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A Topological Degree for Multivalued A-Proper Maps in Banach Spaces.

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Sunto. – *Si dà una teoria del grado topologico per applicazioni multivoche A-proprie che unifica ed estende la teoria del grado topologico per applicazioni A-proprie ad un sol valore, data da F. E. Browder e W. V. Petryshyn in* [2] *e quella per campi vettoriali compatti multivoci, data da A. Cellina e A. Lasota in* [3]. *Inoltre, usando tale teoria, si ottengono sia alcuni già noti teoremi di punto fisso sia teoremi di punto fisso in spazi di Banach parzialmente ordinati che estendono alcuni risultati precedentemente noti.*

Introduction.

The degree theory for singlevalued A-proper maps was developed in [2] by BROWDER and PETRYSHYN. The aim of this note is to give a degree theory for multivalued A-proper maps. Our results unify and extend the degree theory for singlevalued A-proper maps of [2] and the degree theory for multivalued compact vector fields of [3]. The paper is divided into five parts.

In the first part we give the basic definitions and preliminary results to be used in the sequel.

The second part is devoted to the degree theory for multivalued A-proper maps.

In the third part we apply degree to obtain some well known results of fixed-point theory.

In the fourth part we show how our degree can be used to obtain some fixed-point theorems for maps defined on partially ordered Banach spaces. In particular we give some results regarding non-

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zero fixed-points, extending results previously obtained by the other Authors (see [8]).

The last part gathers all the proofs of the statements presented in the second, third and fourth parts of this paper.

1. – Notations, definitions and preliminary results.

Let E and F be Banach spaces and let $E\times F$ be the cartesian product endowed with the maximum norm.

For $U\subset E$, let $d(x, U)=\inf\{\|x-y\|: y\in U\}$. The *separation* of U from V, $d^*(U, V)$ is defined by $d^*(U, V)=\sup\{d(x, V): x\in U\}$. An open ball about x of radius $\varepsilon>0$ is denoted by $B(x, \varepsilon)$. We shall also consider $B(A, \varepsilon)=\{y\in E: d(y, A)<\varepsilon\}$. For any subset $A\subset E$, ∂A denotes the boundary, $\bar{A}$ the closure and $\overline{\text{co}}\, A$ the closed convex hull of A respectively.

A multivalued map $\Gamma: E \multimap F$ is called *upper semicontinuous* (u.s.c.) at $x\in E$ if given $\varepsilon>0$ there exists a $\delta>0$ such that $\Gamma(B(x, \delta))\subset B(\Gamma(x), \varepsilon)$. The map Γ is called u.s.c. on E if it is u.s.c. at each point of E.

The map Γ is called *closed* if its graph is closed in $E\times F$. If $\Gamma: E \multimap F$ is u.s.c. with closed values, then Γ is closed.

A definition of topological degree for multivalued maps in finite dimensional spaces.

Let E and F be finite dimensional normed spaces with $\dim F = \dim E$ and let A be a bounded, open subset of E. If $\Gamma: \bar{A} \multimap F$ is an u.s.c. map with convex values (i.e. $\Gamma(x)$ is convex for any $x\in\bar{A}$), then the following, lemma due to CELLINA [4] holds:

LEMMA 1.1. – *Given $\varepsilon>0$, there exists a continuous single-valued map $f: \bar{A}\to\overline{\text{co}}\, R(\Gamma)$ such that*

$$d^*(G_f, G_\Gamma)<\varepsilon$$

where G_f and G_Γ are the graphs of f and Γ respectively.

DEFINITION 1.1. – We say that a sequence $\{\Gamma_n\}$ of multivalued maps from $\bar{A}$ into F converges to Γ (denoted by $\Gamma_n\to\Gamma$) when

$$d^*(G_{\Gamma_n}, G_\Gamma)\to 0 \qquad \text{as } n\to+\infty .$$

Let $\Gamma: \bar{A} \multimap F$ be an u.s.c. map with compact and convex values by Lemma 1.1. there exists a sequence of continuous maps $\{f_n\}$, $f_n: \bar{A}\to\overline{\text{co}}\, R(\Gamma)$ converging to Γ. Note that $\overline{\text{co}}\, R(\Gamma)$ is compact since Γ sends compact sets into compact sets (see BERGE [1]) and the convex closure of a compact set is compact. The compactness of $\overline{\mathrm{co}}\, R(\Gamma)$ is essential in what follows.

Let $p \notin \Gamma(\partial A)$. We define the *topological degree of Γ at point p* to be.

$$\deg(\Gamma, A, p) = \lim_{n \to +\infty} \deg(f_n, A, p) \tag{1.1}$$

where $\deg(f_n, A, p)$ is the Brouwer topological degree.

The above construction is due to CELLINA and LASOTA [3]. Actually, they defined a topological degree for compact vector fields in locally convex metric spaces.

