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POSITIVE SOLUTIONS OF ELLIPTIC NON-POSITONE PROBLEMS*

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Abstract. We give conditions for the existence or nonexistence of positive solutions of second-order subcritical elliptic nonpositone problems. We do not assume that the problems are radial, nor that they satisfy a variational structure. Our chief tools are Degree Theory, *a priori* estimates, and Maximum Principle arguments.

In this paper we are interested in the existence or non-existence of positive classical solutions for the problem:

$$\left.\begin{aligned} \ell u &= -\Delta u + 2\Sigma b_j D_j u = \lambda f(x,u) \quad \text{in} \quad \Omega \\ u &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{on} \quad \partial\Omega \end{aligned}\right\} \tag{1}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. Here we assume that $b_j \in C^\alpha(\overline{\Omega})$, $f \in C^\alpha_{\ell oc}(\mathbb{R}^{n+1})$ with f superlinear and subcritical: $f(x,\xi) \sim \xi^\gamma$ for $0 < \xi$ large, with $1 < \gamma < (n+2)/(n-2)$. Of specific interest to us is the nonpositone situation, $f(x,0) < 0$ and the prototype equation is:

$$\left.\begin{aligned} -\Delta u &= \lambda[u^\gamma - \varepsilon] \quad \text{in} \quad \Omega \\ u &= 0 \qquad\qquad\quad \text{on} \quad \partial\Omega \end{aligned}\right\}. \tag{2}$$

Unlike in the usual case, $f(x,0) \geq 0$, there seems to be relatively little literature for this situation. We mention in particular the existence criteria of Castro and Shivaji [5, 6], and Smoller and Wasserman [14]. Furthermore, a variational existence result similar to what we shall establish here may be found in Chapter 3 of the thesis of S. Unsurangsie [18]. We thank the referee for bringing this reference to our attention. Nonexistence conditions for λ large may be found in [4] for the radial case. We recall that nonpositone radial problems are of interest if one considers

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"symmetry-breaking" situations [15, 16]. Many of the previous results of which we are aware for problems (1)-(2) are radial (or close to radial) and/or variational with the primitive of f usually playing a significant role in the procedures. While these earlier considerations furnished the motivation for our work, it is clear that they cannot be easily applied to problems such as (1), (2). To illustrate the results which we will obtain we give as a special example:

Corollary 1. *Assume Ω is strictly convex. For any $\varepsilon > 0$ there exist $\lambda_1 > \lambda_0 > 0$ such that problem (2) has a positive solution for $0 < \lambda < \lambda_0$ and no positive solutions for $\lambda > \lambda_1$.*

Despite its very special nature, we believe this Corollary to be new.

We note that our work extends some of the recent radial results given in [4], [5] and thus also answers a conjecture of Smoller and Wasserman [14] – concerning the existence of positive solutions for $n/(n-2) \leq \gamma < (n+2)/(n-2)$ – in the non radial case.

Our chief tools for existence are Degree Theory, a priori estimates of Gidas and Spruck [9], and scaling arguments. The estimates in [9] are particularly selected because they hold under growth conditions which are analogous to the radial ones given in [4] for the "large λ" non existence results. The non existence results presented here, however, are based on the Maximum Principle and ideas of de Figueiredo, Lions and Nussbaum [8], and Gidas, Ni and Nirenberg [10], and hold under different conditions than those given in [4] even in the radial case. They require assumptions only on some boundary patch, so that no global conditions need be placed on problem (1) to ensure nonexistence for λ large.

We emphasize that the solutions we find are strictly positive in Ω. If $f \geq 0$, then nonnegative solutions must be positive by the Maximum Principle, but this is not the case here. Furthermore, we observe as mentioned in [5] that $u = 0$ gives a supersolution [13], thus making upper-lower solution procedures difficult to apply in general, although these have been sometimes used to advantage [14].

