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PERIODIC OSCILLATIONS IN SYSTEMS WITH HYSTERESIS

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This paper is dedicated to Professor Roberto Conti on his 65th birthday.

ABSTRACT. We give precise conditions under which the method of harmonic balance will correctly predict the existence of periodic solutions for a system with relay hysteresis. The equation modeling the system is assumed to be of the form $L_m[y](t) = f[y](t)$, $t \geq 0$, where L_m is a constant coefficient linear differential operator of order $m \geq 2$ and f is a possibly discontinuous operator with hysteresis.

1. Introduction. For a system of the form

$$L_m[y](t) = f[y](t), \qquad t \geq 0, \quad m \geq 2, \tag{1}$$

the method of harmonic balance is a heuristic method for predicting the existence of periodic solutions, see [**1, 7, 21, 23**]. Here, L_m is a constant coefficient linear differential operator of order $m \geq 2$ and $f[y]$ can be any well-defined mapping, not necessarily continuous, operating on functions from an appropriate space. When $f[y](t)$ is for each t just a nonlinear function of the number $y(t)$, there is an extensive literature concerned with general existence, uniqueness, and stability, see [**5, 6, 8, 9, 11, 12, 20, 24, 25, 26**], but only a few papers deal with the existence of periodic solutions, see [**10, 13, 19, 22**]. For an overview of the literature on hysteresis we refer to [**27**].

2. Modeling hysteresis. There are two types of hysteresis, relay (or passive or positive) hysteresis and active (or negative) hysteresis. In

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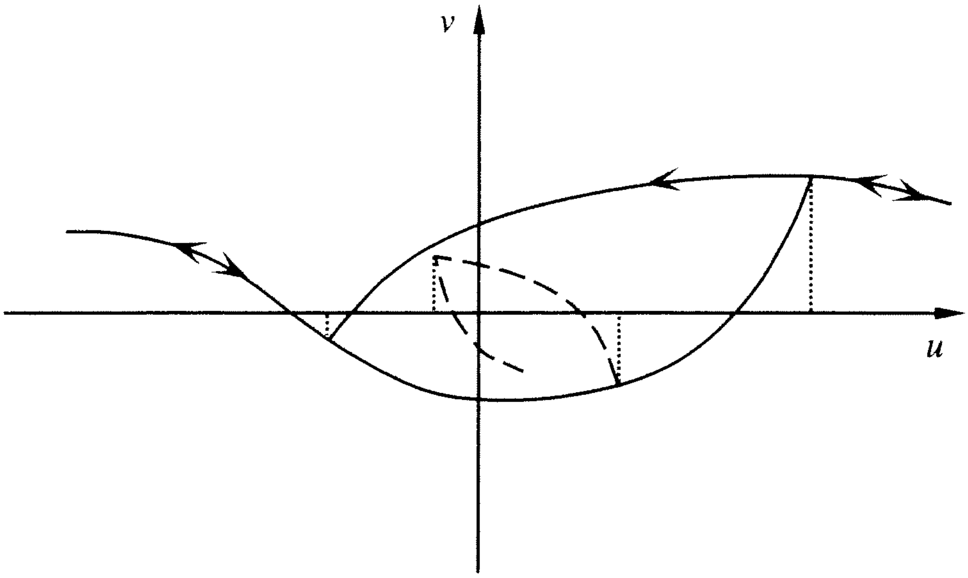

FIGURE 2.

Active hysteresis allows "trajectories" in the interior of the hysteresis region

$$H = \{(u, v) \mid \alpha \leq u \leq \beta, h_c(u) \leq v \leq h_o(u)\}.$$

In active hysteresis, which is perhaps best pictured as representing either magnetization or plastic deformations, if the input $u(t)$ changes from increasing to decreasing as $u(t)$ reaches a point $\gamma \in (\alpha, \beta)$ (see Figure 2), then the pair $(u(t), f[u](t))$ traces out an interior path in H. If $u(t)$ increases after it reaches the value δ, the trajectory $(u(t), f[u](t))$ traces out another interior path (dashed in Figure 2). Clearly this type of hysteresis postulates the existence of two families of curves inside H, one family for decreasing $u(\cdot)$, one family for increasing $u(\cdot)$, with certain reasonable regularity properties. Note that in the case of relay hysteresis, we would have $f[u](t) \equiv h_c(u(t))$ in the given example.