This degree satisfies the *solution, normalization, additivity and homotopy invariance* properties. Further, if A is a symmetric neighborhood of the origin and if $\Gamma \colon \bar{A} \multimap F$ is an u.s.c. map with compact and convex values which is odd on the boundary of A and if $0 \notin \Gamma(\partial A)$, then $\deg(\Gamma, A, 0)$ is odd (in particular different from zero). In fact, the homotopy $H(x, t) = (1/1+t)\,\Gamma(x) + (t/1+t)\cdot(-\Gamma(-x))$ is admissible in the sense of [3], hence

$$\deg(H(\cdot, 0), A, 0) = \deg(H(\cdot, 1), A, 0)$$

where $H(\cdot, 1)$ is an odd map on $\bar{A}$. The fact that $\deg(H(\cdot, 1), A, 0)$ is an odd integer can be shown by constructing a sequence (see [6], Lemma 2.3) of single-valued continuous *odd* maps $\{f_n\}$ $f_n \colon \bar{A} \to \overline{\mathrm{co}}\, R(H(\cdot, 1))$, converging to $H(\cdot, 1)$ in the sense of Definition 1.1.

We are now in a position of defining a topological degree for multivalued A-proper maps.

2. – Definitions of a topological degree for A-proper multivalued maps.

Let (X, Y) be a pair of separable Banach spaces. The triple $\{X_n, Y_n, Q_n\}$ is called an (oriented) approximation scheme of (X, Y) if $\{X_n\}$, $X_n \subset X$ and $\{Y_n\}$, $Y_n \subset Y$ are two monotonically increasing sequences of oriented finite dimensional subspaces with $\dim X_n = \dim Y_n$, $\overline{\bigcup_n X_n} = X$, $\overline{\bigcup_n Y_n} = Y$ and $\{Q_n\}$ is a sequence of linear projections with $Q_n Y = Y_n$ for all n, such that $Q_n y \to y$ as $n \to +\infty$.

This definition slightly differs from the one given in [2]. It is obvious that (X, Y) has an approximation scheme if X and Y have Schauder bases.

The notion of A-proper (singlevalued) maps arises naturally if one tries to characterize the class of maps $T: X \to Y$ for which a solution of the equation $T(x) = y$ for a given $y \in Y$ can be obtained as a strong limit of solutions of the equation

$$T_n(x_n) = Q_n y\,, \qquad \text{where } T_n = (Q_n \circ T)/X_n$$

In solving the same problem in the context of multivalued maps then one is led to the concept of A-proper multivalued maps.

DEFINITION 2.1. – Let (X, Y) be a pair of separable Banach spaces with a given approximation scheme $\{X_n, Y_n, Q_n\}$. Let D be an open bounded subset of X, and $T: \bar{D} \multimap Y$ be a convex-valued map. Then T is said to be A-proper with respect to $\{X_n, Y_n, Q_n\}$ if

(i) $T_n: \bar{D}_n \multimap Y_n$ is an u.s.c. map with compact values for all n, where $D_n = D \cap X_n$,

(ii) for any sequence $\{x_n\}$, $x_n \in D_n$, $n \in N$, the existence of a sequence $\{z_n\}$, $z_n \in T_n(x_n)$ such that $\|z_n - Q_n y\| \to 0$ for some $y \in Y$ implies that there exists a cluster point x of $\{x_n\}$ such that $y \in T(x)$.

The notion of single-valued A-proper maps was introduced by PETRYSHYN in [11].

LEMMA 2.1. – *Let $T: \bar{D} \multimap Y$ be A-proper and let p be a point such that $p \notin T(\partial D)$. Then $Q_n p \notin T_n(\partial D_n)$ for large enough n.*

DEFINITION 2.2. – Let $T: \bar{D} \multimap Y$ be A-proper and p be a point such that $p \notin T(\partial D)$. We define $\mathrm{Deg}(T, D, p)$ the *degree of T on D over p* with respect to a given scheme, as follows. Let $\hat{Z}$ be the set of all integers together with $\{+\infty\}$ and $\{-\infty\}$. Then $\mathrm{Deg}(T, D, p)$ is the subset of $\hat{Z}$ given by:

$\mathrm{Deg}(T, D, p) = \{\gamma \in \hat{Z}/$there exists an infinite sequence $\{n_j\}$ of positive integers with $n_j \to \infty$ such that $\deg(T_{n_j}, D_{n_j}, Q_{n_j} p) \to \gamma\}$, where $\deg(T_{n_j}, D_{n_j}, Q_{n_j} p)$ is the degree defined by (1.1).

