2$ rectangle as in Fig. 15.

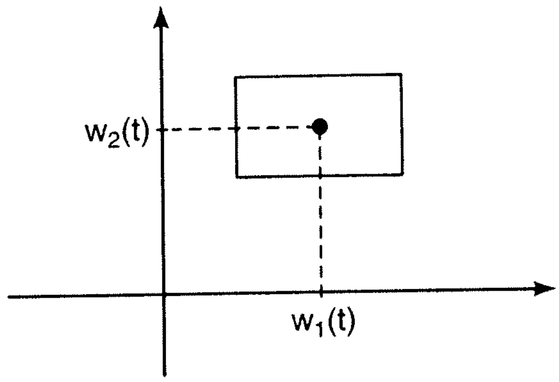

FIG. 15

The coordinates of the center of the rectangle are the output, and we can require

$$w_1(t) = P_{2h_1}[v_1](t), \qquad w_2(t) = P_{2h_2}[v_2](t),$$

i.e., you can only move the rectangle parallel to the axes, and you have slack in both directions. This is exactly the situation described in paragraph 1 of this section in that each component of $w(\cdot)$ exhibits hysteresis dependence on a single component of $v(\cdot)$.

A more general idea is to use a finite closed convex body $Z \subset R^p$. The *play with characteristic* Z is a map $P : v(\cdot) \mapsto w(\cdot)$ with feasible states

$$\Omega(P) = \{(v, w) | v, w \in R^p,\ v - w \in Z\},$$

with the output w defined as a solution of the differential equation (for $v(\cdot) \in C^1$, say)

$$\frac{dw}{dt} = N[K(v(t) - w(t));\ v'(t)], \tag{25}$$

where $K(y)$ is the normal cone to Z at y; $N[K(y);\ v'(t)]$ is the vector in $K(y)$ nearest to $v'(t)$. $(N(\phi) = 0.)$

They show that this defining differential equation has a solution for feasible initial conditions and a piecewise smooth input. Deeper analysis is quite difficult and there are many open questions, however, they are able to establish continuity results for the two cases of polyhedral and smooth convex bodies. We refer the reader to Krasnosel'skii and Pokrovskii [36, Chap. 4].

Wiesen and Charap [61] extend their factorization model (§3.3) to a two-dimensional model that uses scalar Preisach models in orthogonal directions, assuming these models respond to both parallel and orthogonal inputs, i.e., $\vec{w}[\vec{v}(\cdot)](t)$ is a sum of orthogonal components $w_1[\vec{v}(\cdot)](t),\ w_2[\vec{v}(\cdot)](t)$, each of which is a sum of two independent Preisach functionals (Everett integrals), one involving only $v_1(\cdot)$, the other involving only $v_2(\cdot)$. This approach seems to have good potential as a model for recording on tape.

Mayergoyz (see [43]) extended ideas of Stoner and Wohlfarth [53] to suggest the two-dimensional model

$$\vec{F}[\vec{v}\,](t) = \int_0^{2\pi} s(\theta)\vec{e}_\theta F_\theta[\vec{e}_\theta \cdot v(\cdot)](t)d\theta \tag{26}$$

where $\vec{e}_\theta = (\cos\theta, \sin\theta)$ and $F_\theta[\cdot]$ is a scalar Preisach hysteresis operator (7) varying with θ; $s(\theta)$ is a (scalar) function to allow anisotropy. In three dimensions this model would become

$$\vec{F}[\vec{v}\,](t) = \int_0^{\pi}\int_0^{2\pi} s(\theta,\phi)\vec{e}_{\theta,\phi}F_{\theta,\phi}[\vec{e}_{\theta,\phi} \cdot \vec{v}(\cdot)](t)d\theta\, d\phi, \tag{27}$$

and the extension to higher dimensions is clear. In [44] Mayergoyz showed how to analyze the three-dimensional model using an irreducible representation of the rotation group.