In conclusion, we remark that there is a differential inclusion related problem to (1); i.e.,

$$\left.\begin{aligned} \ell u &\in \lambda \tilde{f}(x,u) && \text{in} \quad \Omega \\ u &= 0 && \text{on} \quad \partial\Omega \end{aligned}\right\} \qquad (3)$$

with

$$\tilde{f}(x,u) = \begin{cases} f(x,u) & u > 0 \\ [f(x,0), 0] & u = 0. \end{cases}$$

The connection between problems (1) and (3) is clear: if we can find a solution u to (3) such that $u > 0$ almost everywhere, then u also solves (1). Alternatively, we may consider the problem:

$$\left.\begin{aligned} \ell u &= \lambda f_n(x,u) && \text{in} \quad \Omega \\ u &= 0 && \text{on} \quad \partial\Omega \end{aligned}\right\} \qquad (4)$$

where f_n is a suitable continuous graph approximation to $\hat{f}$ in the Hausdorff distance where

$$\hat{f}(x,u) = \begin{cases} f(x,u) & u > 0 \\ 0 & u = 0. \end{cases}$$

Passing to the limit as $n \to \infty$ again gives a solution to problem (1), if we can show that the limit function is positive almost everywhere. For special cases of Ω, ℓ, f (e.g., f has "upward discontinuities," Ω symmetric) this can actually be done, and the interested reader may find these and related problems discussed in [7], [2], [1]. The difficulty with these approaches is that in general the candidates for solutions of (1) found from (3), (4) may not be positive and hence will not actually solve (1) — a fact which has been known for some time [17]. Consequently, we do not employ these approaches.

Existence results. Our fundamental growth assumption on f is the same as given in [9], and is chosen in analogy with the one used in [4] in the radial case to show nonexistence for λ large. Specifically, we assume

$$\lim_{\xi \to \infty} \frac{f(x,\xi)}{\xi^{\gamma}} = p(x) \tag{5}$$

uniformly in $x \in \overline{\Omega}$, with $0 < p \in C^{\alpha}(\overline{\Omega})$, $1 < \gamma < (n+2)/(n-2)$.

We begin with the following special results. Let $\widetilde{C} = \{\omega | \omega \in C^1(\overline{\Omega}),\ \omega = 0$ on $\partial\Omega\}$ and equip $\widetilde{C}$ with the $\|\ \ \|_{C^1}$ norm.

Lemma 1. *Assume* $f^*(x,\xi) = \begin{cases} p(x)\xi^{\gamma}, & \xi \geq 0 \\ 0, & \xi < 0 \end{cases}$ *and set* $F(u) = \ell^{-1}(f^*(x,u))$; *then* $F : \widetilde{C} \to \widetilde{C}$, $F(u) \geq 0$ *and there exist* $r_1, r_2 > 0$ *such that* $\deg(I - F, B_{r_2} - \overline{B}_{r_1}, 0) \neq 0$, *where* B_{r_i} *denotes the open ball of radius* r_i *in* $\widetilde{C}$.

Proof: This result is well known, but for completeness and convenience in what follows we sketch a proof. Consider the problem:

$$\left.\begin{aligned} \ell u &= f^*(x,u) + tJ \quad &\text{in} \quad &\Omega \\ u &= 0 &\text{on} \quad &\partial\Omega \end{aligned}\right\} \tag{6}$$

with

(i) $t \geq 0$,

(ii) J the normalized positive eigenfunction corresponding to the least eigenvalue of the formal adjoint ℓ^*.

Observe first that any (classical) solution of (6) must be positive by the Maximum Principle [11, p. 33]. Next, following the procedure in [3], we conclude that problem (6) has a solution only for $t \leq t_0$, some $t_0 > 0$. This observation, assumption (5) and the a-priori results in [9] imply that any solution of (6) must be bounded in absolute value. We next apply [11, Corollary 8.36] and conclude that there exists $r_2 > 0$ such that any classical solution u of (6) must satisfy $\|u\|_{C^1} < r_2$. Next, if u is any solution of (6) with $t = 0$ then $\ell u = (p(x)u^{\gamma-1})u$. Applying results in [12, p. 93 and p. 259], we conclude that $p(x_0)u^{\gamma-1}(x_0) > \tau$ for some x_0, where τ denotes the least eigenvalue of ℓ. From this we obtain $\|u\|_{C^1} \geq \|u\|_{C^0} > r_1$, for some $r_1 > 0$ independent of u. We observe that $F : \widetilde{C} \to \widetilde{C}$ is compact, since ℓ^{-1} maps C^1 to C^2 by the Fredholm alternative and [11, p. 107]. Furthermore, our earlier considerations show

(i) $-\mu\ell^{-1}(J) + u \neq F(u)$ for $\mu \geq 0$, $u \in \partial B_{r_2}$;

(ii) $u \neq \nu F(u)$ for $0 \leq \nu \leq 1$, $u \in \partial B_{r_1}$.

