A DIRECT POLYNOMIAL APPROACH TO LQ REGULATION

Edoardo Mosca and Paolo Nistri

Dipartimento di Sistemi e Informatica
Università di Firenze
Via S. Marta 3 - 50139 Firenze, Italy
Tel. +55-4796258 - Telex 580681 UNFING I - Fax +55-4796363

ABSTRACT

The standard LQ output regulation problem is directly approached and solved via spectral factorization and a pair of bilateral Diophantine equations. Stabilizability and/or detectability requirements in the Riccati context are expressed in terms of stability of greatest common left and right divisors of polynomial matrices.

INTRODUCTION

This paper deals with the classic Linear-Quadratic Regulation (LQR) problem. A linear discrete-time, state-space representation for the time-invariant plant to be output regulated is given

$$\begin{cases} x(t+1) = \Phi x(t) + G\,u(t) \\ y(t) = H\,x(t) \end{cases} \tag{1}$$

with $x(0) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^m$.
The problem is to find an input sequence, if it exists, minimizing the quadratic cost

$$J = \sum_{k=0}^{\infty} \left(\|y(k)\|^2_{\psi_y} + \|u(k)\|^2_{\psi_u} \right) \tag{2}$$

for any initial state $x(0)$. In (2) $\psi_y = \psi'_y > 0$ and $\psi_u = \psi'_u > 0$.
As well known, under stabilizability and detectability assumptions on the triplet (Φ,G,H), problem (1)-(2) can be solved in state-feedback form by using the unique nonnegative definite solution of the relevant algebraic Riccati equation. Moreover, the resulting closed-loop system turns out to be asymptotically-stable.
The aim of this paper is to provide a direct matrix-fraction approach to the problem. In this way, the solution is obtained by first solving a spectral factorization problem and next finding the minimum-degree solution with respect to a "dummy" polynomial matrix of a pair of bilateral Diophantine equations. This problem was previously addressed in [1] for the special case of a scalar input and a completely reachable pair (Φ,G). Thanks to these limitative assumptions, in [1] the solution is given in terms of a single Diophantine equation. It will be shown that, in general, two bilateral Diophantine equations must be simultaneously solved as referred above.
One of the reasons for considering a polynomial solution for the standard deterministic LQR problem is to show that Riccati-based and polynomial methods are fully conceptually equivalent, as far as steady-state (semi-infinite horizon) results are concerned. In particular, stabilizability and/or detectability requirements in the Riccati context are expressed in terms of stability of greatest common left and right divisors of polynomial matrices.
The reader is referred to [2], whose notations are adopted hereafter as much as possible, for the basic facts on the theory of bilateral Diophantine equations and its use in the context of LQ stochastic regulation.

MAIN RESULTS

As well known, problem (1)-(2) only depends on a completely observable subsystem of (1) obtainable via Kalman's canonical decomposition. Thus, we assume from the outset that

[A.1] (Φ,H) is a completely observable pair.

The output sequence response $y(d)$ of (1) due to $x(0)$ and the input sequence $u(d)$ is given by

$$y(d) = H\,A^{-1}(d)\,[x(0) + B(d)u(d)] \tag{3}$$

where the indeterminate d stands, as in [2], for the backward shift z^{-1}; and $A(d)$, $B(d)$ are the following polynomial matrices

$$A(d) := I - d\Phi \tag{4}$$

$$B(d) := dG \tag{5}$$

Hereafter, by the sake of simplicity, the argument d will be omitted unless required to avoid possible confusion. The quadratic cost J can be conveniently rewritten as

$$J = \langle y^* \psi_y y + u^* \psi_u u \rangle \tag{6}$$

where $y^*(d) := y'(d^{-1})$ and the prime denotes transpose. Let us also rewrite (3)

$$y(d) = H\,A^{-1}(d)x(0) + H\,B_2(d)A_2^{-1}(d)u(d) \tag{7}$$

where $B_2A_2^{-1}$ is a right coprime (rc) matrix fraction description for $A^{-1}B$. Substituting (7) into (6), one finds

$$\begin{aligned} J = \langle\, & u^* A_2^{-*}(A_2^* \psi_u A_2 + B_2^* H' \psi_y H B_2) A_2^{-1} u + \\ & u^* A_2^{-*} B_2^* H' \psi_y H\,A^{-1} x(0) + x'(0) A^{-*} H' \psi_y H\,B_2\,A_2^{-1} u + \\ & x'(0) A^{-*} H' \psi_y H\,A^{-1} x(0) \,\rangle \end{aligned} \tag{8}$$

