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A note on the approximability property of nonlinear variable structure systems

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Using a definition of approximability based on a suitable class of perturbations it is proved that such a property is verified for nonlinear variable structure systems of general form. To do this we assume the usual conditions of the equivalent control theory and we use the properties of the solution map of the initial-value problems associated with the fully nonlinear control system. A comparison with the results of Bartolini and Zolezzi (1986) is considered.

1. Introduction

We consider the nonlinear control process represented by the differential system

$$\dot{x} = f(t, x, u) \quad \text{for almost all } t \in [0, T] \tag{1}$$

where

$$f\colon [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

satisfies Carathéodory-type conditions that will be specified later. The function $t \to x(t)$ describing the *state* of (1) belongs to the Banach space $AC([0, T], \mathbb{R}^n)$ of absolutely continuous functions defined on $[0, T]$ with values in $\mathbb{R}^n$, while the *control law* $t \to u(t)$ is described by a function u in the Banach space $L^\infty([0, T], \Omega)$ of Lebesgue, measurable, essentially bounded functions defined on $[0, T]$ with values in a given set $\Omega \subset \mathbb{R}^m$. Together with (1) we consider a sliding manifold

$$s(x) = (s_1(x), \ldots, s_m(x))' = 0 \tag{2}$$

where the prime denotes transpose.

Bartolini and Zolezzi (1986) considered the following control problem for system (1).

Control of nonlinear variable structure system: that is, to steer and to keep the state of system (1), during the time interval $[0, T]$, on the given manifold (2) by using feedback control laws

$$u = u(t, x) \in \Omega \tag{3}$$

which are essentially bounded and discontinuous along the surface S given by

$$s_j(x) = 0, \; j = 1, 2, \ldots, m$$

More precisely, the set of admissible feedback control laws $Q \subset L^\infty([0, T], \Omega)$ is

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defined as follows

$Q = \{u\colon [0, T] \times (\mathbb{R}\backslash S) \to \Omega;\ u = u(t, x)$ is a Carathéodory function such that for any $x_0 \in S$ there exists a finite limit $u(t, x_0) = \lim_{x \to x_0} u(t, x)$ with $x \in \mathbb{R}^n\backslash S\}$.

Usually, the control law $u = u(t, x)$ has the form

$$u_j(t, x) = \begin{cases} u_j^+(t, x) & \text{if } s_j(x) > 0 \\ u_j^-(t, x) & \text{if } s_j(x) < 0 \end{cases}$$

where $u_j^+, u_j^-, j = 1, 2, \ldots, m$ are given smooth functions taking values in Ω.

The work of Bartolini and Zolezzi (1986) constitutes the first paper to deal with the controllability of variable structure systems that are described by differential systems which are nonlinear in both variables x and u. Utkin (1978) developed the mathematical theory for the control system

$$\begin{aligned} \dot{x} &= A(t, x) + B(t, x)u \\ s(x) &= 0 \\ u &= u(t, x) \in \Omega \subset \mathbb{R}^m \end{aligned}$$

Specifically, in the paper of Bartolini and Zolezzi (1986), Theorems 1 and 2 of § 1 guarantee that the Filippov solutions of (1) (see Filippov 1964), which correspond to the discontinuous feedback controls $u = u(t, x)$ and satisfying $s(x(t)) = 0$, $t \in [0, T]$, are the same as those corresponding to a suitably defined continuous control law (the equivalent control).

Furthermore, Bartolini and Zolezzi (1986) proposed a definition of approximability for the nonlinear variable structure system (1)–(3). They then showed that this approximability property is verified for some special cases of (1). That is, the cases where

$$f(t, x, u) = A(t, x) + B(t, x)h(u)$$

or

$$z^{(n)} = g(t, z, z', \ldots, z^{(n-1)}, u)$$

The aim of this note is to provide an approximation property for the nonlinear variable structure system (1)–(3), without requiring any special form for the nonlinearity f. This property is defined along the lines given in Bartolini and Zolezzi (1986), but we will require that it is satisfied for a different class of perturbations than the one considered in the above paper.

This allows us to prove, in Theorem 1, that under the usual assumptions of the equivalent control theory, the fully nonlinear control system (1) possesses such a property.