We can now apply Degree Theory arguments by constructing homotopies as in [3, or 19, p. 562] and conclude that $\deg(I - F, B_{r_2}, 0) = 0$, $\deg(I - F, B_{r_1}, 0) = 1$ whence $\deg(I - F, B_{r_2} - \overline{B}_{r_1}, 0) \neq 0$, and the result follows. Observe that here F is actually defined on the whole of $\widetilde{C}$ and the positivity is recovered by Maximum Principle arguments.

Lemma 2. *There exists $\delta > 0$ such that if $F_1 : \widetilde{C} \to \widetilde{C}$ is a compact map with $\|F_1(v)\|_{C^1} < \delta$ for v in B_{r_2}, $v > 0$, then there exists $0 < u \in B_{r_2}$ such that $u = F(u) + F_1(u)$.*

Proof: Using the notation of Lemma 1, let $\mathcal{S}$ denote the set of solutions found for $f(x,\xi) = p(x)\xi^\gamma$ in $B_{r_2} - \overline{B}_{r_1}$. Choose $\varepsilon > 0$ small and set $N_\varepsilon = \bigcup_{u \in \mathcal{S}} B_\varepsilon(u)$ where $B_\varepsilon(u)$ denotes the open ball of radius ε in $\widetilde{C}$ centered at u. Clearly N_ε is open and, by the compactness of F, there exists $\delta > 0$ such that $\|(I - F)(v)\|_{C^1} > \delta$ if $v \in \partial N_\varepsilon$. Next, we observe that if $v \in N_\varepsilon$ and ε is small enough then $v > 0$ in Ω. Otherwise there exists a sequence $\{v_\varepsilon\}$ in $\widetilde{C}$, $\{u_\varepsilon\}$ in $\mathcal{S}$ such that $\|v_\varepsilon - u_\varepsilon\|_{C^1} \to 0$ and $v_\varepsilon \not> 0$ in Ω. Since $\mathcal{S}$ is compact, we assume without loss of generality that $u_\varepsilon \to u$ for some $u \in \mathcal{S}$; i.e., $v_\varepsilon \to u$. Observing that $u \in \mathcal{S}$, we conclude by the Maximum Principle that $u > 0$ in Ω and $\partial u/\partial n < 0$ on $\partial\Omega$, where n denotes the outwards normal. Since $v_\varepsilon = 0$ on $\partial\Omega$ by definition, it follows that $v_\varepsilon > 0$ in Ω for ε small enough and the contradiction establishes the result. Finally, if $\|F_1(v)\|_{C^1} < \delta$ for v in B_{r_2}, we conclude by Lemma 1 that

$$\deg\Big((I - F - F_1), N_\varepsilon, 0\Big) = \deg(I - F, N_\varepsilon, 0) = \deg(I - F, B_{r_2} - \overline{B}_{r_1}, 0) \neq 0.$$

Theorem 1. *Let f satisfy (5). There exists $\lambda_0 > 0$ such that problem (1) has a positive solution for $0 < \lambda < \lambda_0$.*

Proof: Let $\varepsilon > 0$ be given. We observe that from condition (5),

$$\left|\frac{f(x,\xi)}{\xi^\gamma} - p(x)\right| < \varepsilon p(x)$$

if ξ is sufficiently large, whence

$$|f(x,\xi) - p(x)\xi^\gamma| < \varepsilon p(x)\xi^\gamma$$

for such ξ. Since f, p are assumed continuous, we conclude that there exists a constant $k(\varepsilon)$ such that

$$|f(x,\xi) - p(x)\xi^\gamma| < \varepsilon p(x)\xi^\gamma + k(\varepsilon) \tag{7}$$

for all $\xi \geq 0$. Let $\alpha > 0$ be given and set $\xi = \frac{u}{\alpha}$ in (7). We thus have

$$|\alpha^\gamma f(x, \frac{u}{\alpha}) - p(x)u^\gamma| < \varepsilon p(x)u^\gamma + k(\varepsilon)\alpha^\gamma.$$