In the case of active hysteresis, the output of the hysteresis operator is effectively modelled as a particular selection from a compact-convex-valued upper semi-continuous multifunction $\mathcal{H}[u](t)$ which for a given $u(t) \in [\alpha, \beta]$ has the interval $[h_c(u(t)), h_o(u(t))]$ as its value. Thus this type of hysteresis is readily treated by methods of our earlier paper [**15**]. However, in the case of relay hysteresis, our previous results cannot be applied because the associated multifunction does not have convex values.

3. The existence of periodic solutions. In this section we convert the question of the existence of periodic solutions of the equation (1) $L_m[y](t) = f[y](t)$, with $f[y]$ a hysteresis operator of the form (2), into an abstract problem to which we can apply topological methods based on the results in [**14**]. We convert (1) to a vector equation in the usual way, defining $x(t) = [y(t), y'(t), \dots, y^{(m-1)}(t)]^{\mathrm{tr}}$ (tr = transpose),

$$\dot{x}(t) = Ax(t) + F[x](t), \qquad x(0) = x(T), \tag{3}$$

T unknown, $F[x] = [0, \dots, 0, f[x]]^{\mathrm{tr}}$. We normalize the unknown period to 2π by replacing t by $(2\pi/T)t$, resulting in the following problem for $z(t) = x(\omega t)$, $\omega = T/2\pi$:

$$\omega\dot{z}(t) - Az(t) = F[z](t), \qquad 0 \le t \le 2\pi, \qquad z(0) = z(2\pi). \tag{4}$$

We will assume below that in a suitable domain the Green's function exists for the left side of (4) with the given periodic boundary conditions, in which case it is given by

$$G_\omega(t-s) = (1/\omega)[I - e^{2\pi A/\omega}]^{-1} \begin{cases} e^{A(t-s)/\omega}, & 0 \le s < t; \\ e^{A(2\pi+t-s)/\omega}, & t \le s \le 2\pi. \end{cases}$$

In this case (4) is equivalent to

$$z(t) = \int_0^{2\pi} G_\omega(t-s)F[z](s)\,ds \equiv (\mathcal{G}_\omega \circ F[z])(t). \tag{5}$$

Let $W^{1,2}((0,2\pi), R^m)$ be the usual Sobolev space of absolutely continuous R^m-valued functions defined on $[0, 2\pi]$ with derivative in $L^2((0, 2\pi), R^m)$, and let Z denote the subspace of functions $z(\cdot)$ for which $z(0) = z(2\pi)$, extended to R by 2π-periodicity. Let X denote the space of functions which belong to $C([0, 2\pi], R^m)$, extended to R by 2π-periodicity. It is clear that $\mathcal{G}_\omega : L^2((0, 2\pi), R^m) \to Z$, for any $\omega > 0$.

For each $x(\cdot) \in x$ the associated Fourier expansion

$$x(t) \to a_0 + \sum_{k=1}^{\infty} (a_k e^{ikt} + \bar{a}_k e^{-ikt})$$

converges in the $L^2((0,2\pi),R^m)$ norm to $x(\cdot)$, and we can define the projection operator

$$P_n : X \to \operatorname{span}_{\mathbf{C}}\{1, e^{it}, e^{-it}, \dots, e^{\text{int}}, e^{-\text{int}}\},$$
$$P_n : x(t) \mapsto a_0 + \sum_{k=1}^{n}(a_k e^{ikt} + \bar{a}_k e^{-ikt}) \equiv P_n[x](t).$$

For the given hysteresis operator $F[u] = (0,\dots,0,f[u])^{\text{tr}}$ with $f(u)$ defined by (2), we define a set-valued map $\mathcal{F}[u](t)$ which for $u(t) \neq \alpha,\beta$ coincides with $F[u](t)$, i.e., it is the singleton $\{(0,\dots,0,f[u](t))\}$. For $u(t) = \alpha\,(\beta)$, $\mathcal{F}[u](t)$ is the interval $[h_c(\alpha),h_o(\alpha)]$ (respectively, $[h_c(\beta),h_o(\beta)]$). The associated Nemytskii operator $\mathcal{F}[u]$ is defined as all measurable selections $v(\cdot)$ from $\{\mathcal{F}[u](t) \mid 0 \le t < \infty\}$, which satisfy $v(t) = v(t+2\pi)$ a.e. on $[0,\infty]$. Therefore $\mathcal{F} : X \to \{U \mid U \subset L^\infty((0,2\pi),R^m)\}$.