On the other hand, in order to give explicit conditions to verify their definition of approximability, Bartolini and Zolezzi (1986) specialize the form of (1) and they also assume some extra conditions on the resulting dynamics. More precisely, they consider a class of perturbations which is suitable for applying G-convergence arguments (see Patuzzo Greco 1979) to show that the nonlinear variable structure control system (1)–(3) fulfils the approximability property. To do this it is necessary to express explicitly the difference between the dynamics corresponding to the equivalent control and those corresponding to the perturbed equivalent control in terms of the perturbations (see Bartolini and Zolezzi, 1986, Lemma 2).

This requires some restrictions on the dynamics f. Finally, we point out that in some cases our definition of approximability is weaker than the corresponding definition in Bartolini and Zolezzi (1986), (cf. Remark 2, § 2) but, in our opinion, it remains of relevant physical meaning. In fact, as will be shown in Lemmas 2 and 3, we obtain the continuous dependence (in the Hausdorff metric) of the set of solutions of any initial-value problem associated with (1) on the considered set of admissible perturbations. Thus, in particular, we get the stability of the solution, corresponding to the equivalent control, with respect to the perturbations which vanish in the L^1-norm.

2. Definitions, hypotheses and preliminary results

Throughout the paper we will assume the following usual conditions (F), (G), (H) of the equivalent control theory.

Condition F: *$f: [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a Carathéodory function; that is, the map $f(\cdot, p, q)$ is Lebesgue measurable in $[0, T]$ for any $(p, q) \in \mathbb{R}^n \times \mathbb{R}^m$ and the map $f(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m$ for almost all $t \in [0, T]$. Furthermore, for any $\rho > 0$ there exists $\gamma_\rho \in L^1([0, T], \mathbb{R}_+)$ such that*

$$|f(t, p, q)| \leqslant \gamma_\rho(t)$$

for almost all $t \in [0, T]$ and $|p| + |q| < \rho$.

Condition G: *Let $G = \partial s/\partial x$ be the $m \times n$ jacobian matrix of elements $(\partial s_j/\partial x_k)_{\substack{j=1,\ldots,m\\ k=1,\ldots,n}}$. There exists a neighbourhood A of $S = \{x \in \mathbb{R}^n;\ s(x) = 0\}$ such that for almost all $t \in [0, T]$ and $x \in A$ the map $G(x)f(t, x, \cdot)$ is one-to-one on Ω and its range contains 0.*

The unique solution (if any) $u \in \Omega$ of the equation

$$G(x)f(t, x, u) = w$$

for a given $w \in \mathbb{R}^m$ will be denoted by

$$u^* = u^*(t, x, w).$$

For conditions which guarantee that condition G is satisfied, we refer to Partasarathy (1983).

Definition 1: Assume conditions (F)–(G). The equivalent control for the system (1)–(2) is the mapping

$$(t, x) \rightarrow u^*(t, x, 0)$$

with $t \in [0, T]$ and $x \in A$. □

Observe that, in view of the invariance of the domain theorem (see Hurewicz and Wallman 1948), we have that $u^* = u^*(t, x, w)$ is a Carathéodory function on $[0, T] \times A \times W$, where W is the set of the vectors $w \in \mathbb{R}^m$ for which there exists the solution of the above equation, and

$$\lim_{w \rightarrow 0} u^*(t, x, w) = u^*(t, x, 0).$$

We are now in the position of formulating the last condition

Condition H: *Assume conditions (F)–(G). The Cauchy problem*

$$\left.\begin{aligned} \dot{x} &= f(t, x, u^*(t, x, 0)) \\ x(0) &\in S \end{aligned}\right\}$$

has an unique solution defined on all $[0, T]$ *whenever* $x(0) \in S$.

Remark 1: Observe that conditions (F)–(G) and (H) do not require any special form for the nonlinearity f, as can easily be seen in the references quoted above. ☐

Definition 2: Given $M \in L^1([0, T], \mathbb{R}_+)$ and a neighbourhood A of S, let P be the set of all the generalized sequences of functions $a_\varepsilon \in L^1([0, T], \mathbb{R}^m)$ such that $|a_\varepsilon(t)| \leq M(t)$ for any $\varepsilon > 0$ and for almost all $t \in [0, T]$,

$$\lim_{\varepsilon \to 0} a_\varepsilon(t) = 0 \text{ for almost all } t \in [0, T]$$

and such that there exists $u^*(t, x, a_\varepsilon(t))$ for almost all $t \in [0, T]$ and $x \in A$.