We recall that the estimates [11, p. 212] and [11, p. 36] imply

$$\|L^{-1}(g)\|_{C^1} \leq k_1\|g\|_{C^0},$$

for some constant k_1 independent of g. Select δ as in Lemma 2 and now choose ε so that $\varepsilon\|p(x)u^\gamma\|_{C^0} < \delta/(2k_1)$ for $u \in B_{r_2}$, $u > 0$. Finally, choose α so that $\alpha^\gamma k(\varepsilon) < \delta/(2k_1)$. With these choices, we observe that $\|F_1(u)\|_{C^1} < \delta$ for $u \in B_{r_2}$, $u > 0$, where $F_1(u) = \ell^{-1}[\alpha^\gamma f(x, \frac{u}{\alpha}) - p(x)u^\gamma]$. We apply Lemma 2 and conclude the existence of a $u > 0$ such that $u = F_1(u) + F(u)$, where $F(u) = \ell^{-1}[f^*(x,u))$; i.e.,

$$\ell u = \alpha^\gamma f\left(x, \frac{u}{\alpha}\right),$$

for all α small. Finally, setting $v = u/\alpha$ and observing that $\ell v = \alpha^{\gamma-1} f(x,v)$ and $\gamma > 1$, the conclusion follows with $\lambda = \alpha^{\gamma-1}$.

Non-existence results. We consider now problem (1) under only some local conditions of f, b_j and Ω and show that for λ large there are no positive solutions. Specifically, we keep the smoothness assumptions on f, b_j, Ω but now require that for some $\alpha > 0$ and all $x \in \bar{\Omega} \cap S$, for some ball S centered at some point $P \in \partial\Omega$,
(i) $(\xi - \alpha) f(x,\xi) > 0$ for $\xi \neq \alpha$;
(ii) there exists a smooth function $g(x) > 0$ such that

$$f(x,\xi) \geq g(x)(\xi - \alpha) \quad \text{for} \quad \xi \geq \alpha.$$

Observe that condition (i) implies $f(x,\alpha) \equiv 0$. We no longer require condition (5) so that our non-existence results apply, for example, to some cases where $f(x,t) \sim t$ at infinity. We also observe that if condition (5) and assumption (i) hold, then elementary calculus shows that assumption (ii) follows for f smooth, $f_u(x,\alpha) > 0$. The prototype problem (2) is clearly an example of a suitable f. We also require a condition on Ω and f so that there exists a fixed truncated cone K, such that if $x \in S \cap \Omega_\varepsilon$ and $0 < u$ solves (1) for some $\lambda > 0$ then $u(y) \geq u(x)$ for any $y \in \Omega \cap x + K$ where $\Omega_\varepsilon = \{x | x \in \Omega,\ \rho(x, \partial\Omega) < \varepsilon\}$ and $x + K$ denotes a cone obtained by rigid motion from K with vertex at x. Conditions for the existence of such cones have been given by De Figueiredo, Lions and Nussbaum [8], using earlier fundamental arguments on positive solutions of Gidas, Ni and Nirenberg [10]. Note that, unlike the situation in [8], we require the existence of our structure only on some boundary patch rather than everywhere on Ω_ε. For the reader's convenience we recall from [8] that for the existence of such a K it suffices, for example, that:

There exists a smooth strictly convex domain D such that we have $S \cap \partial\Omega \subset \partial D$, $S \cap \Omega \subset D$ and
(iii) $f \in C^1(\overline{D} \times R^1)$ and $\nabla_x f \equiv 0$ in $\overline{D} \times R^1$;
(iv) $b_j \equiv 0$ in D.

Hereafter we assume that K, S exist, but we conjecture that the non existence results may hold even without these assumptions.

Select and fix $R_1 > 0$ such that K contains a ball of radius R_1. The non existence result follows from a series of preliminary results.

Lemma 3. *There exists $\lambda_1 > 0$ such that if $\lambda \geq \lambda_1$, $0 < u$ solves (1) and $B_{R_1} \subset \Omega \cap S$ denotes any ball of radius R_1 then $u(x) \leq \alpha$ for some $x \in B_{R_1}$.*

Proof: Since $f(x,\alpha) = 0$, we have for $u > \alpha$ by condition (ii),

$$\ell(u - \alpha) = \lambda f(x,u) \geq \lambda g(x)(u - \alpha)$$

with $g(x) > 0$. If for some $B_{R_1} \subset \Omega \cap S$ we have $u > \alpha$, we again apply the results of [12] and conclude as before, that $\tau \geq \lambda \inf_{x \in \bar{\Omega}} g(x)$, where τ denotes the least eigenvalue associated with the Dirichlet problem for ℓ in B_{R_1}. Since $g(x) > 0$ in $\bar{\Omega}$, the result follows.