REMARK 2.1. – Notice that this degree is well defined in view of Lemma 2.1. Moreover $\mathrm{Deg}(T, D, p) \neq \emptyset$; since $\hat{Z}$ is compact. In particular if γ is a (finite) integer, then $\deg(T_{n_j}, D_{n_j}, Q_{n_j} p) \to \gamma$ iff $\deg(T_{n_j}, D_{n_j}, Q_{n_j}) = \gamma$ for all but a finite number of n_j.

The following theorem gathers the main properties of the degree for A-proper multivalued maps.

THEOREM 2.1. – *Let D be an open bounded subset of X and let $T: \bar{D} \multimap Y$ be A-proper with respect to a given approximation scheme, with $\|Q_n\| \leqslant M$ for all n. Let $p \notin T(\partial D)$. Then the following hold.*

(i) *If* $\operatorname{Deg}(T, D, p) \neq \{0\}$, *then there exists an element $x \in D$ such that $p \in T(x)$ (solution property).*

(ii) *Let $H: \bar{D} \times [0,1] \multimap Y$ be a multivalued map such that $H_t = H(\cdot, t)$ is a family of A-proper maps which depend continuously on t ($H_{t_n} \to H_t$ in the sense of Definition* 1.1, *when $t_n \to t$). If $p \notin H(\partial D \times [0,1])$, then* $\operatorname{Deg}(H_t, D, p)$ *is independent of t in* $[0,1]$ *(homotopy invariance).*

(iii) *Let $D = D_1 \cup D_2$, $D_1 \cap D_2 = \emptyset$. Put $D' = \partial D_1 \cup \partial D_2$ and suppose that $p \notin T(D')$. Then*

$$\operatorname{Deg}(T, D, p) \subseteq \operatorname{Deg}(T, D_1, p) + \operatorname{Deg}(T, D_2, p)\,,$$

with equality holding if either $\operatorname{Deg}(T, D_1, p)$ *or* $\operatorname{Deg}(T, D_2, p)$ *is a singleton (where for two subsets G_1, G_2 of $\hat{\boldsymbol{Z}}$ we set $G_1 + G_2 = \{\gamma | \gamma = \gamma_1 + \gamma_2\}$ with $\gamma_1 \in G_1$ and $\gamma_2 \in G_2$ and apply the convention that $+\infty + (-\infty) = \gamma$ for every $\gamma \in \hat{\boldsymbol{Z}}$).*

(iv) *If D is symmetric and T is a map, odd on ∂D, then* $\operatorname{Deg}(T, D, 0)$ *is odd (i.e., $2m \notin$* $\operatorname{Deg}(T, D, 0)$ *for any integer m). In particular* $\{0\} \neq \operatorname{Deg}(T, D, 0)$ *so that the equation $0 \in T(x)$ has a solution in D.*

(v) *Let $T, S: \bar{D} \multimap Y$ be A-proper maps, such that $T(x) = S(x)$ for all $x \in \partial D$. Let $p \in Y$ be a point such that $p \notin T(\partial D) = S(\partial D)$. Then* $\operatorname{Deg}(T, D, p) = \operatorname{Deg}(S, D, p)$.

A uniqueness theorem for the generalized degree.

In this section we would like to point out that if Φ is a compact vector field with closed, convex values, i.e., $\Phi = I - \Gamma$, where Γ is an u.s.c. closed-convex valued map and $\overline{R(\Gamma)}$ is compact, then the following theorem holds:

THEOREM 2.2. – *Let $\{X_n, Q_n\}$, with $\|Q_n\| \leqslant M$ for all n, be an approximation scheme of X and let D be an open bounded subset of X. If $\Phi: \bar{D} \multimap X$ is as above, then Φ is an A-proper map. Further if $p \notin \Phi(\partial D)$, then* $\operatorname{Deg}(\Phi, D, p)$ *is the singleton* $\{\deg_{\mathrm{C.L.}}(\Phi, D, p)\}$ *where* $\deg_{\mathrm{C.L.}}(\Phi, D, p)$ *is the Cellina-Lasota topological degree* [3].

REMARK 2.2. – Theorem 2.2 above can be extended in several ways.

(i) Let $D \subset X$ be open and bounded and let $T: \bar{D} \multimap X$ be an u.s.c. condensing map with closed and convex values.

Then $I - T$ is A-proper with respect to any approximation scheme $\{X_n, Q_n\}$, $\|Q_n\| = 1$ for all $n \in N$. In this case if $p \notin (I - T)(\partial D)$, then $\mathrm{Deg}(I - T, D, p)$ is defined and coincides with the singleton $\{\deg_w(I - T, D, p)\}$, where $\deg_w(I - T, D, p)$ is the topological degree for u.s.c. maps with closed and convex values as defined by WEBB in [13].