From an engineering point of view, of paramount importance is the *identification problem* of finding the measure $d\mu$, given $\vec{F}[\vec{v}\,](t)$ for a large number of diverse inputs $\vec{v}(\cdot)$. Mayergoyz and Friedman showed that when $s(\theta) \equiv 1$ in the two-dimensional model, then

$$2\int_0^{\pi/2} \cos\theta \left(\int\int_{\hat{T}(a,b,\theta)} d\mu(\rho_1,\rho_2)\right) d\theta = \frac{w_b - w_{ba}}{2} \equiv H(b,a), \tag{28}$$

where w_b, w_{ba} were defined in the discussion leading to (19), while $\hat{T}(a,b,\theta)$ is the triangle $T(b\cos\theta, a\cos\theta)$ defined as in Fig. 9, and as before $\partial^2 H/\partial b\partial a = d\mu(a,b)$. They were able to convert this identification problem to an integral equation. First we write

$$2\int_0^{a} \frac{x}{b\sqrt{b^2 - x^2}} \left(\int\int_{T(x,\lambda x)} d\mu(\rho_1,\rho_2)\right) dx = \frac{bH(b,\lambda b)}{2}, \tag{29}$$

with $\lambda = a/b,\ x = b\cos\phi$. Then, defining $N(x) = x\int\int_{T(x,\lambda x)} d\mu(\rho_1,\rho_2),\ R(b) = bH(b,\lambda b)/2$, we have

$$\int_0^{b} N(x)(b^2 - x^2)^{-1/2}dx = R(b). \tag{30}$$

Using standard techniques, we obtain

$$N(b) = \frac{2}{\pi}\frac{d}{db}\int_0^b tR(t)(b^2 - t^2)^{-1/2}dt,$$

so

$$\iint_{T(b,\lambda b)} d\mu = \frac{1}{\pi}\int_0^b \left[H(r,\lambda r) - r\frac{d}{dr}H(r,\lambda r)\right](b^2 - r^2)^{-1/2}dr. \tag{31}$$

By varying b and a (e.g., λ) over a sufficiently wide range we can experimentally determine H, and then (30) will allow the determination of the measure $d\mu$.

Damlamian and Visintin in [15] proposed two different models for two-dimensional vector hysteresis, and established a few basic properties. If $\vec{a}_\theta = (\cos\theta, \sin\theta)$, and $\vec{v}(\cdot) \in C([0,T], R^2)$, then an elementary hysteresis operator is defined by

$$\vec{F}_{\alpha,\beta,\theta}[\vec{v}(\cdot)](t) = \widehat{F}_{\alpha,\beta}[\vec{v}(\cdot)\cdot\vec{a}_\theta](t)\,\vec{a}_\theta,$$

where $\widehat{F}_{\alpha\beta}$ is a basic scalar operator defined, for example, below (6). This is exactly the elementary operator in Mayergoyz's formulation (26). In fact we have to include initial values as discussed in the first section, but this is a simple addition. Then we can average over (α,β) in P to get a Preisach operator, in fact exactly (26) with $s(\theta) \equiv 1$.

Their second model is based on a convex compact set K in R^2. The threshold values of an input are defined by

$$I_t = \{\tau \in [0,t] | \vec{v}(\tau) \in \partial_{\mathrm{rel}}K\},$$

where $\partial_{\mathrm{rel}}K$ is the relative boundary of K. If $I_t = \emptyset$, then $\vec{w}[\vec{v}(\cdot)](t) = \vec{w}(0)$. If $I_t \neq \phi$, then $\vec{w}[\vec{v}(\cdot)](t) = \mathrm{Proj}_K\ \vec{v}(\tau_M)$, where $\tau_M = \sup_{I_t} \tau$ (i.e., the last threshold). Very little is known of the properties or applications of this model.

There is a need for a systematic and complete vector hysteresis model, since ferromagnetic materials are generally two (e.g., recording tape) or three-dimensional. Such a model needs to be both theoretically sound, i.e., able to predict and model the full range of possible behavior, and also practical in the sense that, for example, the identification problem can be numerically solved. The monograph of Mayergoyz [43] should provide a benchmark for further work.

