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VARIABLE STRUCTURE CONTROL PROBLEMS AND THE THEORY OF SINGULAR PERTURBATIONS

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Abstract *We use the classical theory of singularly perturbed systems to design a dynamical feedback controller which allows us to solve control problems, involving slidings manifolds, within prescribed approximation error. The dynamical feedback controller is the solution of a differential equation containing a small parameter $\epsilon > 0$. The equation is derived from the data of the considered problem. The controller turns out to be an absolutely continuous function and so the chattering phenomenon is practically eliminated. We investigate the relationship between the controller and the equivalent control. We also study the behaviour of the controlled response of the system to disturbances.*

§1. Introduction

We consider a nonlinear control system described by the differential equations

$$\dot{x} = f(t,x,u) \tag{1}$$

where f satisfies Carathéodory-type conditions to be specified later. The state variable x belongs to $\mathbb{R}^n$ and the control variable u belongs to U, where U is a given subset of $\mathbb{R}^m$ and $m \leq n$.

An important class of control systems modelled by equations (1) is given by so-called variable structure control problems. The idea is the following. Consider a sliding manifold

$$S = \{x \in \mathbb{R}^n \mid s(x) = 0\} \tag{2}$$

where the function $s:\mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable function. Given a time interval $[0,T]$ with $0 < T \leq \infty$, we want to steer and then hold the state vector x on the sliding manifold S by using feedback control laws $u = u(t,x) \in U$ which are discontinuous along the surfaces

$$s_j(x) = 0 \quad (j = 1, 2, \ldots, m).$$

Such control systems have very good properties; they exhibit stable behaviour, accurate tracking, robust performance, and insensitivity with respect to disturbances and variation of plant parameters.

Interesting applications to the control of dynamical systems with deterministic uncertainty have been considered in [1], [2], [7], and [8]. The main drawbacks of these systems are the chattering phenomenon and the necessity of considering a generalized notion of solution for (1) (usually that due to Filippov [4]) instead of the classical solution concept of Carathéodory.

Indeed, extra work and extra conditions on f are required when we have to relate the Carathéodory conditions corresponding to the "equivalent control" (which is a Carathéodory function: see Definition 2.1 below) to the Filippov solutions which lie on the sliding manifold S and correspond to discontinuous feedback controls (see [3]).

In this paper we introduce a different approach to the control problem (1)-(2): we aim at solving it "approximately". This approach is based on the consideration of a different class of feedback controls which we will introduce via the theory of singularly perturbed ordinary differential equations. Such controls will depend on a small parameter $\epsilon > 0$ whose corresponding states will not realize, in general, exactly the sliding condition (2). However, for any prescribed neighborhood of S, we can determine values of the parameter ϵ for which the corresponding trajectories of equations (1) belong to that neighborhood. Furthermore, by using this class of controls, we eliminate both chattering and the necessity of considering Filippov solutions.

In [2] and [12], two different methods are proposed to eliminate the chattering phenomenon. In [2] the problem of asymptotic tracking of a deterministically uncertain system is solved using differential inequalities and variable structure control methods.

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One considers a suitable augmented system for which, via application of the method outlined in [7] and [10], one obtains the desired asymptotic behaviour without chattering. One uses a continuous feedback control law which solves a differential inequality of Hamilton-Jacobi type.

In [12] on the other hand, Sira Ramirez uses Fliess' s generalized observability canonical form to obtain an implicit ordinary differential equation with discontinuous right-hand side for the dynamical feedback control law. Thus the control law turns out to be at least absolutely continuous.

This control law solves the problem of making the output tracking error function (relative to a nonlinear dynamical system) tend asymptotically to zero. Therefore, since the controller is at least absolutely continuous, the chattering of the controlled responses is considerably reduced.

Returning to the class of controls which we introduce, another of their features is that, upon passing to the limit as $\epsilon \to 0$, they converge to the equivalent control as defined (for different classes of control problems) in [3] and [13].
Moreover, under suitable conditions, the corresponding states converge to the so-called ideal state determined by the equivalent control. By means of the equivalent control, we also give a suitable definition of the approximability property for the fully nonlinear system (1). This property describes the behaviour of system (1), controlled by the dynamical feedback control, with respect to the presence of a certain class of perturbations.

We remark that [3] contains a definition of approximability for system (1)-(2), with $u(t,x) \in U$, and some particular cases of systems (1) which fulfill the approximability property are presented. In [9] it is shown that, if we restrict the class of admissible perturbations to an appropriate subclass, then the approximability property of [3] is satisfied also by the fully nonlinear control system (1).