For our purposes, the input $u(\cdot)$ will take on the values α,β at most a finite number of times and $u'(t)$ will be nonzero at these instants. In this case any function in $\mathcal{F}[u](\cdot)$ will consist of a fixed finite number of pieces of $h_o(\cdot)$ and $h_c(\cdot)$, with arbitrary values from $[h_c(t),h_o(t)]$ at the jump points. Thus, $\mathcal{G}_\omega \circ \mathcal{F}[u]$ will be in fact piecewise C^1 for $m \ge 2$.

Continuing our abstract formulation from (5), we define $T_\omega = i \circ \mathcal{G}_\omega \circ \mathcal{F} : X \to 2^X$ where i is the natural imbedding of Z into X. We note that i is a compact imbedding, and we will not bother to write it in what follows. We can now rewrite (5) as

$$0 \in (I - T_\omega)[z]. \tag{6}$$

If we set $X_n = P_nX$, $X^* = (I-P_n)X$, $z_n(t) = P_n[z](t)$, $z^*(t) = (I-P_n)[z](t)$, then we can create the following homotopy for $z \in X$, $0 \le \lambda \le 1$:

(6_λ)
$$\begin{cases} \text{(a)} \quad 0_n \in (I - P_nT_\omega)P_n[z] + \lambda\{P_n(I-T_\omega) - (I-P_nT_\omega)P_n\}[z], \\ \text{(b)} \quad 0^* \in (I-P_n)(I-\lambda T_\omega)[z]. \end{cases}$$

When $\lambda = 0$, (6_0) is the equation of harmonic balance, while for $\lambda = 1$ we obtain an equation which is homotopic to the original problem (5) (see [**15**, Remark 3]).

Our approach is to assume that the harmonic balance equation has a solution $(\tilde{\omega}, \tilde{z}_n)$ with nonzero degree, then place assumptions on the problem (see (A1)–(A5) below) which allow us to use the homotopy (6_λ) to conclude that the degree is nonzero when $\lambda = 1$, so the original problem has a solution.

There are two technical complications which must be addressed. The first stems from the fact that the problem is autonomous, thus the time origin is not fixed—if $z(t)$ solves the system (4), for some $\omega > 0$, so does $z(t+\theta)$ for any $\theta \in R$. This means that the Fourier coefficients a_k are not uniquely determined, i.e., if $(a_0, a_1, \dots, a_n)$ are the coefficients of $z(t)$, then $(a_0, a_1 e^{i\theta}, \dots, a_n e^{i\theta})$ are the coefficients of $z(t+\theta)$. To select a single-valued branch of solutions for $0 \le \lambda \le 1$, we can arbitrarily (without loss of generality) select a coefficient a_{i_0} $(i_0 \ge 1)$, which has one component, $a_{i_0 j_0}$ with $\mathrm{Im}\,(a_{i_0 j_0}) = 0$, $\mathrm{Re}\,(a_{i_0 j_0}) \ne 0$. We then "normalize" by keeping $a_{i_0 j_0}$ real-valued as (ω, x_n) vary by introducing the map $V_\lambda : R_+ \times X_n \times X^* \to R \times 2^{X_n} \times 2^{X^*}$ defined by $V_\lambda = (V_\lambda^a, V_\lambda^b)$, $0 \le \lambda \le 1$, where

$$V_\lambda^a = (\arg a_{i_0 j_0}(\omega, z_n), \text{ right side of } (6_\lambda)(a))),$$
$$V_\lambda^b = \text{right side of } (6_\lambda(b)),$$

and replace (6_λ) by

$$(H_\lambda) \qquad \begin{cases} \text{(a)} & 0 \in V_\lambda^a[\omega, z_n, z^*], \\ \text{(b)} & 0 \in V_\lambda^b[\omega, z_n, z^*]. \end{cases}$$

We will ask that $\deg(V_0^a, \Omega_n, 0) \ne 0$ for an appropriate region $\Omega_n \subset R_+ \times X_n$ of (ω, z_n)-space containing $(\tilde{\omega}, \tilde{z}_n)$.