(ii) Let $\{X_n, Y_n, Q_n\}$, $\|Q_n\| \leqslant M$, $n \in N$, be an approximation scheme for (X, Y) such that if F is any given finite-dimensional subspace of Y, then there exists an integer $n_F \geqslant 1$ such that Q_n is injective on F for each $n \geqslant n_F$, and let $H: \bar{D} \to Y$ be an A-proper homeomorphism from an open bounded subset $D \subset X$ onto an open subset $H(D) \subset Y$. Assume the following

1) H maps $\bar{D}$ homeomorphically onto $\overline{H(D)}$;

2) for each n the map H_n is an orientation preserving homeomorphism of D_n onto $H_n(D_n)$ and maps $\bar{D}_n$ homeomorphically onto $\overline{H_n(D_n)}$;

3) there exists a continuous function $\alpha(r)$ from R^+ into itself such that $r_i \to 0$ whenever $\alpha(r_i) \to 0$ as $i \to +\infty$ for which

$$\|H_n x - H_n y\| \geqslant \alpha(\|x - y\|)\,, \qquad \forall x, y \in D\,.$$

Then if $\Gamma: \bar{D} \multimap Y$ is u.s.c. compact with closed and convex values the map $T = H - \Gamma$ is A-proper. Moreover, if $p \notin T(\partial D)$ then $\mathrm{Deg}(T, D, p) = \{\deg_{\mathrm{C.L.}}(I - \Gamma H^{-1}, H(D), p)\}$.

Let us add in passing that (i) extends Webb's result [12, Theorem 1.2] to the multivalued case and (ii) represents a multivalued version of Theorem 2 given in [2].

3. - P-compact multi-valued maps.

In this part we apply our degree for multivalued A-proper maps to projectionally-compact (P-compact) maps introduced in [10].

An approximation scheme $\{X_n, P_n\}$ of X, with $\|P_n\| \leqslant M$, for all n, is called a *projective approximation scheme* [2].

DEFINITION 3.1. - Let D be an open subset of X. The map $T: \bar{D} \multimap X$ is said to be *P-compact* iff the map $T + \lambda I$ is A-proper with respect to the projectional scheme for each $\lambda \geqslant 0$.

Let D be a bounded subset of X with $0 \in D$.

THEOREM 3.1. – *Let* $T: \bar{D} \multimap X$ *be a bounded P-compact map. If* $-\lambda x \notin T(x)$ $\forall x \in \partial D$ *and* $\lambda \geqslant 0$, *then* $\mathrm{Deg}(T, D, 0) = \{1\}$ *and in particular there exists* $x_0 \in D$ *such that* $0 \in T(x_0)$.

Theorem 3.1 can be stated in terms of the existence of a fixed point for a multivalued map. More precisely the following result holds true.

THEOREM 3.1′. – *Let* $I - S: \bar{D} \multimap X$ *be a bounded P-compact map. If* $\lambda x \notin S(x)$ $\forall x \in \partial D$ *and* $\lambda \geqslant 1$, *then* $\mathrm{Deg}(I - S, D, 0) = \{1\}$ *and in particular* S *has a fixed-point in* D, *i.e., there exists* $x_0 \in D$ *such that* $x_0 \in S(x_0)$.

Theorem 3.1 and Theorem 3.1′ represent extensions of Theorem 5 and Corollary 4 of [2] respectively.

4. – *P*-compact maps in partially ordered Banach spaces.

In what follows $\boldsymbol{R}$ will denote the real numers and X will stand for a Banach space over $\boldsymbol{R}$ with norm $\|\cdot\|$. A subset K of X is called a *cone* in X provided it satisfies the following properties

(i) $x + ty \in K$, whenever $x, y \in K$ and $t \geqslant 0$;

(ii) $K \cap \{x \in K: -x \in X\} = \{0\}$;

(iii) K is closed and $K \neq \{0\}$.

We may introduce a partial ordering in X which is compatible with the vector space structure of X by writing $x \leqslant y$ iff $y - x \in K$. If X is partially ordered by a cone K then X is called a partially ordered Banach spaces (O.B.S.). An O.B.S. will be denoted by the symbol (X, K).

In [9] Krasnosel'skij gives the following theorem.

Let (X, K) *be an* O.B.S. *and let* A *be a compact map of* K *into itself such that* $A(0) = 0$. *Suppose there are numbers* r *and* R *with* $0 < r < R$ *such that for all* $\varepsilon > 0$ *we have*

$$(a) \qquad \begin{cases} A(x) - (1+\varepsilon)x \notin K & \textit{whenever } x \in K \textit{ and } 0 < \|x\| \leqslant r, \textit{ and} \\ x - A(x) \notin K & \textit{whenever } x \in K \textit{ and } \|x\| > R \end{cases}$$

Then A *must have a non zero fixed-point in* K.

In [8] Hamilton gives a version of this theorem replacing the condition that A is compact by the condition that $T = I - A$ is P-compact and condition (a) above by more general ones.