We will say that system (1)–(2) fulfils the *approximability property* if and only if conditions (F)–(H) hold, P is non-empty and for every $a_\varepsilon \in P$, if x_ε, $\varepsilon > 0$ is almost everywhere a (Carathéodory) solution in $[0, T]$ of

$$\left.\begin{aligned} \dot{x} &= f(t, x, u^*(t, x, a_\varepsilon(t))) \\ s(x_\varepsilon(0)) &= \alpha_\varepsilon \end{aligned}\right\}$$

with $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$, if y is almost everywhere a solution in $[0, T]$ of the Cauchy problem

$$\left.\begin{aligned} \dot{x} &= f(t, x, u^*(t, x, 0)) \\ s(y(0)) &= 0 \end{aligned}\right\}$$

then $x_\varepsilon(0) \to y(0)$ implies $x_\varepsilon \to y$ uniformly in $[0, T]$ as $\varepsilon \to 0$. ☐

Therefore, the approximability property holds for the nonlinear variable structure system (1)–(3) if and only if

(*a*) any sliding state corresponding to the equivalent control (ideal state) can be approximated by real states x_ε realizing only approximately the sliding condition, in the sense that $\mathrm{d}/\mathrm{d}t\, s(x_\varepsilon(t)) = a_\varepsilon(t)$ for almost all $t \in [0, T]$, where $a_\varepsilon \to 0$ in the $L^1([0, T], \mathbb{R}^m)$ norm;

(*b*) the real states x_ε converge to an ideal state when the disturbances $a_\varepsilon(t)$ tend to zero for almost all $t \in [0, T]$ and the initial states tend to the sliding manifold.

Remark 2: Observe that if $P \subset L^p([0, T], \mathbb{R}^m)$, for $p > 1$, then $P \subset H$, where H is the set of the disturbances as defined in the paper of Bartolini and Zolezzi (1986). Therefore, in this case, our definition of approximability is weaker than the corresponding definition of that paper. ☐

Definition 3: Let L be the linear space of all Carathéodory functions $g: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ such that for any $g \in L$ and for any $\rho > 0$ there exists $\eta_\rho \in L^1([0, T], \mathbb{R}_+)$ for which

$$|g(t, x)| \leq \eta_\rho(t) \quad \text{for any } |x| \leq \rho$$

Let us define on L, for any $K \subset \mathbb{R}^n$ compact, a seminorm $p_K(\cdot)$ as follows:

$$p_K(g) = \int_0^T \sup_{x \in K} |g(t, x)| \, \mathrm{d}t \quad \text{for any } g \in L$$

Then we obtain that L is metrizable (see Schaefer 1967). □

Definition 4: Let $\varphi: L \times [0, T] \times \mathbb{R}^n \to C([0, T], \mathbb{R}^n)$ be the multivalued map defined as follows

$$\varphi(g, \tau, \alpha) = \{x \in C([0, T], \mathbb{R}^n): \dot{x}(t) = g(t, x(t)) \text{ for almost all } t \in [0, T] \text{ and } x(\tau) = \alpha\} \tag{4}$$

In order to guarantee that $\varphi(g, \tau, \alpha) \neq \emptyset$ we assume the extension of the solutions of the Cauchy problems to the interval $[0, T]$. □

Definition 5: Let X, Y be two metric spaces. A multivalued map $\psi: X \to Y$, such that $\psi(x)$ is a non-empty closed subset of Y for every $x \in X$, is called upper semicontinuous (u.s.c.) at the point $x_0 \in X$ if for each open set U containing $\psi(x_0)$ there exists a neighbourhood V of x_0 such that $\psi(x) \subset U$ for any $x \in V$. While ψ is said to be lower semicontinuous (l.s.c) at the point $x_0 \in X$ if for each open set U such that $\psi(x_0) \cap U \neq \emptyset$ there exists a neighbourhood V of x_0 such that $\psi(x) \cap U \neq \emptyset$ for any $x \in V$. If the map ψ is u.s.c. (l.s.c.) at any point $x \in X$ we will say that ψ is u.s.c. (l.s.c.) in X. Moreover, if ψ is both u.s.c. and l.s.c. at the point x_0 then ψ is continuous at x_0 in the sense of the Hausdorff metric. □

Definition 6: A compact, metric space X is called an R_δ-set provided there exists a decreasing sequence $\{A_n\}$ of compact, metric absolute retracts such that

$$X = \bigcap_n A_n$$ □

3. Approximability property

We recall the following result stated by Furi *et al.* (1985) which will be used later.