where $A_2^{-*} := (A_2^{-1})^*$.
Let $E(d)$ be the Hurwitz polynomial matrix solving the following spectral factorization problem

$$E^*E = A_2^*\psi_u A_2 + B_2^* H'\psi_y H\, B_2 \tag{9}$$

Using (9) into (8), one finds

$$J = J_1 + J_2 \tag{10}$$

with

$$J_1 := \langle L^*L \rangle$$

$$J_2 := \langle x'(0)A^{-*}(I - H'\psi_y HB_2E^{-1}E^{-*}B_2^*)H'\psi_y HA^{-1}x(0) \rangle$$

$$L := E^{-*}B_2^*H'\psi_y HA^{-1}x(0) + E\,A_2^{-1}u$$

Note that J_2 does not depend on u. Let

$$p := \max(\partial E, \partial B_2, \partial A_2) \tag{11}$$

where ∂E denotes the degree of $E(d)$. Next define:

$$\bar{E} := d^p E^*; \quad \bar{B}_2 := d^p B_2^*; \quad \bar{A}_2 := d^p A_2^* \tag{12}$$

Consequently,

$$L = \bar{E}^{-1}\bar{B}_2 H'\psi_y H A^{-1} x(0) + EA_2^{-1}u \tag{13}$$

In order to simplify the above expression, let us consider the following bilateral Diophantine equation

$$\bar{E}Y + ZA = \bar{B}_2 H'\psi_y H \tag{14}$$

It is temporarily assumed that a solution (Y,Z) of (14) exists with $\partial Z < \partial\bar{E}$. Under this assumption, (13) becomes

$$L = Yx + (E - YB_2)A_2^{-1}u + \bar{E}^{-1}Z\, x(0) \;.$$

Since $\partial Z < \partial \bar{E}$, $\bar{E}^{-1}Z\,x(0)$ is strictly anticipative. Consequently,

$$J_1 = J_3 + J_4 \tag{15}$$

where

$$J_3 := \langle x'(0)Z^*\bar{E}^{-1}\bar{E}^{-*}Z\,x(0) \rangle$$

does not depend on u, and

$$J_4 := \langle (Yx + Xu)^*(Yx + Xu) \rangle \;.$$

In the above equation,

$$X := (E - YB_2)A_2^{-1}$$

which yields

$$XA_2 + YB_2 = E \tag{16}$$

Premultiplying both sides of (16) by $\bar{E}$ and taking into account (14), one gets

$$\bar{E}X - ZB = \bar{A}_2\psi_u \tag{17}$$

The following lemma sums up the above results.

Lemma 1
Provided that:

i) equations (14) and (17) [or (14) and (16)] have a solution (X,Y,Z) with $\partial Z < \partial \bar{E}$; and

ii) $J_2 + J_3$ is bounded;

the LQ output regulation problem is solved by

$$u(d) = -X^{-1}Y\,x(d) \tag{18}$$

with X and Y, constant matrices, specified in i), and, correspondingly,

$$J_{\min} = J_2 + J_3 \;.$$

Proof.
It remains to be shown that $\partial X = \partial Y = 0$ and that X is nonsingular. From $A^{-1}B = B_2A_2^{-1}$, or $dGA_2 = (I - d\Phi)B_2$, it follows that $\partial A_2 = \partial B_2$ and, consequently from (9),

$$\partial E = \partial A_2 = \partial B_2$$

Thus, (16) implies that $\partial X = \partial Y$, and, hence, $\partial X = \partial Y = 0$ that follows from (17).
Next, nonsingularity of X is implied by (16). □

A condition under which i) of Lemma 1 is fulfilled is given by the next lemma whose proof closely follows similar proofs in [2].

Lemma 2
Let the greatest common left divisors (GCLD) of A and B be stable. Thus, there is a unique solution (X,Y,Z) of (14) and (17) [or (14) and (16)] such that $\partial Z < \partial \bar{E}$.

