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Brief Paper

Adaptive Predictive Control with Mean-Square Input Constraint*

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Key Words—Adaptive control; predictive control; input constraints; convergence analysis; ODE method; LQ control.

Abstract—Convergence properties of a self-tuning regulator incorporating an input mean-square constraint are studied. An algorithm, derived from the long-range controller MUSMAR, is considered. For this algorithm, using the ODE method for analysing stochastic recursive algorithms and singularly perturbed ODE theory, a local convergence result is proved. This result characterizes possible convergence points of the algorithm as the constrained minima of the underlying steady-state quadratic cost. The actual convergence of the algorithm to the possible equilibrium points predicted by theory is verified by means of simulation examples including unmodelled plant dynamics.

1. *Introduction*

In all control applications, the actuator power is limited. It is therefore important to explicitly take into account such a restriction in the controller design specifications. This can be done by adopting either a hard-limit input constraint or a mean-square (MS) input constraint approach. These are two possible alternatives and the most convenient use of which depends on the application at hand. A hard-limit input constraint leads to a difficult non-linear optimization problem. In this connection, approximate solutions are proposed in Tsang and Clarke (1988), Toivonen (1983a), and Bohm (1985). In Tsang and Clarke (1988) an adaptive GPC yielding an approximate solution to a Quadratic Programming problem is considered. In Toivonen (1983a) an approximation to the probability density function of the plant input is used. In Bohm (1985) spread-in-time Riccati iterations are considered. For the alternative approach, Toivonen (1983b) considered an input variance constraint. Specifically, an algorithm was proposed by combining the classic self-tuning regulator (STURE) (Clarke and Gawthrop, 1975) with a stochastic approximation scheme, in order to ensure the Kuhn–Tucker complementary condition. Though appealing for its simplicity, the algorithm of Toivonen (1983b) has drawbacks (non-minimum phase, unstable plants, plants with a time-varying I/O delay), inherited from the one-step ahead cost.

This paper studies an MS input constrained adaptive control algorithm whose underlying control law is capable of overcoming the STURE drawbacks. The algorithm is obtained by combining conveniently the MUSMAR algorithm presented in Menga and Mosca (1979) and Greco *et al.* (1984) with the stochastic approximation scheme of Toivonen (1983b). Hereafter this algorithm is referred as CMUSMAR (Constrained MUSMAR). The main interest is in the convergence properties of CMUSMAR. In the sequel, local convergence results are obtained. The strongest of them asserts that the isolated *constrained* minima of the underlying steady-state quadratic cost are possible convergence points of CMUSMAR. This conclusion holds also true in the presence of plant unmodelled dynamics and unknown I/O transport delay. The study is carried out by using ODE convergence analysis as given in Ljung (1977) and singular perturbation theory of ODEs (Wasov, 1965; Kokotović *et al.*, 1986). The actual convergence of CMUSMAR to the possible equilibrium gains predicted by the theory is explored by means of simulation examples. Since the explicit criterion adaptive controller of Trulsson and Ljung (1985), under model matching conditions, asymptotically satisfies the same ODE as MUSMAR, the conclusions that we obtain for the latter controller apply also to the first, provided that no unmodelled dynamics are present.

2. *Formulation of the problem*

Consider the ARMAX plant

$$A(d)y(t) = B(d)u(t) + C(d)e(t), \tag{1}$$

in which A, B and C are polynomials in the backward-shift operator d, such that A and C are monic, viz. $A(0) = C(0) = 1$, $B(0) = 0$, C, as well as the greatest common divisor of A and B, is strictly Hurwitz, and $\{e(t)\}$ is a sequence of zero mean, independent identically distributed (i.i.d.) random variables such that all moments exist. A linear control regulation law

$$R(d)u(t) = -S(d)y(t), \tag{2}$$

is considered for the plant (1). In (2) R and S are polynomials in d, with R monic. Equation (2) can be equivalently rewritten as

$$u(t) = f's(t), \tag{3}$$

where f is a vector whose entries are the coefficients of R and S, and $s(t)$ is a vector, hereafter referred to as the *pseudostate*, given by

$$s(t) = [y(t) \cdots y(t-n)u(t-1) \cdots u(t-m)]'. \tag{4}$$

The following problem is considered.