5. Periodic oscillations in systems with hysteresis. Electrical engineers are interested in the oscillatory behavior of circuits with hysteresis, a subject which turns out to be very difficult. The response $w(\cdot)$ to a periodic input $v(\cdot)$ for the basic active or passive hysteron as sketched in Figs. 1 and 2 is easily seen to be eventually periodic. However, a circuit with a hysteresis element might be modelled by

$$L_m[y](t) = F[y](t) + p(t)$$

with L_m an mth-order differential operator and $p(t)$ a periodic forcing term. Also of interest is the unforced case when $p(t)$ is absent. In either case the prediction of periodic oscillations is a difficult problem. We describe the small number of known results for such problems.

Bouc [8], [9] proves the existence of periodic solutions to linear second-order equations involving the Duhem hysteresis operator (§2.1, §3.1):

$$x''(t) + \mathcal{F}[x(\cdot)](t) = p(t),$$

$p(t)$ periodic, using the Schauder fixed point theorem.

The *Method of Harmonic Balance* is a standard method for predicting oscillations in nonlinear circuits, and this heuristic method has been successful to some extent in predicting periodic solutions for systems with hysteresis—however, as Tsypkin has pointed out, the method can make erroneous predictions in such systems [55]. In this approach, we have a system

$$\dot{z}(t) - Az = F[z(\cdot)], \quad z(t) \in R^m, \tag{32}$$

with periodic boundary conditions $z(0) = z(T)$, with A an $m \times m$ constant matrix, F a nonlinear term (perhaps with hysteresis), and T unknown. We assume a "solution" of the form

$$z_n(t) = \sum_{k=0}^{n} (a_k e^{ik\omega t} + a_k^* e^{-ik\omega t}), \qquad a_k \in \mathbb{C}^n \tag{33}$$

(a_k^* stands for the complex conjugate of a_k), i.e., a truncated Fourier series with vector coefficients, with $\omega = 2\pi/T$. Putting this in (31), we obtain

$$\dot{z}_n(t) - Az_n(t) = F[z_n(\cdot)]. \tag{34}$$

Replacing z_n and $\dot{z}_n$ by using (33), and replacing the function $F[z_n(\cdot)]$ by its truncated Fourier series $F[z_n(\cdot)] = \sum_{k=0}^{n} c_k e^{ik\omega t} + c_k^* e^{-ik\omega t}$ with $c_k = c_k(a_1, \ldots, a_n)$, (34) becomes the equation of harmonic balance:

$$\sum_{k=0}^{n} (ik\omega I - A) a_k e^{ik\omega t} + (-ik\omega I - A) a_k^* e^{-ik\omega t} = \sum_{k=0}^{n} c_k e^{ik\omega t} + c_k^* e^{-ik\omega t}. \tag{35}$$

Equating coefficients of each term $e^{ik\omega t}$, we get a set of algebraic equations for the unknown a_k's and ω. If, however, this set of equations has a nontrivial solution $\{\bar{\omega} \neq 0,\ \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_n$ with $\sum_0^n |\bar{a}_k|^2 > 0\}$, then under certain conditions we can assert that the original system (31) has a periodic or "close to periodic" solution. One of the key assumptions is that the linear circuit described by $M[z] = \dot{z} - Az$ be a "low-pass filter," that is, the inverse of M with periodic boundary conditions should sharply attenuate the "tail" of any Fourier series input: the function

$$M^{-1}\left[\sum_{k=n+1}^{\infty} (a_k e^{ik\omega t} + a_k^* e^{-ik\omega t})\right]$$

should be small in an appropriate norm when compared with the norm of

$$\sum_{k=n+1}^{\infty} (a_k e^{ik\omega t} + a_k^* e^{-ik\omega t}).$$

For a full discussion of the heuristic use of the method see [56].

Bergen, Mees, and Franks in various combinations and with others have shown how to justify the use of the method in nonlinear systems without hysteresis [4], [5], [45]. Braverman, Meerkov, and Pyatnitskii [10] have justified its use to predict "close-to-periodic" oscillations in the presence of hysteresis under certain assumptions. Macki, Nistri, and Zecca [40], [41] have given conditions under which the method can be used to predict periodic solutions in the presence of hysteresis. We now describe some of these results.