Proof.
Let D be a GCLD of A and B. Then, there is a unimodular matrix U such that

$$[-B \quad A]\,U = [D \quad 0] \;.$$

Next rearrange (14) and (17) as follows

$$\bar{E}\,[X \quad Y] + Z\,[-B \quad A] = [\bar{A}_2\psi_u \quad \bar{B}_2 H'\psi_y H]$$

Postmultiplying the latter equation by U and setting $\underline{X} := XU$ and $\underline{Y} := YU$, one gets

$$\bar{E}\underline{X} + ZD = \bar{A}_2\psi_u U_{11} + \bar{B}_2 H'\psi_y H\, U_{21} \tag{19}$$

$$\bar{E}\underline{Y} = \bar{A}_2\psi_u U_{12} + \bar{B}_2 H'\psi_y H\, U_{22} \tag{20}$$

$$0 = -BU_{12} + AU_{22} = -BA_2 + AB_2$$

Last equation shows that there is a D_2 such that $U_{12} = A_2D_2$ and $U_{22} = B_2D_2$. Thus (20) reduces to

$$\underline{Y} = ED_2 \;. \tag{20'}$$

Further, (20) is solvable since E is Hurwitz and D stable and hence $\bar{E}$ and D do not share any nonunit invariant polynomial. Let one solution of (19) and (20') be $(\underline{X}_o, \underline{Y}_o, Z_o)$. The equations

$$\bar{E}\underline{X} + ZD = 0 \qquad \text{and} \qquad \bar{E}\underline{Y} = 0$$

have the general solution $\underline{X} = TD$, $\underline{Y} = 0$, $Z = -\bar{E}T$ with T any polynomial matrix of compatible dimensions. Correspondingly,

$$[X \quad Y]\,U = [TD \quad 0] = [-TB \quad TA]\,U$$

Therefore all solutions of (14) and (17) are:

$$X = X_o - TB \;; \quad Y = Y_o + TA \;; \quad Z = Z_o - \bar{E}T$$

Since $\bar{E}$ is proper [2], it follows that there is a unique solution (X,Y,Z) such that $\partial Z < \partial \bar{E}$.
In [3] it is shown that this solution coincides

with the one of (14) and (16). □

Boundedness of $J_2 + J_3$ is clearly guaranteed by the stability of the closed-loop system, that, in turn, if the plant has no unstable hidden modes, is fulfilled if and only if E is strictly Hurwitz. In fact, if the latter is true, the control law (18) can be rewritten as $E^{-1}Xu = -E^{-1}Yx$ with both $E^{-1}X$ and $E^{-1}Y$ stable sequence-matrices such that $E^{-1}XA_2 + E^{-1}YB_2 = I$.

Lemma 3

If [A.1] holds then the spectral factor E is strictly Hurwitz.

Proof.

We first observe that if HB_2 and A_2 are rc then $\det E(e^{j\omega}) \neq 0$, $\forall\ \omega \in [0, 2\pi)$, and we conclude that E is strictly Hurwitz. It thus sufficies to show that complete observability of (Φ, H) implies that HB_2 and A_2 are rc. In order to prove this, we begin by noting that by the PBH test [4], the complete observability of (Φ, H) is equivalent to H and A being rc. This, in turn, implies that H and A_1 are rc, if $A^{-1}B = A_1^{-1}B_1$ with A_1 and B_1 left coprime (lc). In fact, $A = \Delta A_1$ and $B = \Delta B_1$ with Δ a GCLD of A and B. Hence

$$I = XA + YH = (X\Delta)A_1 + YH .$$

We finally show that if H and A_1 are rc then HB_2 and A_2 are rc. In fact, consider the transfer functions

$$HA_1^{-1}B_1 = HB_2A_2^{-1} \quad \text{for which} \quad \partial\det A_1 = \partial\det A_2$$

For $i = 1,2$ we set

$$q_i = \max\{\partial A_i, \partial B_i\} \quad \text{and let}$$

$$\hat{A}_i(z) = d^{-q_i}A_i(d) , \qquad \hat{B}_i(z) = d^{-q_i}B_i(d)$$

where $z = d^{-1}$.

We can represent the above transfer functions in terms of $\hat{A}_i$ and $\hat{B}_i$, $i = 1,2$, that is

$$H\hat{A}_1^{-1}\hat{B}_1 = H\hat{B}_2\hat{A}_2^{-1} \quad \text{with} \quad \partial\det\hat{A}_1 = \partial\det\hat{A}_2$$

The expression on the LHS can be minimally realized in state-space form with a state of dimension equal to $\partial\det\hat{A}_1$ since H and A_1 are rc iff H and $\hat{A}_1$ are rc. Thus $H\hat{B}_2\hat{A}_2^{-1}$ can be also minimally realized with the same state dimension. We conclude that $H\hat{B}_2$ and $\hat{A}_2$ are rc and so also HB_2 and A_2 . □

The previous three lemmas show that the LQ output regulation problem can be solved provided that (Φ, H) is a completely observable pair and the GCLD's of A and B are stable. This is the same as assuming that the given plant (Φ, G, H), which in general need not be completely observable, has all its observable-unreachable eigenvalues stable. All the results are summed up in the following