Problem 1. Given $c^2 > 0$, find an f as in (3), solving

$$\min_f \lim_{t\to\infty} E[y^2(t)], \tag{5}$$

subject to the constraint

$$\lim_{t\to\infty} E[u^2(t)] \le c^2. \tag{6}$$

According to the Kuhn–Tucker theorem (Luenberger, 1969), Problem 1 is converted to the following *unconstrained* minimization problem.

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Problem 2. Given $c^2 > 0$, find an f solving

$$\min_f \mathscr{L}(f, \rho), \tag{7}$$

where the Lagrangian function $\mathscr{L}$ is given by the unconditional cost

$$\mathscr{L}(f, \rho) := \lim_{t\to\infty} E[y^2(t) + \rho u^2(t)], \tag{8}$$

and the Lagrange multiplier ρ satisfies the Kuhn–Tucker complementary condition

$$\rho\Big(\lim_{t\to\infty} E[u^2(t)] - c^2\Big) = 0. \tag{9}$$

For an unknown plant, Problem 1, or equivalently Problem 2, is to be solved by an adaptive control algorithm capable of selecting ρ so as to satisfy (9) and approximate an f minimizing (8).

Remark. Let r be the output set point and $\tilde{y}(t) := y(t) - r$ the tracking error whose MS value $E[\tilde{y}^2(t)]$ has to be minimized in stochastic steady-state (s.s.s.) under constraint (6). This problem can be cast into the formulation above changing $y(t)$ into $\tilde{y}(t)$ and using the enlarged pseudostate $s_r(t) := [s'(t)r']$ and $u(t) = f's_r(t)$ instead of (3). In case the circumstances are such that $E[\delta u^2(t)] \le c^2$, $\delta u(t) := u(t) - u(t-1)$, is more suitable than (6), one can use the pseudostate $s_\delta(t) := [\tilde{y}(t) \cdots \tilde{y}(t-n), \delta u(t-1) \cdots \delta u(t-n)]'$, the control $\delta u(t) = f's_\delta(t)$, for an ARIMAX plant $A(d)\,\Delta(d)\tilde{y}(t) = B(d)\,\delta u(t) + C(d)e(t)$, $\Delta(d) := 1 - d$. This is an integral action variant of (1)–(6) with $y(t)$ changed into $\tilde{y}(t)$ and $s(t)$ into $s_\delta(t)$, and insures in s.s.s. rejection of constant disturbances.

3. *MS input constrained MUSMAR self-tuner*

As a candidate algorithm for solving the problem stated in Section 2, the stochastic approximation approach proposed in Toivonen (1983b), combined with the MUSMAR algorithm, is considered.

At each sampling time t, the MUSMAR algorithm selects $u(t)$ so as to minimize in s.s.s. the multistep quadratic cost

$$J_T = \frac{1}{T} E\Big[\sum_{i=1}^{T} y^2(t+i) + \rho(t)u^2(t+i-1) \mid y^t, u^{t-1}\Big], \tag{10}$$

where $x^t := \{x(t), x(t-1), \ldots\}$.

Next, the Lagrange multiplier $\rho = \rho(t)$ is updated via the following recurrent scheme:

$$\rho(t) = \rho(t-1) + \varepsilon\gamma(t)\rho(t-1)(u^2(t) - c^2), \tag{11}$$

in which ε is a positive real and $\{\gamma(t)\}$ a sequence of real numbers, whose selection will be made clear in the sequel. As detailed below, CMUSMAR (Constrained MUSMAR) is obtained.

CMUSMAR algorithm. At each step t, recursively execute the following steps:

(1) Update Recursive Least Squares (RLS) estimates of the closed-loop system parameters θ_i, μ_i, ψ_i and ϕ_i, $i = 1, \ldots, T$ as follows (Greco *et al.*, 1984)

$$\begin{bmatrix}\theta_i(t)\\ \psi_i(t)\end{bmatrix} = \begin{bmatrix}\theta_i(t-1)\\ \psi_i(t-1)\end{bmatrix} + K(t-T) \times [y(t-T+i+1) - \theta_i(t-1)u(t-T) - \psi_i'(t-1)s(t-T)],$$
$$i = 0, \ldots, T-1. \tag{12}$$

$$\begin{bmatrix}\mu_i(t)\\ \varphi_i(t)\end{bmatrix} = \begin{bmatrix}\mu_i(t-1)\\ \varphi_i(t-1)\end{bmatrix} + K(t-T) \times [u(t-T+i) - \mu_i(t-1)u(t-T) - \varphi_i'(t-1)s(t-T)]$$
$$i = 1, \ldots, T-1. \tag{13}$$

In (12) and (13) $K(t)$ denotes the Kalman gain associated with the regressor sequence $\{z(j), j = 1, \ldots, t\}$

$$z(j) := [u(j)s'(j)]', \tag{14}$$

and $\mu_0(t) = 1$ and $\varphi_0(t) = 0$.