Braverman, Meerkov, and Pyatnitskii treat a system of the form

$$\dot{x}(t) = Ax + b\big[F[x_1(\cdot)](t) + c\ \sin \omega t\big] \tag{36}$$

with $x(t) \in R^n$; c and $\omega \geq 0$ are given scalar constants; $b \in R^n$ is a constant vector, A is a constant $n \times n$ matrix, and $x_1(\cdot)$ is the first component of $x(\cdot)$. They assume the real-valued operator $F[\cdot]$ is a passive hysteresis operator (Fig. 2) without jumps at α and β.

They apply the harmonic balance method by assuming that the harmonic balance equation (35) has a solution of the form

$$\bar{x}(t) = \sum_{k=0}^{p} \bar{a}_k \sin(k\bar{\omega}t + \gamma_k), \tag{37}$$

$\bar{a}_k$ a vector, $p \geq 1$ fixed. Their assumptions are:

(A1) $n \geq 2$ and A is a stable matrix (all eigenvalues have negative real part).

(A2) $h_L(\cdot)$ and $h_U(\cdot)$ in Fig. 2 satisfy a Lipschitz condition with constant L.

(A3) The linear system is a low-pass filter; they phrase this assumption in the language of Laplace transforms as follows.

If we take the Laplace transform of (36), we can solve for the transform of $x(\cdot)$,

$$\hat{x}(s) = W(s)b\mathcal{L}\big[F[x_1(\cdot)] + c \sin \omega t\big], \tag{38}$$

and the first component can be written as $\hat{x}_1(s) = \tilde{W}(s)\mathcal{L}[F[x_1(\cdot)] + c \sin \omega t]$, where $\tilde{W}$ is a rational scalar function of s.

They then define the following numbers:

$$\varepsilon_i = \frac{\int_{|\nu|>\bar{\omega}(s+\frac{1}{2})} |\frac{d^i}{d\nu^i}\, \tilde{W}(j\nu)| d\nu}{\int_{|\nu|\leq\bar{\omega}(s+\frac{1}{2})} |\frac{d^i}{d\nu^i}\, \tilde{W}(j\nu)| d\nu}, \qquad \varepsilon = \max(\varepsilon_0, \varepsilon_1, \varepsilon_2). \tag{39}$$

Define also the constants

$$R = \left[\max_{|\lambda|\leq\sum_{k=0}^{p} \bar{a}_k} |F[\bar{x}_1(\cdot)](\lambda)| + c\right] \int_{-\infty}^{\infty} (|\tilde{W}| + 2|\tilde{W}'| + |\tilde{W}''|) d\nu \tag{40}$$

$$S = \left(5L \int_{-\infty}^{\infty} |\tilde{W}| d\nu\right)^{-1}, \qquad \psi(\bar{\omega}) = \max\left(\frac{4}{\bar{\omega}}, \frac{28}{\bar{\omega}^2}, 1\right) \tag{41}$$

with $j = \sqrt{-1}$, $\tilde{W} = \tilde{W}(j\nu)$, $\tilde{W}' = \tilde{W}'(j\nu)$, etc.

(A4) They assume the following.

(a) If $\bar{x}(t)$ satisfies $\inf_t \bar{x}_1(t) > \beta$, then

$$\inf_t \bar{x}_1(t) \notin [\beta - \varepsilon^{1/2} R\psi(\bar{\omega}), \beta + \varepsilon^{1/2} R\psi(\bar{\omega})], \tag{42}$$

(b) If $\bar{x}(t)$ satisfies $\sup \bar{x}_1(t) < \alpha$, then

$$\sup_t \bar{x}_1(t) \notin [\alpha - \varepsilon^{1/2} R\psi(\bar{\omega}),\ \alpha + \varepsilon^{1/2} R\psi(\bar{\omega})]; \tag{43}$$

(c) If $\bar{x}(t)$ satisfies $\inf_t \ \bar{x}_1(t) < \beta < \alpha < \sup_t \ \bar{x}_1(t)$, then both (42) and (43) hold. A function $\bar{x}_1(\cdot)$ satisfying (a) (respectively, (b),(c)) is called a "motion of the first kind" (respectively, "second kind," "third kind"), and the assumption ensures that the first component of any function in a neighborhood of the approximation $\bar{x}(\cdot)$ will pass through the same thresholds as $\bar{x}(\cdot)$, in the same order.