The remaining technical difficulty is due to the fact that the equations $0 \in V_1$, (6_λ) and (6) are not entirely equivalent because we are dealing with inclusions rather than equations (e.g., $A - A$ is not in general $\{0\}$ for a set A). However, an easy argument shows that if the degree is nonzero for any one of them, then it is nonzero for the others [**15**, Remark 3].

Below, $\|\cdot\|$ will denote the norm in $L^\infty((0, 2\pi), R^m)$. Obviously, it reduces to the sup norm on X. We are now in a position to prove the following:

Theorem. *If Assumptions* (A1)–(A5) *below hold, then the nonlinear system* $L_m[y](t) = f[y](t)$, $t \geq 0$, $m \geq 2$, *with* f *the relay hysteresis operator defined by* (2), *has a nontrivial periodic solution.*

(A1) *The continuous functions* $h_o(u)$, $h_c(u)$ *in* (2) *both satisfy* $|h(u)| \leq \eta_1|u| + \eta_2$ *for some constants* $\eta_1 > 0$, $\eta_2 \geq 0$;

(A2) *Let* $\mathcal{A}_n \subset R_+ \times X_n$ *be the open set satisfying the following assumptions*

(h_1) *For all* $(\omega, z_n) \in \mathcal{A}_n$ *there exists* $r_1 = r_1(\omega, z_n) > 0$ *such that*

$$\sqrt{2\pi}\left[\sum_{|k|\geq n+1} |\hat{G}_\omega(ik)|^2\right]^{1/2} \sup_{||z^*||<r_1} \sup_{y\in\mathcal{F}[z_n+z^*]} ||(I - P_n)y|| < r_1(\omega, z_n)$$

where $\hat{G}_\omega(ik)$ *is the Fourier transform of the matrix* $G_\omega(t)$ *evaluated at* ik.

(h_2) *For all* $(\omega, z_n) \in \mathcal{A}_n$ *we have* $||(I - P_nT_\omega)[z_n]|| < \sigma(\omega, z_n)$ *where*

$$0 < \sigma(\omega, z_n) = \sqrt{2\pi}\left[\sum_{|k|=0}^{n} |\hat{G}_\omega(ik)|^2\right]^{1/2} \sup_{||z^*||<r_1} ||P_n\mathcal{F}[z_n + z^*] - P_n\mathcal{F}[z_n]||$$

and equality holds on $\partial\mathcal{A}_n$.

The term $|\hat{G}_\omega(ik)|^2$ *is the sum of the squares of the entries, and the last "set-valued norm" on the right is defined by the usual convention:*

$$\sup ||w - w_n|| \mid w \in P_n\mathcal{F}[z_n + z^*],\ w_n \in P_n\mathcal{F}[z_n]\}.$$

Note that $(I - P_nT_\omega)[z_n]$ *is set-valued. Its norm is defined by the same convention.*

Let Ω_n *be the connected component of* $\mathcal{A}_n$ *containing the solution* $(\tilde{\omega}, \tilde{z}_n)$ *of the harmonic balance equation*

$$(6_0) \qquad O_n \in (I - P_nT_\omega)[z_n],$$

for which $\arg a_{i_0j_0}(\bar{\omega}, \tilde{z}_n) = 0$. Let $\Omega = \{(\omega, z_n, z^*) \mid (\omega, z_n) \in \Omega_n, ||z^*|| < r_1\}$.

(A3)(i) $I - e^{2\pi A/\omega}$ *is invertible whenever* ω *is such that*

$$(\omega, z_n) \in \Omega_n \quad \text{for some } z_n;$$

(ii) $(\omega, 0) \notin \overline{\Omega}_n$ *for any* $\omega \in R_+$. *Moreover, the function* $(\omega, z_n) \to a_{i_0 j_0}(\omega, z_n)$ *has real part different from zero in* Ω_n.

(iii) $\deg(V_0^a, \Omega_n, 0)$ *is well defined and different from zero.*

Here $V_0^a = (\arg a_{i_0 j_0}(\omega, z_n), (I - P_n T_\omega)[z_n])$.