(2) Update the control cost weight, $\rho(t)$ by using (11) with

$$\gamma(t) = [K'(t-T)K(t-T)]^{1/2}. \tag{15}$$

(3) Update the vector of feedback gains f by

$$\alpha(t) = \sum_{i=1}^{T} \theta_i^2(t) + \rho(t)\Big(1 + \sum_{i=1}^{T-1} \mu_i^2(t)\Big), \tag{16}$$

$$f(t) = -\frac{1}{\alpha(t)}\Big(\sum_{i=1}^{T} \theta_i(t)\psi_i(t) + \rho(t)\sum_{i=1}^{T-1} \mu_i(t)\phi_i(t)\Big). \tag{17}$$

(4) Apply to the plant a control given by

$$u(t) = f'(t)s(t) + \eta(t), \tag{18}$$

where $\{\eta(t)\}$ is a zero mean low variance i.i.d. dither noise independent of $\{e(t)\}$.

The dither presence is introduced so as to guarantee persistency of excitation. Note that, for $T = 1$, the above algorithm reduces to the constrained MV self-tuner given in Toivonen (1983b).

4. *ODE convergence analysis*

The algorithm introduced in Section 3 is now analyzed using the ODE method presented in Ljung (1977). It is to be noticed that the applicability of the ODE approach relies on the following assumption.

The sequence of the regulator parameters $[f'(t)\rho(t)]' \in \mathcal{M}$ and $\|[u(t)s'(t)]'\|$ is bounded for infinitely many t with probability 1. Here $\mathcal{M}$ is a compact subset of the region where the regulator gain f and the control weight ρ define an asymptotically stable closed-loop system.

We can associate to CMUSMAR the following set of ODEs as in Mosca *et al.* (1989): ($i = 0, \ldots, T-1$ and $j = 1, \ldots, T-1$)

$$\begin{bmatrix}\dot\theta_i(\tau)\\ \dot\psi_i(\tau)\end{bmatrix} = R_z^{-1}(\tau)E[z(t)(y(t+i) - (\theta_i(\tau)u(t) + \psi_i'(\tau)s(t)))], \tag{19}$$

$$\begin{bmatrix}\dot\mu_j(\tau)\\ \dot\phi_j(\tau)\end{bmatrix} = R_z^{-1}(\tau)E[z(t)(u(t+j) - \mu_j(\tau)u(t) + \phi_j'(\tau)s(t)))], \tag{20}$$

$$\dot R_z(\tau) = E[z(t)z'(t)] - R_z(\tau), \tag{21}$$

$$\dot\rho(\tau) = \varepsilon\rho(\tau)\{E[u^2(t)] - c^2\}, \tag{22}$$

where $z(t)$ is as in (14). In (19) and (20), a dot denotes derivative with respect to τ, and $E[.]$ the expectation w.r.t. the probability density function induced on $\{u(t)\}$ and $\{y(t)\}$ by $\{e(t)\}$ and $\{\eta(t)\}$, assuming that the system is in s.s.s. corresponding to the constant control law

$$u(t) = f'(\tau)s(t) + \eta(t). \tag{23}$$

Hereafter, the variable τ will be omitted so as to simplify the notation. In order to obtain a differential equation for f, differentiate (17) with respect to τ,

$$\dot f = \dot f_0 - \dot\rho\frac{1}{\alpha}\Big[f\sum_i \mu_i^2 + \sum_i \mu_i\phi_i\Big], \tag{24}$$

where $\dot f_0$ denotes the derivative of f assuming ρ to be constant. In Mosca *et al.* (1989) it is shown that the following ODE holds

$$\dot f_0 = -\frac{1}{\alpha}R_s^{-1}\nabla_T\mathscr{L} + o(|\tilde f|), \tag{25}$$

where $\tilde f := f_0 - f^*$; f^* denotes any equilibrium point; $o(|x|)$ is such that $\lim_{x\to 0}[o(|x|)/|x|] = 0$; $R_s := E[s(t)s'(t)]$. Finally, $\nabla_T\mathscr{L}$ is an approximation to the gradient of $\mathscr{L}$ w.r.t. f_0 which becomes increasingly higher as $T \to \infty$.