THEOREM (Braverman, Meerkov, and Pyatnitskii [10]). *Under the above assumptions, the system* (36) *has a "close to periodic" solution satisfying*

$$(44) \qquad |x(t) - \bar{x}(t)| \leq \varepsilon^{1/2} R\psi(\bar{\omega}) \quad \text{for } t \in [t_0, t_0 + S \ \log(1/\varepsilon^{1/2})].$$

They also consider the system (36) with a relay hysteresis operator as defined by Fig. 1(b) with $h_L(v) = -M$, $h_U(v) = +M$. In this case, with A stable, the only interesting case is for solutions of the third kind (if eventually $F[x_1(t)] \equiv M$, or $\equiv -M$, as happens with motions of the first and second kind, the equation can be integrated). In this case, let $k(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{W}(j\omega)e^{j\omega t}d\omega$ be the pulse transition function. Assume the following.

(A5) There exists $t^* > 0$ such that $\operatorname{sgn} k(t) = -1$ for $0 \leq t \leq t^*$.

THEOREM (Braverman, Meerkov, and Pyatnitskii [10]). *Assume* (A1), (A4), *and* (A5) *hold for the relay hysteresis operator, and assume the method of harmonic balance with* $p = 1$ *yields a function* $\bar{x}(t) = \bar{a}\sin(\bar{\omega}t + \gamma)$. *If* $L \leq \bar{a}_1$ *and* ε *as defined in* (A3) *satisfies*

$$\varepsilon < \min\left\{-L + \bar{a}_1 \sin\left(\frac{\pi}{4} + \frac{1}{2}\arcsin\left(\frac{L}{a_1}\right)\right), \quad \bar{a}_1\bar{\omega}\cos\left(\frac{\pi}{4} + \frac{1}{2}\arcsin\left(\frac{L}{a_1}\right)\right)\right\},$$

then (36) *has a "close to periodic" solution* $x(\cdot)$ *satisfying* $|x(t) - \bar{x}(t)| \leq \varepsilon^{1/2}$ *for*

$$0 \leq t \leq \left[\frac{\log(\varepsilon^{-1/2})}{2\log[R(1+K)]} - 1\right]\frac{\pi}{\bar{\omega}},$$

where

$$K = 1 + \frac{2M}{\bar{a}_1\bar{\omega}\cos(\frac{\pi}{4} + \frac{\phi}{2})}\int_{-\infty}^{\infty} |\tilde{W}(j\omega)|d\omega.$$

Kamachkin [33] proves existence and uniqueness of a periodic solution for a forced linear system with a hysteresis term appended; Miller, Michel, and Krenz [46] prove existence for a forced nonlinear system with hysteresis.

Macki, Nistri, and Zecca [40], [41] have applied the method of harmonic balance to systems with discontinuities and relay hysteresis as sketched in Fig. 1(a). Their approach is based in part on that developed by Bergen, Mees, and Franks [4], [5], [45] and their collaborators. The first step is to normalize the unknown frequency ω in (32) to 2π by writing $z(t) = x(2\pi t/T)$ so (32) becomes

$$(45) \qquad \omega\dot{x} - Ax = F[x](t), \qquad x(0) = x(2\pi).$$

We invert $(\omega d/dt - A)$ with the periodic boundary condition to get an operator equation on an appropriate space of periodic functions:

$$(46) \qquad x(t) = T_\omega[x](t), \qquad T_\omega = \left(\omega\,\frac{d}{dt} - A\right)^{-1} \circ F.$$