(A4) *There is a solution in* Ω_n *of the harmonic balance equation,* $(\tilde{\omega}, \tilde{z}_n)$ *such that, if we denote the* m*th component* $(\tilde{z}_n(t))_m$ *by* $\tilde{y}(t)$,

(a) $\inf_{[0,2\pi]} \tilde{y}(t) < \alpha < \beta < \sup_{[0,2\pi]} \tilde{y}(t)$

(b) *For all* t_i *such that* $\tilde{y}(t_i) = \alpha$ *or* β, $\tilde{y}'(t_i) \neq 0$.

(A5) *There exists* $\varepsilon_0 > 0$ *such that*

(a) $\Omega \subset R_+ \times \operatorname{cl} B(\tilde{z}_m, \varepsilon_0)$ *where* $\operatorname{cl} B$ *is the closure of the ball of radius* ε_0 *about* $\tilde{z}_n$ *in* X.

(b) $z \in \operatorname{cl} B(\tilde{z}_n, \varepsilon_0) \Rightarrow y(t) \equiv (z(t))_m$ *satisfies* (A4)(a),(b).

Remark 1. (A1) guarantees that $\mathcal{F}$ satisfies the linear growth condition, (A2) is the "low-pass filter" assumption [**2**], i.e., it implies that the linear part of the system as represented by $\mathcal{G}_\omega$ attenuates high frequencies.

Remark 2. The assumption (A3)(i) is not essential, since any given system can be "pole shifted" to an equivalent system satisfying (A3)(i). To see this, note that the eigenvalues of $(e^{2\pi A/\omega} - I)$ are $(e^{2\pi\lambda/\omega} - 1)$ where λ is an eigenvalue of A. Thus, (A3)(i) will hold if $\operatorname{Re}\lambda_k \neq 0$ for all the eigenvalues λ_k of A. This can always be achieved by using the equivalent system $\dot{x}(t) - (A - \gamma I)[x](t) \in \mathcal{F}[x](t) + \gamma I x(t)$. For some real γ this system will satisfy (A3)(i) and our hypotheses will not be affected. In fact, in most real-world situations, one has $\operatorname{Re}\lambda_k < 0$ for all λ_k. Assumption (A3)(ii) guarantees that the trivial solution is excluded when we apply a fixed point theorem or degree theory. Assumption (A3)(iii) implies in particular that when $\lambda = 0$ there is a solution $(\tilde{\omega}, \tilde{z}_n)$ of (H_0). Therefore, if the homotopy (H_λ) is admissible in $\overline{\Omega}$, where $\overline{\Omega}$ is the closure of the set Ω, (i.e., the set $S = \{(\omega, z) \in R_+ \times X \mid (\omega, z)$ is a solution of (H_λ) for some $\lambda \in [0, 1)\}$ does not meet $\partial\Omega$), then

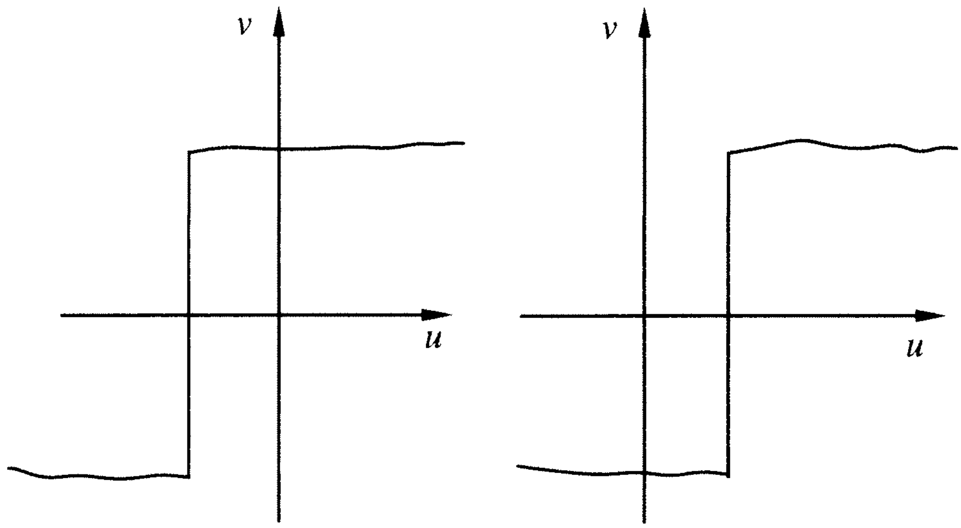

FIGURE 3.

the homotopy invariance property and the solution property of the topological degree guarantee the conclusion of the Theorem.