Thus, the ODE associated to f for CMUSMAR is

$$\dot f_0 = -\frac{1}{\alpha}R_s^{-1}\nabla_T\mathscr{L} - \dot\rho\frac{1}{\alpha}\Big[f\sum_i \mu_i^2 + \sum_i \mu_i\phi_i\Big] + o(|\tilde f|), \tag{26}$$

and
$$\dot{\rho} = \varepsilon\rho\{E[u^2(t)] - c^2\}. \tag{27}$$

Since, if f converges to a stabilizing control law R_s converges to a strictly positive, definite bounded matrix, it can be shown also that the parameter estimates $\Gamma(t) := \{\theta_i(t), \psi_i(t), \mu_i(t), \varphi_i(t)\}$ converge. Therefore, the only possible convergence points of CMUSMAR are given by the stable equilibrium points of (26) and (27). Therefore, the equilibria of (26) and (27) are given by:

(A) $\qquad \nabla\mathscr{L} = 0, \quad \rho = 0,$

(B) $\qquad \nabla\mathscr{L} = 0, \quad E[u^2(t)] - c^2 = 0.$

(A) Equilibria correspond to the extrema of the MS output cost for which the corresponding MS input is less than c^2. (B) Equilibria correspond to the extrema of the MS output cost on the boundary of the feasibility region $E[u^2(t)] \le c^2$.

4.1. *Stable* (A)*-equilibria.* We have the following result.

Proposition 1. Consider CMUSMAR with *any* controller complexity and *any* plant i/o transport delay smaller than T. Then, if T is large enough, among the (A)-equilibria, the only possible convergence points of CMUSMAR are the minima of the MS output value in the interior of the feasibility region $E[u^2(t)] \le c^2$.

Proof. The equations (26) and (27) are of the form
$$\dot{f} = G(f, \rho), \tag{28}$$
$$\dot{\rho} = \varepsilon H(f, \rho). \tag{29}$$

In order to find the locally stable equilibria of (28) and (29) the following Jacobian matrix is considered
$$J = \begin{bmatrix} \dfrac{\partial G}{\partial f} & \dfrac{\partial G}{\partial \rho} \\ \varepsilon\dfrac{\partial H}{\partial f} & \varepsilon\dfrac{\partial H}{\partial \rho} \end{bmatrix}. \tag{30}$$

the entries of the Jacobian matrix at the (A)-equilibria are given by
$$J_{|A} = \begin{bmatrix} -\dfrac{1}{\alpha}R_s^{-1}\nabla^2\mathscr{L} & \dfrac{\partial G}{\partial \rho}\Big|_A \\ 0 & \varepsilon\{E[u^2(t)] - c^2\} \end{bmatrix}.$$

This, being upper block triangular with $\alpha > 0$, $\varepsilon > 0$ and $R_s > 0$, turns out to be a stability matrix, whenever $\nabla^2\mathscr{L} > 0$ and $E[u^2(t)] < c^2$. □

4.2. *Stable* (B)*-equilibria.* Stability analysis of (B)-equilibria appears to be a difficult task since, in this case, the Jacobian matrix (30) need not be block diagonal. In such a case, we consider (28) and (29) for small, positive real ε. Then, (28) and (29) can be regarded as a singularly perturbed differential system (Wasov, 1965; Kokotović *et al.*, 1986) of which (28) and (29) describe the 'fast' and, respectively, the 'slow' states.

Hereafter, the interest is directed to the (B)-equilibria at which $\nabla^2\mathscr{L} > 0$, *viz.* (B)-equilibria which are isolated minima of the cost (8) for a fixed ρ_0. Any such a (B)-equilibrium point will be denoted by $\beta = [f_0'\rho_0]'$.

Since the plant to be regulated is linear and time invariant, and, at every β-point, the closed loop system is asymptotically stable, the next condition holds.

Condition 1. The functions G and H in (28) and, respectively, (29) are continuously differentiable in a neighbourhood of β. □

Since, for every β

$-\frac{1}{\alpha}R_s^{-1}$
$$\frac{\partial G}{\partial f}\Big|_\beta = \nabla^2\mathscr{L}|_\beta + O(\varepsilon),$$

where $\lim_{\varepsilon\to 0} O(\varepsilon) = 0$, for ε small enough,
$$\frac{\partial G}{\partial f}\Big|_\beta \cancel{>} 0. \tag{31}$$
$<$

Then Dini's Theorem assures that the following condition is satisfied.