The associated harmonic balance equation is

$$(47) \qquad (P_n x)(t) \equiv x_n(t) = P_n T_\omega[x_n](t),$$

where $P_n x$ is the projection of $x(\cdot)$ onto the first $2n+1$ terms of its (complex) Fourier expansion. We then decompose (46) into a pair of equations:

$$(46') \qquad \begin{array}{ll} \text{(a)} & x_n(t) = P_n T_\omega[x_n + x^*](t), \\ \text{(b)} & x^*(t) = (I - P_n) T_\omega[x_n + x^*](t), \end{array}$$

where $x^* = (I - P_n)x$. Note that when $x^* = 0$, (46′) (a) is just the equation of harmonic balance (47). We then create a homotopy between (46′) and (47) which, when combined with a judicious use of fixed point theory and/or degree theory, allows one to argue that (46′) has a solution if (47) has a solution. Because $F[x]$ has discontinuities and/or relay hysteresis, we must replace (46′) and (47) by differential inclusions because the discontinuities can be treated by making $F[x]$ a set-valued mapping (multifunction). This is easy to see in the case when $F[x](t)$ is a nonlinear function of $x(t)$ with a jump discontinuity at x_0 (Fig. 16).

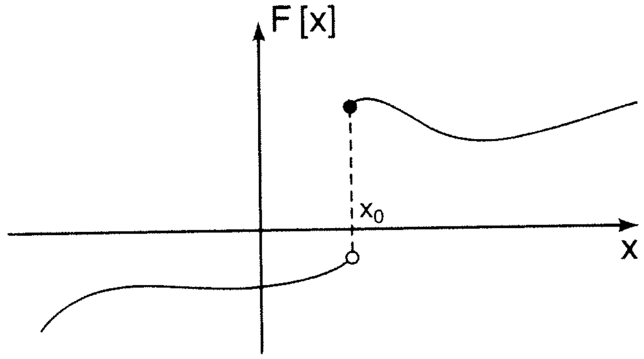

FIG. 16

We then define the set-valued map $\tilde{F}$:

$$\tilde{F}(x) = \text{ convex hull of } \{F(x^-), F(x^+)\}.$$

Under reasonable conditions on the multifunction T_ω, which we will not repeat here (see [41]), they prove that there exists a solution of the inclusion $\dot{x}(t) \in T_\omega[x(\cdot)](t)$, hence there exists a periodic solution of (45). They use both the theory of Leray–Schauder degree and the Schauder fixed point theorem for multifunctions.

Krejči proves the existence of periodic solutions to *partial* differential equations with an Ishlinskii hysteresis term (see [37] and the references therein).

6. Hysteresis in biological problems. There are models in biology which involve thresholds in such a way that a hysteresis operator is suggested. As an example, Hoppensteadt, Jäger, and Pöppe [29] describe the growth of bacteria in a petri dish by means of a system of partial differential equations. We present a summary of their model. Let $B(r,t)$ denote the bacterial population at radius r, with $B(r,0) = B_0$, $0 \le r \le R$. A drop of histidine solution is placed in the center of the dish, so if the histidine level is denoted $H(r,t)$, then $H(r,0) = H^\circ$ for $0 \le r \le R'$, $H(r,0) = 0$ for $r > R'$. The histidine diffuses, is taken up by the bacteria, and is also neutralized by acids produced as a byproduct of cell growth. The bacteria are fixed on an agar gel which contains a buffer concentration $G(r,t)$ needed for growth.

The model they use is

$$\frac{\partial B}{\partial t} = \alpha V B, \quad \frac{\partial H}{\partial t} = D\Delta H - \beta V B, \quad \frac{\partial G}{\partial t} = D'\Delta G - \gamma V B,$$

where $\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$ is the Laplacian with circular symmetry; D and D' are diffusion coefficients for histidine and buffer, respectively. The "viability" term V is a hysteresis term representing the dependence of rate of growth of the bacterial population on the pH of the acid-buffer mix. The bacteria stop growing ($V = 0$) if the pH is less than a level α, and will not start growing again unless the pH exceeds a larger value β (in which case $V = 1$) (Fig. 17 (a)).