Lemma 1: *Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence in L. Let $\{(\tau_n, \alpha_n)\}_{n \in \mathbb{N}}$ be a sequence in $[0, T] \times \mathbb{R}^n$ converging to a point $(\bar{\tau}, \bar{\alpha})$; and let x_n be any non-continuable, absolutely continuous solution of the initial value problem*

$$\left.\begin{aligned} \dot{x} &= g_n(t, x) \text{ for almost all } t \in [0, T] \\ x(\tau_n) &= \alpha_n \end{aligned}\right\}$$

Assume that there exists a function $g: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, such that

(a) for almost all $t \in [0, T]$, the sequence $\{x \to g_n(t, x)\}_{n \in \mathbb{N}}$ converges uniformly to $x \to g(t, x)$ on any compact subset of $\mathbb{R}^n$;

(b) any absolutely continuous solution of the Cauchy problem

$$\left.\begin{aligned} \dot{x} &= g(t, x) \text{ for almost all } t \in [0, T] \\ x(\bar{\tau}) &= \bar{\alpha} \end{aligned}\right\} \tag{5}$$

is defined on $[0, T]$, *moreover there exists a constant* $C > 0$ *such that* $\max_{t \in [0,T]} |x(t)| \leq C$ *for all the solutions of* (5). *Then there exists* $\bar{n} \in \mathbb{N}$ *such that, for any* $n \geq \bar{n}$, x_n *is defined on* $[0, T]$ *and the sequence* $\{x_n\}_{n \in \mathbb{N}}$ *has a subsequence converging uniformly in* $[0, T]$ *to a solution of* (5).

We are in a position to prove the following

Lemma 2: *The set* $\varphi(g, \tau, \alpha)$ *defined by* (4) *is an* R_δ*-set for any* $(g, \tau, \alpha) \in L \times [0, T] \times \mathbb{R}^n$. *Moreover, the map* $\varphi: L \times [0, T] \times \mathbb{R}^n \to C([0, T], \mathbb{R}^n)$ *is upper semicontinuous on* $L \times [0, T] \times \mathbb{R}^n$.

Proof: The first assertion follows from a result of Aronszajn (1942). We prove now by contradiction that φ is upper semicontinuous on $L \times [0, T] \times \mathbb{R}^n$.

Let $(g_0, \tau_0, \alpha_0) \in L \times [0, T] \times \mathbb{R}^n$. Assume that the map φ is not upper semicontinuous at this point. Therefore, there exists a neighbourhood U_0 of $\varphi(g_0, \tau_0, \alpha_0)$ in $C([0, T], \mathbb{R}^n)$ such that for any neighbourhood V of (g_0, τ_0, α_0) in the space $L \times [0, T] \times \mathbb{R}^n$, equipped with the usual product metric d, there exists a point $(\tilde{g}, \tilde{\tau}, \tilde{\alpha}) \in V$ such that $\varphi(\tilde{g}, \tilde{\tau}, \tilde{\alpha}) \not\subset U_0$.

If we define, for any $n \in \mathbb{N}$, the set

$$V_n = \{(g, \tau, \alpha) \in L \times [0, T] \times \mathbb{R}^n;\ d((g, \tau, \alpha), (g_0, \tau_0, \alpha_0)) < 1/n\}$$

Then, for any $n \in \mathbb{N}$, there exists $(g_n, \tau_n, \alpha_n) \in V_n$ and a point $y_n \in \varphi(g_n, \tau_n, \alpha_n)$ such that $y_n \notin U_0$.

In other words, we have that

$$\left.\begin{aligned} \dot{y}_n(t) &= g_n(t, y_n(t)) \text{ for almost all } t \in [0, T] \\ y_n(\tau_n) &= \alpha_n \end{aligned}\right\}$$

and $y_n \notin U_0$.

It is easy to see that passing to a subsequence we have that

$$\tau_n \to \tau_0 \in [0, T], \quad \alpha_n \to \alpha_0 \in \mathbb{R}^n \quad \text{and for almost all } t \in [0, T]$$

$$g_n(t, x) \to g_0(t, x) \quad \text{uniformly on compact set of } \mathbb{R}^n.$$

Thus all the assumptions of Lemma 1 are satisfied, so that there exists a subsequence of $\{y_n\}_{n \in \mathbb{N}}$ converging uniformly to a function y_0 such that

$$\left.\begin{aligned} \dot{y}_0(t) &= g_0(t, y_0(t)) \quad \text{for almost all } t \in [0, T] \\ y_0(\tau_0) &= \alpha_0 \end{aligned}\right\}$$

That is, $y_0 \in \varphi(g_0, \tau_0, \alpha_0)$, contradicting the fact that $y_n \notin U_0 \supset \varphi(g_0, \tau_0, \alpha_0)$ for any $n \in \mathbb{N}$. ☐