Condition 2. In a neighbourhood of ρ_0, the equation $G(f, \rho) = 0$ has an isolated solution $f = f(\rho)$ and $f(.)$ is continuously differentiable. □

Setting $t := \varepsilon\tau$, (28) and (29) become:
$$\varepsilon\frac{\mathrm{d}f}{\mathrm{d}t} = G(f, \rho) \quad \text{and} \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = H(f, \rho). \tag{32}$$

Condition 3. Consider the *reduced system*
$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = H(f(\rho), \rho) = \rho[\bar{u}^2(\rho) - c^2]. \tag{33}$$

Then ρ_0, such that $\bar{u}^2(\rho_0) = c^2$, is an isolated equilibrium point at which (33) is exponentially stable. □

In order to prove Condition 3, it will be shown by Lemma 1 and Condition 1 that the following MS input *monotonicity property* holds
$$\frac{\partial H}{\partial \rho}\Big|_{\rho_0} = \frac{\partial \bar{u}^2(\rho)}{\partial \rho}\Big|_{\rho_0} < 0.$$

Lemma 1. Let $\bar{u}^2(\rho)$ be the MS input $\bar{u}^2(\rho) := E[u^2(\rho)]$ corresponding to an isolated minimum of the s.s.s. quadratic cost $\mathscr{L}(f, \rho)$ for a given ρ. Then, $\bar{u}^2(\rho)$ is a strictly decreasing function of ρ in a neighbourhood of ρ_0 as specified in Condition 2.

Proof. Let ρ_1 and $\rho_2, \rho_2 > \rho_1$, be in a suitably small neighbourhood of ρ_0. Let, according to Condition 2, $f_i = f(\rho_i) = \operatorname{argmin}_f \mathscr{L}(f, \rho_i)$, $i = 1, 2$. Further let $\bar{u}_i^2$, $\bar{y}_i^2$ denote the corresponding s.s.s. MS values of the input and the output, respectively. Then, one has
$$\bar{y}_1^2 + \rho_1\bar{u}_1^2 < \bar{y}_2^2 + \rho_1\bar{u}_2^2,$$
$$\bar{y}_2^2 + \rho_2\bar{u}_2^2 < \bar{y}_1^2 + \rho_2\bar{u}_1^2,$$
or equivalently,
$$\rho_2(\bar{u}_2^2 - \bar{u}_1^2) < \bar{y}_1^2 - \bar{y}_2^2,$$
$$\bar{y}_1^2 - \bar{y}_2^2 < \rho_1(\bar{u}_2^2 - \bar{u}_1^2).$$
Thus
$$\rho_2(\bar{u}_2^2 - \bar{u}_1^2) < \rho_1(\bar{u}_2^2 - \bar{u}_1^2).$$
Since, $\rho_2 > \rho_1$, it follows that $\bar{u}_2^2 - \bar{u}_1^2 < 0$. □

Condition 4. Consider, for fixed ρ, the boundary layer system
$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = G(f, \rho). \tag{34}$$

Then, (34) is exponentially stable at $f = f(\rho)$ uniformly in ρ in a suitable neighbourhood of ρ_0. □

Condition 4 is fulfilled by virtue of (31). In fact, (31) implies (Brauer and Nohel, 1966, Theorem 9.3) exponential stability at $f_0 = f(\rho)$. Next, (31), together with Condition 2 implies that there exists a suitably small neighbourhood of ρ_0 where the exponential stability referred to above is uniform in ρ.

Taking into account conditions 1–4, stability theory of singularly perturbed ODEs (Kokotović *et al.*, 1986, Corollary 7.2.3) yields the following conclusion.

Theorem 1. Let the control horizon T of CMUSMAR be large enough w.r.t. the time constants of the closed loop system, and $s(t)$ chosen so as to yield isolated minima of the cost. Then, there exists an $\bar{\varepsilon} > 0$ such that, for every positive $\varepsilon < \bar{\varepsilon}$, any feedback-gain solving Problem 1 (*viz.* minimizing the MS output value inside or along the boundary of the feasibility region $[Eu^2(t)] \le c^2$) is a possible convergence point of CMUSMAR.