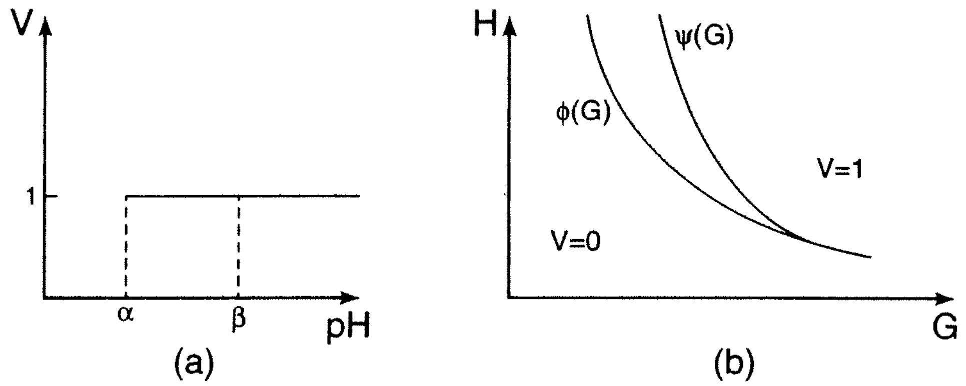

FIG. 17. *This is taken from* [25].

In fact, the threshold of Fig. 17 (a) depends on adequate histidine, that is, there is a similar threshold phenomenon involving histidine levels. The best way to incorporate both thresholds is to forego the model in Fig. 17 (a) for a more complicated model (Fig. 17 (b)). The lower curve $H = \phi(G)$ is the "off" curve and the upper curve $H = \psi(G)$ is the "on" curve. At a given fixed radius r, we track the curve $\big(G(r,t), H(r,t)\big)$ as t increases, using crossings of the "on" and "off" curves as hysteresis-type thresholds.

Experimentally (and numerically) it has been shown (see Hoppensteadt and Jäger [28]) that the growth pattern stabilizes in time as a set of concentric rings of high B-values, with relatively low values in between—a Liesegang phenomenon. There is at present no theoretical derivation of this phenomenon from the model; it represents an interesting open question. Visintin [60] and Jäger [30] have shown that the system has a positive solution, but the prediction of Liesegang rings is still open.

As a general observation both Krasnosel'skii and Pokrovskii [36, §28.6] and Hoppensteadt, Jäger, and Pöppe note that a model involving a fold catastrophe (Fig. 18 (a)) can be replaced by a model involving hysteresis (Fig. 18 (b)). (In fact this observation occurs, for example, in mechanics at least as early as the work of Prandtl [50], and also in the work of Everett from the 1950s [21]–[25] in the context of adsorption hysteresis.)

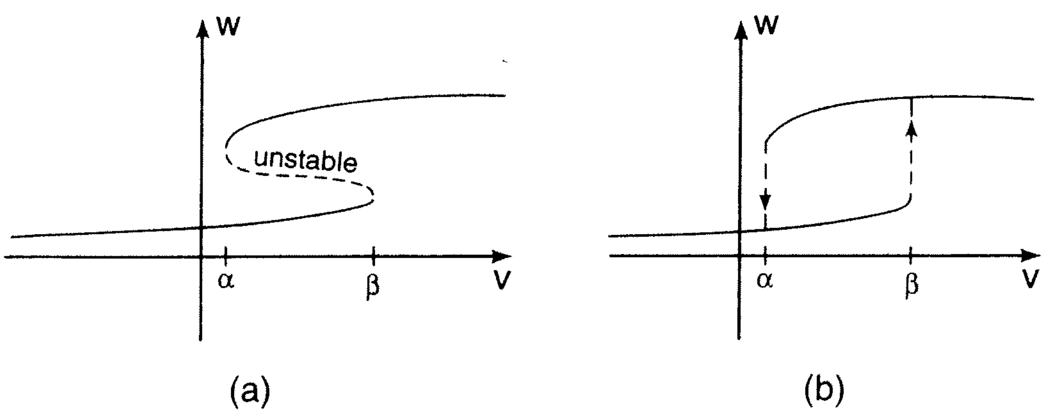

FIG. 18

7. References. We make no pretense of listing every paper we have consulted. In general we have listed the most important and the most recent papers by authors cited. References to their other work are contained in the cited papers. We mention the useful extensive bibliographies in [59], [36], [43]. In addition, Visintin has prepared a bibliography with 110 entries, which appears as "A Collection of References on Hysteresis," Istituto di Analisi Numerica del Consiglio Nazionale delle Ricerche, Pub. No. 621, Pavia, 1988.

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