Lemma 3: *The map* φ *defined by* (4) *is continuous at any point* $(\tilde{g}, \tilde{\tau}, \tilde{\alpha}) \in L \times [0, T] \times \mathbb{R}^n$ *for which* $\varphi(\tilde{g}, \tilde{\tau}, \tilde{\alpha})$ *is a singleton.*

Proof: It is enough to show that if $\varphi(\tilde{g}, \tilde{\tau}, \tilde{\alpha})$ is a singleton then φ is lower semicontinuous at $(\tilde{g}, \tilde{\tau}, \tilde{\alpha})$. In fact, let U be an open set of $C([0, T], \mathbb{R}^n)$ such that $\varphi(\tilde{g}, \tilde{\tau}, \tilde{\alpha}) \cap U \neq \emptyset$, and thus $\varphi(\tilde{g}, \tilde{\tau}, \tilde{\alpha}) \subset U$. On the other hand φ is upper semicontinuous at $(\tilde{g}, \tilde{\tau}, \tilde{\alpha})$ by virtue of Lemma 2; thus, there exists a neighbourhood V of $(\tilde{g}, \tilde{\tau}, \tilde{\alpha})$ such that $\varphi(\tilde{g}, \tilde{\tau}, \tilde{\alpha}) \subset U$ for any $(g, \tau, \alpha) \in V$.

Therefore, $\varphi(g, \tau, \alpha) \cap U \neq \varnothing$ for any $(g, \tau, \alpha) \in V$, so that φ is lower semicontinuous at $(\tilde{g}, \tilde{\tau}, \tilde{\alpha})$. $\square$

We prove now the following theorem.

Theorem 1: *Under the assumptions (F)–(H) the nonlinear variable structure system (1)–(3) possesses the approximability property.*

Proof: Let $\{a_\varepsilon\}_{\varepsilon>0}$ be a generalized sequence in P. Since, for any $\varepsilon > 0$, the control $u^* = u^*(t, x, a_\varepsilon(t))$ is a Carathéodory function on $[0, T] \times A \times \mathbb{R}^m$ we obtain the $\lim_{\varepsilon \to 0} u^*(t, x, a_\varepsilon(t)) = u^*(t, x, 0)$ for almost all $t \in [0, T]$ and uniformly on the compact sets of $\mathbb{R}^n$. Consider now the following Cauchy problems for $\varepsilon > 0$

$$\left.\begin{aligned} \dot{x} &= g_\varepsilon(t, x) \\ x_\varepsilon(0) &= \alpha_\varepsilon \end{aligned}\right\}$$

and

$$\left.\begin{aligned} \dot{x} &= \tilde{g}(t, x) \\ x(0) &= \tilde{\alpha} \end{aligned}\right\}$$

where $\tilde{\alpha} = \lim_{\varepsilon \to 0} \alpha_\varepsilon$, $\tilde{\alpha} \in S$, $g_\varepsilon(t, x) = f(t, x, u^*(t, x, a_\varepsilon(t)))$ and $\tilde{g}(t, x) = f(t, x, u^*(t, x, 0))$. We have that for almost all $t \in [0, T]$, $g_\varepsilon(t, x) \to \tilde{g}(t, x)$ uniformly on the compact sets of $\mathbb{R}^n$.

From assumptions (F)–(G) on f and on u^* respectively we get that g_ε, $\tilde{g} \in L$ and they are uniformly bounded in $L^1([0, T], \mathbb{R}^n)$ for any compact set $K \subset \mathbb{R}^n$. On the other hand, by virtue of (H) and Lemma 3 the solution map φ is continuous at the point $(\tilde{g}, 0, \tilde{\alpha})$. Therefore, $\varphi(g_\varepsilon, 0, \alpha_\varepsilon) \to \varphi(\tilde{g}, 0, \tilde{\alpha})$ in the Hausdorff distance of $C([0, T], \mathbb{R}^n)$, where $\varphi(\tilde{g}, 0, \tilde{\alpha})$ is a singleton. This concludes the proof. $\square$

4. Conclusions

In this paper, under the usual assumptions of the equivalent control theory, we have provided an approximation property for nonlinear variable structure systems, where the nonlinearity f is not necessarily restricted to have a particular form. The method is based on the topological properties of the set of solutions of the initial-value problems associated with (1). A discussion on the relationship of our results with those of Bartolini and Zolezzi (1986) has also been carried out.

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