Lemma 4. *There exists an open ball $B_\varepsilon(P)$ centered at $P \in \partial\Omega$ such that if $\lambda \geq \lambda_1$ then $u \leq \alpha$ in $B_\varepsilon(P) \cap \Omega$.*

Proof: Let K, S denote the cone and sphere introduced above and let $B_\varepsilon(P) \subset S$ with $0 < \varepsilon$ small. If $u(x) > \alpha$ for some $x \in B_\varepsilon(P) \cap \Omega$, then $u(y) > \alpha$ for any $y \in x + K$ and, in particular, for any $y \in B_{R_1}(\xi)$ for some $\xi \in x + K$. The result follows by Lemma 3 applied to $B_{R_1}(\xi)$.

Lemma 5. *Let $\lambda \geq \lambda_1$ and $0 < u$ solve (1). Select a sphere S_0 centered at P with $\overline{S}_0 \subset B_\varepsilon(P)$. Let v solve:*

$$\begin{aligned} \ell v &= 0 \quad && \text{in} \quad Z \\ v &= u \quad && \text{on} \quad \partial Z \end{aligned}$$

where Z is a smooth domain, $S_0 \cap \Omega \subset Z \subset B_\varepsilon(P) \cap \Omega$, $S_0 \cap \partial\Omega \subset \partial Z$. Then for some β, $\|v\|_{C^{1+\beta}(S_0 \cap \bar{\Omega})}$ is bounded independently of λ, u. In particular, $|\nabla v| <$ const. on $S_0 \cap \bar{\Omega}$.

Proof: Observe that by the Maximum Principle, v is positive and bounded by α, since $0 < u \leq \alpha$ on $B_\varepsilon(P) \cap \Omega$. Furthermore, $v \in C^{1+\beta}(S_0 \cap \bar{\Omega})$ and

$$\|v\|_{C^{1+\beta}(S_0 \cap \bar{\Omega})} \leq k[\|v\|_{C^0(B_\varepsilon(P) \cap \Omega)}].$$

again by [11, p. 212].

Lemma 6. *Let δ be given with $0 < \delta < \alpha$. There exists a sphere $S_1 \subset \Omega$, independent of $\lambda \geq \lambda_1$ such that $|u| < \delta$ in S_1.*

Proof: Recall that $u \leq \alpha$ in $B_\varepsilon(P) \cap \Omega$ by Lemma 4 whence $\ell u \leq 0$. Let v be the solution found in Lemma 5 and observe that $\ell(v - u) \geq 0$ in Z and $v \geq u$ on ∂Z. We conclude $v \geq u \geq 0$ in Z. Since $|\nabla v| <$ const. on $S_0 \cap \Omega$ and $v = 0$ on $\partial\Omega$ we select $\varepsilon > 0$ so that $u < v < \delta$ in $S_0 \cap \Omega_\varepsilon$ and then choose any $S_1 \subset S_0 \cap \Omega_\varepsilon$.

Collecting the above results, we have:

Theorem 2. *Problem (1) has no positive solutions for λ large.*

Proof: For $\lambda \geq \lambda_1$, we have shown that $0 < u < \delta < \alpha$ in some fixed sphere $S_1 \subset \Omega$. We recall that by assumption (i) and continuity,

$$\inf_{(\xi,x) \in (0,\delta) \times (\overline{\Omega} \cap S_1)} (\xi - \alpha) f(x, \xi) = \theta > 0;$$

i.e., $f(x, u) < -\theta/\alpha$ in S_1. In summary.

$$\begin{aligned} \ell u &\leq -\lambda\theta/\alpha \quad && \text{in} \quad S_1 \\ u &\leq \delta \quad && \text{on} \quad \partial S_1. \end{aligned}$$

From this we conclude, again by the Maximum Principle, that if λ is large enough then $u < 0$ somewhere in S_1 and the contradiction establishes the theorem.

In conclusion we observe that Corollary 1 is merely a special case of Theorems 1,2.

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