We will extend Hamilton's theorem to multivalued A-proper maps. Our proof will be based upon the degree developed in the previous sections.

Let (X, K) be an O.B.S. and let $\{X_n, P_n\}$ be a projective approximation scheme of (X, K), i.e.

(i) $\{X_n, P_n\}$ is a projective approximation scheme of X;

(ii) P_n maps K into $K_n = K \cap X_n$ for all n.

THEOREM 4.1. – *Let* (X, K) *be an* O.B.S. *and* $\{X_n, P_n\}$ *be a projective approximation scheme of* (X, K). *Let* $r_1 \neq r_2 \in (0, +\infty)$ *and* $r = \max\{r_1, r_2\}$. *If* $T\colon \bar{B}(0, r) \multimap X$ *is a P-compact map such that* $(I - T)(\bar{B}(0, \Gamma)) \subset K$ *and*

(i) *there is some non zero* $p \in K$ *such that*

$$\lambda p \notin T(x) \qquad \forall x \in K, \ \textit{with } \|x\| = r_1, \ \textit{and } \forall \lambda \geq 0,$$

(ii) $-\lambda x \notin T(x)$ $\forall x \in K$, *with* $\|x\| = r_2$, *and* $\forall \lambda \geq 0$,

then there is an $x_0 \in K$, *with* $\min\{r_1, r_2\} < \|x_0\| < \max\{r_1, r_2\}$, *such that* $0 \in T(x_0)$.

We would like to point out that conditions (i)-(ii) have been used in [7] to prove a fixed-point theorem for multivalued condensing maps sending a cone into itself. The following conditions used in [5]

(i′) $A(x) - x \notin K$ if $x \in K$ with $\|x\| = r_1$,

(ii′) $x - A(x) \notin K$ if $x \in K$ with $\|x\| = r_2$,

are less general than conditions (i)-(ii), where A is an α-contraction (k-set-contraction) sending the whole space X into K.

5. – Proofs.

PROOF OF LEMMA 2.1. – Assume the contrary, i.e., assume that there exists a sequence of integers $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$, such that $Q_{n_k} p \in T_{n_k}(x_{n_k})$, $x_{n_k} \in \partial D_{n_k}$.

Since $Q_{n_k} p \to p$ and T is an A-proper map, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_k}\}$ which converges to some $x \in \partial D$ and $p \in T(x)$. Therefore $p \in T(\partial D)$ contradicting the fact that $p \notin T(\partial D)$.

PROOF OF THEOREM 2.1. – (i) (*Solution property*). If

$$\mathrm{Deg}(T, D, p) \neq \{0\}$$

then there exists a sequence $\{n_j\}$ with $n_j \to \infty$ such that $\deg(T_{n_j}, D_{n_j}, Q_{n_j}p) \neq 0$, for all n_j. Therefore there exists $x_{n_j} \in D_{n_j}$ such that $Q_{n_j}p \in T_{n_j}(x_{n_j})$ for all n_j.

Since T is A-proper and $Q_{n_j}p \to p$ there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ such that $x_{n_{j_k}} \to x \in D$ and $p \in T(x)$. Notice that $x \notin \partial D$ since $p \notin T(\partial D)$.

(ii) (*Homotopy invariance*). Clearly $\mathrm{Deg}(H_t, D, p)$ is well defined for any $t \in [0, 1]$. In order to prove (ii) it sufficies to show that there exists $n_0 \in N$ such that for any $n \geqslant n_0$ $Q_n p \notin Q_n H_t(\partial D_n)$ for all $t \in [0, 1]$. Indeed, by the homotopy invariance property of the degree defined by (1.1) $\deg(Q_n H_t, D_n, Q_n p)$ is independent on $t \in [0, 1]$ for all $n \geqslant n_0$ and we are done. Assume the contrary, then there exist $\{n_j\}$, $\{x_{n_j}\}$ and $\{t_j\} \subset [0, 1]$ with $n_j \to \infty$ as $j \to \infty$, $t_j \to t_0 \in [0, 1]$, $x_{n_j} \in \partial D_{n_j}$ and $Q_{n_j}p \in Q_{n_j}H_{t_j}(x_{n_j})$.