Theorem

Consider the LQ output regulation problem (1)-(2) for the arbitrary plant $\Sigma = (\Phi, G, H)$. Then,

i) The problem is solvable if and only if the GCLD's of A_o and B_o are stable, where $A_o := I - d\Phi_o$ and $B_o := dG_o$ and $\Sigma_o = (\Phi_o, G_o, H_o)$ is a completely observable subsystem of Σ obtained via Kalman's canonical observability decomposition of Σ .

ii) Provided that the solvability condition is fulfilled, the optimal input sequence is given as in (18) where X and Y are obtained by first solving the spectral factorization problem (9) and next finding the minimum degree solution w.r.t. Z of the pair of bilateral Diophantine equations (14) and (17) [or (14) and (16)].

iii) The LQ optimal feedback-gain matrix

$$F := -X^{-1}Y \tag{21}$$

is a root of the bilinear matrix polynomial equation

$$Z(A-BF) - \bar{A}_2\psi_u F = \bar{B}_2 H'\psi_y H . \tag{22}$$

with $\partial Z < \partial\bar{E}$.

iv) The overall closed-loop system is asymptotically stable if and only if the plant Σ has no unstable hidden modes.

Proof

Eq. (22) is readily obtained by rewriting (17) as $\bar{E} = (ZB + \bar{A}_2\psi_u)X^{-1}$ and using it into (14). □

CONCLUSIONS AND FURTHER REMARKS

1 - There are, in general, two bilateral Diophantine equations, viz. (14) and (17) [or, as shown in [3], equivalently (14) and (16)], must be solved with $\partial Z < \partial\bar{E}$ in order to finding the LQ optimal feedback-gain matrix $F = -X^{-1}Y$. In particular, the only use of (14) does not in general suffice . The reader is referred to [3] and [5] where the above issue is thoroughly studied. Nevertheless, for the sake of better understanding, an example is discussed hereafter.

Example 1

Let: $\Phi = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$; $G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $H = [1 \quad 1]$; $\psi_u = 2$; $\psi_y = 1$. Notice that (Φ, G) is not completely reachable, whereas (Φ, H) is completely observable. We find $A = \begin{bmatrix} 1-d & 0 \\ 0 & 1-d/2 \end{bmatrix}$; $B = \begin{bmatrix} d \\ 0 \end{bmatrix}$.

A GCLD of A and B is $\begin{bmatrix} 1 & 0 \\ 0 & 1-d/2 \end{bmatrix}$ which is stable.

Thus, according to Theorem - Part i) the problem is solvable, and, according to Part iv) the resulting LQ optimal feedback stabilizes the plant, being the only plant hidden eigenvalue $\lambda = 1/2$ stable.

We also find: $A_2 = 1-d$; $B_2 = [d \quad 0]'$; and, via spectral factorization, $E = 2(1-d/2)$. Further: $\bar{A}_2 = -1+d$; $\bar{B}_2 = [1 \quad 0]$; $\bar{E} = -(1-2d)$ which implies $\partial Z = 0$

Eq. (14) and (17), or (14) and (16), give $X = 2$, $Y = [y_1{=}1 \quad y_2{=}1/3]$ together with $Z = [z_1{=}2 \quad z_2{=}4/3]$.

Hence, $F = -[1/2 \quad 1/6]$ and $\Phi + GF = \begin{bmatrix} 1/2 & -1/6 \\ 0 & 1/2 \end{bmatrix}$.

Eq. (16) alone yields $X = 2$, $y_1 = 1$, $z_1 = 2$.

However, it does not yield any information on y_2 and z_2 . □

2 - Eq. (22) is an interesting and somewhat unexpected equation in that it could be directly used to compute the LQ optimal feedback-gain matrix without solving the spectral factorization problem (9) except for the easy task of determining $\partial\bar{E}$,

provided that together with the condition $\partial Z < \partial \bar{E}$ it yields only a single stabilizing F among all its solutions whenever the GCLD's of A and B are stable. This can be admitedly advanced at time only as a conjecture.

Example 2
Consider again the LQ output regulation problem of Ex. 1. For this case, (22) yields for $\partial Z = 0$ two solutions (Z,F). The first coincides with the one found in Ex. 1. The second one is given by $Z = [-1 \quad 1/2]$ and $F = [1 \quad 1/4]$. Note that the latter yields $\Phi + GF = \begin{bmatrix} 2 & 1/4 \\ 0 & 1/2 \end{bmatrix}$ and, hence, an unstable closed-loop system.

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