Finally, we will show how the proposed approach works for several control problems modelled by (1). Specifically, we will consider both controllability problems in a finite time interval and tracking problems in an infinite time interval.

§2. The Dynamical Feedback Control

We make the following assumption on the dynamics f.

(f1) For each $(p,q) \in \mathbb{R}^n \times \mathbb{R}^m$, the map $t \to f(t,p,q)$ is Lebesgue measurable on $[0,T]$. In addition, for almost all $t \in [0,T]$, the map $(p,q) \to f(t,p,q)$ is continuous in $\mathbb{R}^n \times \mathbb{R}^m$.

(f2) For each $\rho > 0$, there exists $\gamma_\rho \in L^1([0,T],\mathbb{R}_+)$ such that, for almost all $t \in [0,T]$ and every (p,q) with $|p| + |q| \leq \rho$ one has

$$|f(t,p,q)| \leq \gamma_\rho(t) .$$

For our purposes, it is convenient to introduce a sliding manifold S which depends also on the time $t \in [0,T]$. That is, we consider

$$S = \{(t,x) \in [0,T] \times \mathbf{R}^n \mid s(t,x) = 0\} .$$

where $s : [0,T] \times \mathbf{R}^n \to \mathbf{R}^m$, $m \leq n$, is a continuously differentiable function. We assume that

(H1) for each $t \in [0,T]$, there exists $x \in \mathbf{R}^n$ such that $(t,x) \in S$.

For each $\epsilon > 0$, consider the following set of differential equations:

$$\begin{aligned} \dot{x} &= f(t,x,u) \\ \epsilon \dot{u} &= \frac{\partial}{\partial t} s(t,x) + \frac{\partial}{\partial x} s(t,x)\, f(t,x,u) \end{aligned} \tag{3}$$

where $t \in [0,T]$.
Define

$$g(t,x,u) = \frac{\partial}{\partial t} s(t,x) + \frac{\partial}{\partial x} s(t,x)\, f(t,x,u) .$$

We make the following additional assumption.

(H2) There exists a neighborhood I of the manifold S such that, for every $(t,x) \in I$, the map

$$u \to g(t,x,u)$$

is one-to-one on U and its range contains zero.

Definition 2.1. The unique solution (if it exists) $u \in U$ of the algebraic equation

$$g(t,x,u) = w$$

for a given $w \in \mathbb{R}^n$ will be denoted by $u^*(t,x,w)$. When $w = 0$, the map $(t,x) \to u^*(t,x,0)$ or simply $u^*(t,x)$ is called the equivalent control for the system (1)-(2).

The notion of equivalent control was introduced in [13] and was recast in the above form in [3].

Observe that, by hypothesis (H2), for each $(t,x) \in I$ the equilibrium point $u^* = u^*(t,x)$ is isolated.

Furthermore we assume

(H3) there exists $\mu > 0$ such that, if $(t,x) \in I$, $|v - u^*(t,x)| < \mu$, and $v \neq u^*(t,x)$, then $g(t,x,v) \neq 0$;

(H4) for any $(x_0,u_0) \in \mathbb{R}^n \times \mathbb{R}^m$ and any $\epsilon \geq 0$, system (3) has a unique solution $(x(t,\epsilon),u(t,\epsilon))$ defined in the interval $[0,T]$ such that $(x(0,\epsilon),u(0,\epsilon)) = (x_0,u_0)$;

(H5) the equilibrium point $\bar{u}_0 = u^*(t_0, x_0)$ of the equation $\epsilon\dot{u} = g(t,x,u)$ is asymptotically stable for all $(t_0, x_0) \in I$. In other words, the solution of the differential equation

$$\dot{z} = g(t_0, x_0, z)$$

corresponding to the initial condition $z(0) = u_0$ converges asymptotically to the point $\bar{u}_0 = u^*(t_0, x_0)$ whenever u_0 is sufficiently close to $\bar{u}_0$ and $(t_0, x_0) \in I$. Moreover we assume that the asymptotic stability is uniform in $(t_0, x_0) \in I$.

Definition 2.2. A point $(t_0, x_0, u_0) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m$ such that the solution of the Cauchy problem

$$\begin{aligned} \dot{z} &= g(t_0, x_0, z) \\ z(0) &= u_0 \end{aligned}$$

satisfies (H5) is said to belong to the domain of influence of $\bar{u}_0 = u^*(t_0, x_0)$.