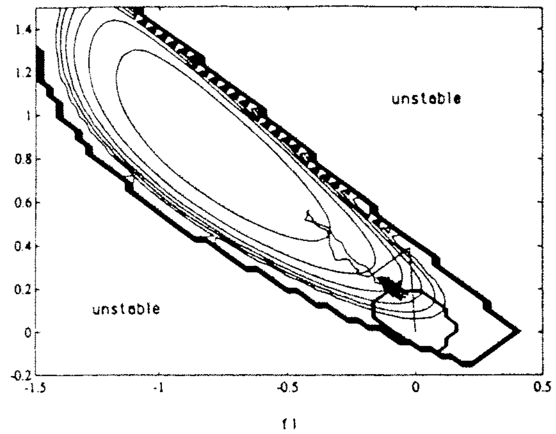

FIG. 1. Example 1. Superposition of the feedback with the level curves of $E[y^2(t)]$ and the allowed boundary $E[u^2(t)] = 0.1$.

5. *Simulation results*

Proposition 1 together with Theorem 1 suggest that CMUSMAR *may* possess good convergence properties. However, there is no guarantee that CMUSMAR will actually converge to the desired points. In order to explore this point, it is essential to resort to simulation experiments. In all the experiments $e(t)$ is a zero-mean, Gaussian, stationary sequence with $E[e^2] = 1$.

Example 1. CMUSMAR convergence properties are studied when the constrained minimum is different from the unconstrained one. Consider the nonminimum-phase stable fourth order plant, representing an approximate model of a flexible robot-arm, which keeps the first two resonance modes:

$$y(t+4) - 0.167)y(t+3) - 0.74y(t+2) - 0.132y(t+1) + 0.87y(t) = 0.132u(t+3) + 0.545u(t+2) + 1.117u(t+1) + 0.262u(t) + e(t+4),$$

and the restricted complexity controller

$$u(t) = f_1 y(t) + f_2 y(t-1).$$

The Lagrange multiplier ρ is initialized at a small value ($\rho = 0.0001$) and $T = 5$ is the control horizon used. Since ρ grows slowly, the feedback gains initially approach the unconstrained minimum of $E[y^2(t)]$. As ρ converges to its final value, the gains converge to a point close to the constrained minimum.

Figure 1 shows the superposition of the feedback gains with the level curves of $E[y^2(t)]$ and the boundary of the region defined by the restriction (6) with $c^2 = 0.1$. The adaptation gain $\varepsilon\gamma(t)$ was made to vary as $\gamma(t) = 1/(1 + t/N)$ with $N = 100$, for $t > 2000$.

Example 2. As referred before, to ensure that CMUSMAR has the constrained local minima of the steady-state (s.s.) LQ stochastic regulation cost as possible convergence points, the horizon T must be large enough. In this example a plant is used for which the control based on a single-step cost functional ($T = 1$) greatly differs from the s.s. LQ stochastic regulation. Consider the plant

$$y(t+3) - 2.75y(t+2) + 2.61y(t+1) + 0.855y(t) = u(t+2) - 0.5u(t+1) + e(t+3) - 0.2e(t+2) + 0.5e(t+1) - 0.1e(t),$$

and the full complexity controller defined by

$$s(t) = [y(t)y(t-1)y(t-2)u(t-1)u(t-2)u(t-3)]'.$$

As shown in Möden and Söderström (1982), using a *one-step* ahead control ($T = 1$) and taking ρ as a parameter, this plant gives rise to a relationship between $E[u^2(t)]$, and $E[y^2(t)]$, which is not monotone (Fig. 2). Instead, for LQ stochastic regulation the relationship is monotone as guaranteed by Lemma 1. It can be seen from Fig. 2 that, in this example, the single-step ahead constrained self-tuner has two possible equilibrium points denoted B and D in Fig. 2. Both of them correspond to the same value of the input variance but to quite different values of the output variance. These equilibria are attained depending on the initial conditions (Fig. 3).

This unpleasant phenomenon is eliminated by increasing the value of T in CMUSMAR (Fig. 4). In fact, the dotted line in Fig. 2, which exhibits a monotonic behaviour, is already obtained for $T = 2$.