Since $t_j \to t_0$, by the continuity of H on t_0 we have

$$d^*(G_{H_{t_j}}, G_{H_{t_0}}) \to 0 \qquad \text{as } j \to \infty .$$

As consequence, since $\|Q_n\| \leqslant M$ for all n, given $\varepsilon > 0$ there exists $j_0(\varepsilon)$ such that

$$d^*(G_{Q_{n_j}H_{t_j}}, G_{Q_{n_j}H_{t_0}}) < \varepsilon \qquad \text{for all } j > j_0(\varepsilon)$$

This implies that there exist $\bar{x}_{n_j} \in \bar{D}$ and $z_{n_j} \in Q_{n_j}H_{t_0}(\bar{x}_{n_j})$ such that $\|x_{n_j} - \bar{x}_{n_j}\| < \varepsilon$ and $\|Q_{n_j}p - z_{n_j}\| < \varepsilon$ for all $j \geqslant j_0(\varepsilon)$. Since $Q_{n_j}p \to p$ by the above inequality we have that $z_{n_j} \to p$. Then by the A-properness of H_{t_0} there exists a subsequence $\{\bar{x}_{n_{j_k}}\}$ of $\{\bar{x}_{n_j}\}$ converging to some x such that $p \in H_{t_0}(x)$. Since $\|x_{n_j} - \bar{x}_{n_j}\| < \varepsilon$ and $\{x_{n_j}\} \subset \partial D$, we get $x \in \partial D$, contradicting the fact that $p \notin H(\partial D \times [0, 1])$.

The proofs of (iii) and (iv) are quite similar to those of [2] and therefore we omit them.

(v) — $\deg(T_n, D_n, Q_n p)$ and $\deg(S_n, D_n, Q_n p)$ are defined for n sufficiently large, say for $n \geqslant \bar{n}$. The homotopy $H_n(x, t) = (1 - t) \cdot T_n(x) + tS_n(x)$ is admissible in the sense of [3] for all n, since $T_n(\partial D_n) = S_n(\partial D_n)$. Therefore

$$\deg(T_n, D_n, Q_n p) = \deg(S_n, D_n, Q_n p)$$

for $n \geqslant \bar{n}$. Hence $\mathrm{Deg}(T, D, p) = \mathrm{Deg}(S, D, p)$.

PROOF OF THEOREM 2.2. — Let $\{x_n\}$ be a sequence such that $x_n \in D_n$ for all n. Let $z_n = x_n - Q_n y_n$, with $y_n \in \Gamma(x_n)$, be such that

$z_n \to z \in X$. Since $\Gamma(\{x_n\})$ is a relatively compact of X, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to some $y \in X$. Therefore the sequence $\{x_{n_k}\}$, with $x_{n_k} = z_{n_k} + Q_{n_k} y_{n_k}$ converges to $x = z + y$. Since Γ is a closed map, we get that $y \in \Gamma(x)$, hence $z = x - y \in \in x - \Gamma(x) = \Phi(x)$, i.e., Φ is an A-proper.

To prove that $\operatorname{Deg}(\Phi, D, p)$ is the singleton $\{\deg_{\text{C.L.}}(\Phi, D, p)\}$ we need the following Lemma.

LEMMA. – *Let $\Gamma\colon X \multimap X$ be an u.s.c. closed-convex valued map with $\overline{R(\Gamma)}$ compact. There exist an $\varepsilon_0 > 0$ and $n_0 \in N$ such that if $f\colon \overline{D} \to X$ is a compact map with $R(f) \subset \overline{\text{co}}\, R(\Gamma)$ and $d^*(G_f, G_\Gamma) < \varepsilon_0$ then*

$$(a) \qquad \deg_{\text{L.S.}}(\varphi, D, p) = \deg_{\text{C.L.}}(\Phi, D, p)$$

where $\varphi = I - f$ and $\Phi = I - \Gamma$.

$$(b) \qquad \deg(\varphi_n, D_n, Q_n p) = \deg(\Phi_n, D_n, Q_n p)$$

where $\varphi_n = Q_n \circ \varphi/\overline{D}_n$ and $\Phi_n = Q_n \circ \Phi/\overline{D}_n$ for all $n \geqslant n_0$.

PROOF OF THE LEMMA. – (a) is implicity contained in [3]. (b) Assume the contrary, i.e., assume that there exist sequences $\{f^{(k)}\}$, $\{\varepsilon_k\}$, $\{n_k\}$, with $\varepsilon_k \to 0$, $n_k \to \infty$ and $d^*(G_f(k), G_\Gamma) < \varepsilon_k$ such that (*) $\deg(\varphi^{(k)}_{n_k}, D_{n_k}, Q_{n_k} p) \neq \deg(\Phi_{n_k}, D_{n_k}, Q_{n_k} p)$.