Remark 3. (A4) and (A5) guarantee that $\mathcal{F}$ is well-behaved on Ω. (A4) postulates that the trigonometric polynomial $\tilde{y}(t)$ crosses the thresholds whenever it attains them. Thus there will certainly exist $\varepsilon_0 > 0$ such that $z \in B(\tilde{z}_n, \varepsilon_0)$ will have the same property. Assumption (A5) postulates that ε_0 can be chosen large enough to ensure that the projection of Ω into X is contained in $\mathrm{cl}\, B(\tilde{z}_n, \varepsilon_0)$.

Proof. For simplicity, we assume $\tilde{y}(0) < \alpha$, so the hysteresis operator is easy to picture for all $t \geq 0$. Since the values of $\mathcal{F}$ are selected from the two functions in Figure 3, it is easy to show that $i \circ \mathcal{G}_\omega \circ \mathcal{F}$ is upper semicontinuous and compact (takes bounded sets from $\mathrm{cl}\, B(\tilde{z}_n, \varepsilon_0)$ into compact sets in X).

Let $z \in \mathrm{cl}\, B(\tilde{z}_n, \varepsilon_0)$. Then $(z(0))_m < \alpha$ and $(z(t))_m$ will cross a threshold α or β at most a finite number of times, say $t_1 < \cdots < t_p$, for example, $(z(t_1))_m = \alpha$, $(z(t_2))_m = \beta$ (jump), $(z(t_3))_m = \beta$ (no jump), $(z(t_4))_m = \alpha$ (jump), and so on. Then every $v(\cdot) \in \mathcal{F}[z]$ will have the form

$$v(t) = \begin{cases} h_c((z(t_1))_m), & 0 \le t < t_2 \\ p_1, & p_1 \in [h_c(\beta), h_o(\beta)], \quad t = t_2, \\ h_o((z(t_1))_m), & t_2 < t < t_4; \\ p_2, & p_2 \in [h_c(\alpha), h_o(\alpha)], \quad t = t_4; \\ \text{etc.} \ldots . \end{cases}$$

It is clear that the set of all possible such $v(\cdot)$ is closed and convex in X. Therefore, the operator $T_\omega = i \circ \mathcal{G}_\omega \circ \mathcal{F}$ maps X into 2^X and is compact, upper semicontinuous with closed, convex values and satisfies a linear growth condition (see Remark 1).

We will now prove existence by a two-stage argument. First, for each $(\omega, z_n) \in \Omega_n$ and each $0 \le \lambda \le 1$, we will show that we can apply the Schauder fixed point theorem for set-valued maps in the ball $B(0, r_1)$ to get a solution $z^*_\lambda(\omega, z_n)$ of $H_\lambda(b)$. Thus, $H_\lambda(a)$ becomes an equation in ω and z_n, when we replace z^* by $z^*_\lambda(\omega, z_n)$. We will then show that the original equation (1) has a solution by showing that H_λ represents an admissible homotopy in $\overline{\Omega}$ (see Remark 2).

We turn to the first stage. It is clear that for a given (ω, z_n) in Ω_n, $H_\lambda(b)$ represents a fixed-point problem $z^* \in M_\lambda[z^*]$ for the map $M_\lambda : z^* \mapsto \lambda(I - P_n)T_\omega[z^* + z_n]$ on $(I - P_n)X$. If $||z^*|| < r_1$, then (recalling that for a multifunction $H[u]$, $||H[u]|| = \sup_{v \in H(u)} ||v||$):

$$\begin{aligned} ||M_\lambda[z^*]|| &\equiv ||\lambda(I - P_n)T_\omega[z^* + z_n]|| \le ||\mathcal{G}_\omega(I - P_n)\mathcal{F}[z^* + z_n]|| \\ &\le \sqrt{2\pi}\left\{ \sum_{|k| \ge n+1} |\hat{G}_\omega(ik)|^2 \right\}^{1/2} \sup_{||z^*|| \le r_1} \ \sup_{v \in \mathcal{F}[z^* + z_n]} ||(I - P_n)[v]|| \\ &< r_1 \end{aligned}$$

by (A2)(h_1), so $M_\lambda : B(0, r_1) \to 2^{B(0,r_1)}$ in $(I - P_n)X$. Now the Schauder fixed-point theorem for multivalued maps is valid for any upper semi-continuous, compact multifunction with convex compact values, and as we have already noted, M_λ is such a map for any $\lambda \in [0, 1]$. Therefore, there exists a fixed point $z^*_\lambda(\omega, z_n)$ of the map M_λ, for each $(\omega, z_n) \in \Omega_n$, $0 \le \lambda \le 1$.