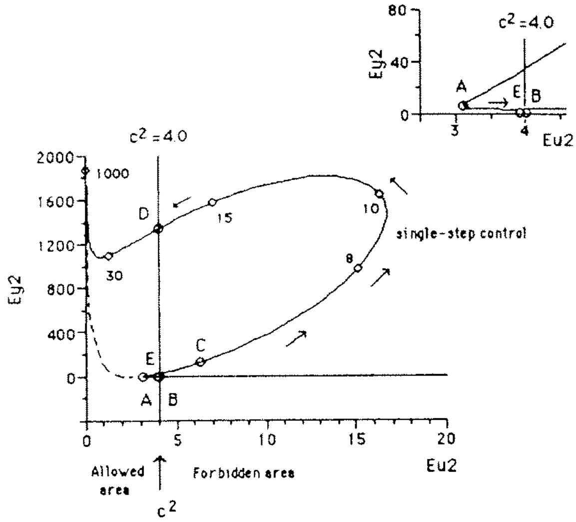

FIG. 2. Example 2. The relation between the input and output variances parametrized by ρ.

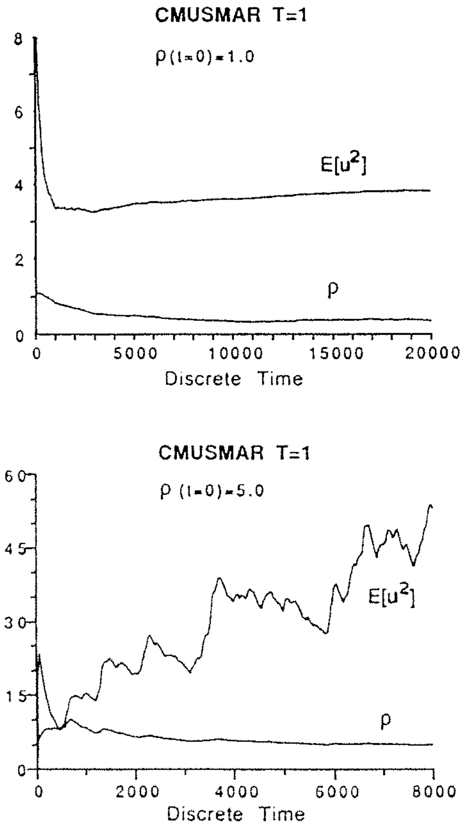

FIG. 3. Example 2. Time evolution of ρ and $E[u^2(t)]$ with two different initializations of ρ and single-step control.

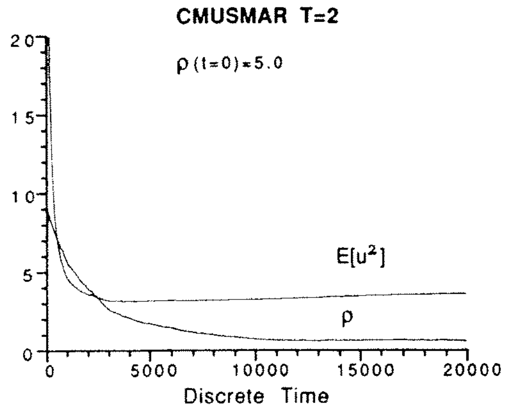

FIG. 4. Example 2. Time evolution of ρ and $E[u^2(t)]$ with ρ initialized at 5.0 and CMUSMAR with $T = 2$.

6. *Conclusions*

An adaptive control algorithm, the CMUSMAR (Constrained Multistep Multivariable Adaptive Regulator), for the minimization of the plant MS output value under the constraint of an upper bound on the MS input value, is considered. It results from the combination of MUSMAR, an adaptive control algorithm for the minimization of the steady-state LQ stochastic regulation cost, with a stochastic approximation scheme for adjusting the control weight such that the Kuhn–Tucker complementary condition is verified. Furthermore, CMUSMAR includes, as a special case, $T = 1$, the classic self-tuner based on the one-step-ahead constrained cost.

The above idea for obtaining constrained self-tuners can be extended to other adaptive predictive controllers as well. If model matching conditions are assumed, the present analysis can cover an MS input constrained version of the adaptive explicit criterion minimizer. The interest of considering CMUSMAR is that its convergence properties can be analysed via the ODE method. In this work it is proved that, when suitable provisions are taken, the local *constrained* minima of the underlying s.s.s. quadratic cost are possible convergence points of the algorithm. Note that unmodelled plant dynamics are allowed to be present. This leads to an *adaptive robustness* result.

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