Let $H^{(k)}(t, x) = x - [(1-t) f^{(k)}(x) + t\Gamma(x)]$ for each $t \in [0, 1]$ and $x \in \overline{D}$. For each k, $H^{(k)}(t, \cdot)$ is a family of A-proper maps which depend continuously on t, therefore by our assumption (*) there exist $\{t_k\}$, $t_k \in [0, 1]$ and $\{x_{n_k}\}$, $x_{n_k} \in \partial D_{n_k}$ such that

$$Q_{n_k} p \in Q_{n_k} H^{(k)}(t_k, x_{n_k})$$

i.e.,

$$Q_{n_k} p = x_{n_k} - [(1-t_k) f^{(k)}_{n_k}(x_{n_k}) + t_k y_{n_k}] \quad \text{with } y_{n_k} \in \Gamma_{n_k}(x_{n_k}) .$$

Clearly $y_{n_k} = Q_{n_k} u_{n_k}$ with $u_{n_k} \in \Gamma(x_{n_k})$. In virtue of the compactness of $\overline{\text{co}}\, R(\Gamma)$, taking subsequences if necessary, we can assume $t_k \to t_0$, $u_{n_k} \to u$, $f^{(k)}(x_{n_k}) \to v$. Therefore $y_{n_k} = Q_{n_k} u_{n_k} \to u$ and $f^{(k)}_{n_k}(x_{n_k}) \to v$, since $\|Q_n\| \leqslant M$ for all n. Hence

$$x_{n_k} \to p + (1-t_0) v + t_0 u = x_0, \qquad \text{where } x_0 \in \partial D \text{ since } \{x_{n_k}\} \subset \partial D .$$

By the closedness of G_Γ, it follows that $u \in \Gamma(x_0)$. Moreover, we have

$$d^*(G_{f^{(k)}_{n_k}}, G_{\Gamma_{n_k}}) < \varepsilon_k M$$

and this means that there exist two sequences $\{x'_{n_k}\}$ and $\{y'_{n_k}\}$ with $y'_{n_k} \in \Gamma_{n_k}(x'_{n_k})$ such that

$$\|x_{n_k} - x'_{n_k}\| < \varepsilon_k M \qquad \text{and} \qquad \|f^{(k)}_{n_k}(x_{n_k}) - y'_{n_k}\| < \varepsilon_k M$$

that is $x'_{n_k} \to x_0$, $y'_{n_k} \to v$. In virtue of the closedness of G_Γ it follows that $v \in \Gamma(x_0)$. Finally, we get

$$p = x_0 - [(1-t_0)v + t_0 u] \in x_0 - \Gamma(x_0) = \Phi(x_0)$$

since $\Gamma(x_0)$ is a convex set. Contradicting the fact that $p \notin \Phi(\partial D)$.

Now we are in a position of proving Theorem 2.2. In fact, by Cellina's approximation theorem [4], there exists a compact map $f\colon \bar{D} \to X$ with $R(f) \subset \overline{\mathrm{co}}\, R(\Gamma)$ and $d^*(G_f, G_\Gamma) < \varepsilon_0$, where $\varepsilon_0 > 0$ is that of the Lemma. Theorefore, by the Lemma we get

$$\deg_{\mathrm{C.L.}}(\Phi, D, p) = \deg_{\mathrm{L.S.}}(\varphi, D, p)\,.$$

On the other hand, since $\varphi = I - f$ is a compact single-valued vector field there exists an $n_1 \in N$ such that

$$\deg_{\mathrm{L.S.}}(\varphi, D, p) = \deg(\varphi_n, D_n, Q_n p)$$

for all $n \geq n_1$ (see Theorem 2 of [2]).

Therefore, for all $n \geq \max\{n_0, n_1\}$, where $n_0 \in N$ is that of the Lemma, we have

$$\deg_{\mathrm{L.S.}}(\varphi, D, p) = \deg(\varphi_n, D_n, Q_n p) = \deg(\Phi_n, D_n, Q_n p)\,.$$

Hence

$$\mathrm{Deg}(\Phi, D, p) \text{ is the singleton } \{\deg_{\mathrm{C.L.}}(\Phi; D, p)\}\,.$$

Proof of Theorem 3.1. – Let us define on $D \times [0,1]$ the mapping $H_t(x) = (1-t)\,T(x) + tx$ we want to prove that this mapping is an admissible homotopy in the sense of (ii) of Theorem 2.1. we show first that

$$d^*(G_{H_t}, G_{H_s}) \to 0 \qquad \text{as } t \to s\,.$$

i.e. for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|t - s| < \delta(\varepsilon)$ implies $d^*(G_{H_t}, G_{H_s}) < \varepsilon$. Let $\varepsilon > 0$, $x \in \bar{D}$ and $z \in H_t(x)$ be fixed. Clearly z can be written as $z = (1-t)v + tx$, where $v \in T(x)$. Let

$z' = (1 - s)v + sx$. We have

$$\|z - z'\| = \|v - x\|\,|s - t|\,v \in T(x)$$

From the boundedness of T we get

$$\|z - z'\| \leqslant c|s - t|\,,$$

where c is a constant independent on $x \in \bar{D}$. Thus by setting $\delta(\varepsilon) = \varepsilon/c$, we have $\|z - z'\| < \varepsilon$, or equivalently

$$d^*(G_{H_t}, G_{H_s}) < \varepsilon \qquad \text{for } |t - s| < \delta(\varepsilon)\,.$$

Now, let us show that H_t is A-proper for any $t \in [0, 1]$. Indeed, $H_t = (1 - t)[T + (t/1 - t)I]$ for $t \in [0, 1)$ is A-proper since T is P-compact and $H_1 = I$ is A-proper.