Under our assumptions the topological degree of the compact convex-valued vector field $V_\lambda : (\omega, z) \mapsto V_\lambda[(\omega, z)]$ on Ω is defined for $\lambda \in [0, 1]$.

We can now show that $0 \notin V_\lambda[(\omega, z)]$ for $(\omega, z) \in \partial\Omega$, $0 \leq \lambda < 1$, thus by the homotopy invariance property of the topological degree $0 \neq \deg(V_0, \Omega, 0) = \deg(V_\lambda, \Omega, 0)$ for $\lambda \in [0, 1)$, which implies the existence of a nontrivial $2\pi\omega$ periodic solution of (1) in $\overline{\Omega}$.

Suppose there exists $(\omega, z_n, z^*) \in \partial\Omega$ such that $0 \in V_\lambda[\omega, z_n, z^*]$ for some $\lambda \in [0, 1)$ where z^* stands for $z^*_\lambda(\omega, z_n)$. Then, in particular, we would have selections $v_n \in \mathcal{F}[z_n]$ and $v, \bar{v} \in \mathcal{F}[z]$ such that at least one of the following inequalities holds as equality:

$$(7a) \qquad ||z_n - P_n \mathcal{G}_\omega v_n|| \leq \sigma(\omega, z_n)$$

$$(7b) \qquad ||z^*|| \leq r_1(\omega, z_n)$$

and for which

$$(8a) \qquad 0_n = (z_n - P_n \mathcal{G}_\omega v_n) + \lambda[(z_n - P_n \mathcal{G}_\omega v) - (z_n - P_n \mathcal{G}_\omega v_n)]$$

$$(8b) \qquad 0^* = (I - P_n)(z - \lambda \mathcal{G}_\omega \bar{v}).$$

Equations (8a)–(8b) imply that

$$0 \geq ||z_n - P_n \mathcal{G}_\omega v_n|| - \lambda ||P_n \mathcal{G}_\omega (v - v_n)||$$
$$0 \geq ||z^*|| - \lambda ||(I - P_n) \mathcal{G}_\omega \bar{v}||,$$

respectively. From these inequalities we obtain

(9a)

$$0 \geq ||z_n - P_n \mathcal{G}_\omega v_n|| - \lambda ||P_n \mathcal{G}_\omega (v - v_n)|| \geq ||z_n - P_n \mathcal{G}_\omega v_n|| - \lambda \sigma(\omega, z_n)$$

$$(9b) \qquad 0 \geq ||z^*|| - \lambda r_1(\omega, z_n)$$

where the last estimates on the right hand side of (9a) and (9b) are obtained by the usual Fourier expansion techniques, i.e.,

$$||P_n \mathcal{G}_\omega (v - v_n)|| \leq \sqrt{2\pi} \left[\sum_{|k|=0}^{n} |\hat{G}_\omega(ik)|^2 \right]^{1/2} ||v - v_n|| \leq \sigma(\omega, z_n)$$

$$||(I - P_n) \mathcal{G}_\omega \bar{v}|| \leq \sqrt{2\pi} \left[\sum_{|k| \geq n+1} |\hat{G}_\omega(ik\omega)|^2 \right]^{1/2} ||\bar{v}|| \leq r_1(\omega, z_n)$$

and by using (A2)(h_2) and (A2)(h_1), respectively. But $\lambda \in [0,1)$, hence from (9a)–(9b) we obtain

$$0 > \|z_n - P_n \mathcal{G}_\omega v_n\| - \sigma(\omega, z_n)$$
$$0 > \|z^*\| - r_1(\omega, z_n).$$

Therefore, neither (7a) nor (7b) can reach equality, contradicting the fact that $(\omega, z_n, z^*) \in \partial\Omega$ for some $\lambda \in [0,1)$. □

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