Finally, let us prove that $0 \notin H_t(x)$ on $\partial D \times [0, 1]$. Assume this is not the case i.e. there exist $(x, t) \in \partial D \times [0, 1]$ such that

$$0 \in (1 - t)\,T(x) + tx\,.$$

If $t \in [0, 1)$, then $-(t/1 - t)x \in T(x)$ contradicting our assumption. If $t = 1$ we get $0 \in \partial D$ contradicting the fact that D is open and $0 \in D$. Therefore H_t is an admissible homotopy and $\mathrm{Deg}(T, D, 0) = \mathrm{Deg}(I, D, 0) = \{1\}$.

PROOF OF THEOREM 4.1. – We will first show that $\mathrm{Deg}(T, B_{r_1}, 0) = \{0\}$ and $\mathrm{Deg}(T, B_{r_2}, 0) \neq \{0\}$ where $B_{r_i} = B(0, r_i)$, $i = 1, 2$. Let ∂B_{r_i} denotes the boundary of B_{r_i}, $i = 1, 2$.

Let p the non zero vector of K for which condition (i) is satisfied. If $\lambda p \in T(x)$ for some $x \in \partial B_{r_1}$ and $\lambda \geqslant 0$, then $x \in K$, since $x - \lambda p \in (I - T)(x) \subset K$, contradicting (i). Hence

$$(*) \qquad \lambda p \notin T(x)\,, \qquad \forall x \in \partial B_{r_1} \text{ and } \lambda \geqslant 0\,.$$

In particular $0 \notin T(\partial B_{r_1})$ and consequentiy $\mathrm{Deg}(T, B_{r_1}, 0)$ is defined. Moreover, since $p \in K$ then $-p \notin K$, there is a $\varrho_0 \in \boldsymbol{R}$ such that $-p + y \notin K$ for all $y \in X$ with $\|y\| < \varrho_0$. Let n be an integer such that $r_1/\varrho_0 < n$. Define the map $H: \bar{B}_{r_1} \times [0, 1] \to X$ by $H(x, t) = T(x) - tnp$ and assume that $0 \in H(x, t)$ for some $x \in \partial B_{r_1}$ and some $t \in [0, 1]$. Then $tnp \in T(x)$ which is impossible by (*). It follows that H is an admissibie A-proper homotopy, and so $\mathrm{Deg}(T, B_{r_1}, 0) = \mathrm{Deg}(T - np, B_{r_1}, 0)$; If $0 \in T(x_0) - np$ for some $x_0 \in B_{r_1}$ then

$x_0 - np \in (I - T)(x_0) \subset K$, and from this we see that $x_0/n - p \in K$ and $\|x_0/n\| < r_1/n$. This contradicts our choice of n and we must have that $np \notin T(\bar{B}_{r_1})$. Hence

$$\{0\} = \mathrm{Deg}(T - np, B_{r_1}, 0) = \mathrm{Deg}(T, B_{r_1}, 0) .$$

If $0 \in T(x)$ for some $x \in \partial B_{r_2}$, then $x \in (I - T)(x) \subset K$ which is impossible by (ii), consequently $\mathrm{Deg}(T, B_{r_2}, 0)$ is defined. Consider now the homotopy $H: \bar{B}_{r_2} \times [0, 1] \to X$ defined by $H(x, t) = tx + (1 - t) T(x)$. We want to show that $0 \notin tx + (1 - t) T(x)$ for all $x \in B_{r_2}$ and $t \in [0, 1]$.

This is obvious for $t = 1$. If $0 \in tx + (1 - t) T(x)$ for some $x \in B_{r_2}$ and some $t \in [0, 1)$ then $-(t/1 - t) x \in T(x)$ for $t/1 - t \geqslant 0$, which implies $x \in K$ contradicting (ii). Thus H is an admissible A-proper homotopy and

$$\{1\} = \mathrm{Deg}(I, B_{r_2}, 0) = \mathrm{Deg}(T, B_{r_2}, 0) .$$

From the additivity of the degree over domains it follows that $\mathrm{Deg}(T, B_r \setminus \bar{B}_{\tilde{r}}, 0) \neq \{0\}$, where

$$r = \max\{r_1, r_2\} \quad \text{and} \quad \tilde{r} = \min\{r_1, r_2\}$$

and so there exists an $x_0 \in K$, $\tilde{r} < \|x_0\| < r$ such that $0 \in T(x_0)$.

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