Remark 2.3. Observe that, if $(x_0(t), u_0(t))$ $(0 \le t \le T)$ is a solution of system (3) when $\epsilon = 0$, then

$$g(t, x_0(t), u_0(t)) = \frac{d}{dt} s(t, x_0(t)) = 0$$

where $u_0(t) = u^*(t, x_0(t))$ and $0 \le t \le T$. Therefore, if $s(t_0, x_0(t_0)) = 0$ for some $t_0 \in [0,T]$, then $s(t, x_0(t)) = 0$ for all $t \in [0,T]$, and consequently the graph of $x_0 = x_0(t)$ lies on the manifold S.

From the classical theory of singular perturbations we have the following result (e.g. [14]).

Theorem 2.4 Suppose that assumptions (f1)-(f2) and (H1)-(H5) are satisfied. Let $(0, x_0, u_0) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m$ be a point in the domain of influence of $u_0 = u^*(t_0, x_0)$. Then the solution $(x(t,\epsilon), u(t,\epsilon))$ of the Cauchy problem

$$\begin{aligned} (\dot{x}, \epsilon\dot{u}) &= (f(t,x,u), g(t,x,u)) \\ (x(0), u(0)) &= (x_0, u_0) \end{aligned}$$

has the following properties:

$$\lim_{\epsilon \to 0} x(t,\epsilon) = x_0(t) \qquad \text{uniformly in } [0,T] ,$$

$$\lim_{\epsilon \to 0} u(t,\epsilon) = u_0(t) \qquad \text{uniformly in } [t_1,T]$$

whenever $0 < t_1 < T$. Here $(x_0(t), u_0(t))$ is the solution of the reduced system

$$\begin{aligned} \dot{x} &= f(t,x,u) \\ x(0) &= x_0 \\ 0 &= g(t,x,u) \end{aligned}$$

so that in particular $u_0(t) = u^*(t, x_0(t))$.

In the sequel we will refer to $u(t,\epsilon)$ $(\epsilon > 0)$ as the dynamical feedback control law corresponding to the state $x(t,\epsilon)$.

Explicit conditions ensuring that (H2) is satisfied can be found in [11], while conditions ensuring (H5) are given in [6]. Specifically, if we wish to employ the first or second Lyapounov method, we can give explicit conditions on g in order to obtain the uniform asymptotic stability of the equilibrium points $u^*(t,x)$ required by hypothesis (H5). Perhaps the simplest such condition is that $g(t,x,z)$ be differentiable with respect to z and that the eigenvalues $\lambda(t,x)$ of the matrix $\frac{\partial g}{\partial z}(t,x,u^*(t,x))$ satisfy $\lambda(t,x) \le -\lambda_0 < 0$ for all $(t,x) \in I$. Finally, hypotheses (H1)-(H3) guarantee suitable properties of the manifold S and the map $u^*(t,x)$, while (H4) is satisfied under conditions well-known from the classical theory of ordinary differentail equations. Theorem 2.4 can also be proved in the case when solutions of the Cauchy problem considered in (H4) are not unique. In this case one has to use results on the continuous dependence of the solutions on the data; see [5].

§3. Approximability Property

An important concept related to the equivalent control $u = u^*(t,x)$ is that of approximability property. It has been illustrated through meaningful examples by Utkin [13] for control systems of the form

$$\begin{aligned} \dot{x} &= A(t,x) + B(t,x)\,u \\ s(x) &= 0 \\ u &= u(t,x) \in \Omega \subseteq \mathbf{R}^n . \end{aligned}$$

Using Utkin's discussion as a starting point, a definition of approximability for the fully nonlinear variable structure system

$$\begin{aligned} \dot{x} &= f(t,x,u) \\ s(x) &= 0 \\ u &= u(t,x) \in \Omega \subseteq \mathbf{R}^m . \end{aligned}$$

was proposed in [3]. There it was proved by means of the notion of G-convergence that the approximability property is satisfied for special cases of the nonlinear equation $\dot{x} = f(t,x,u)$.

Finally, in [9] it was shown that the approximability property is also fulfilled for the fully nonlinear variable structure system, provided that it is satisfied for a subclass of the set of perturbations considered in [3].

For the reader's convenience we recall the definition of the approximability property given in [3]. Recall that, in this case,

$S = \{x \in \mathbf{R}^n \mid s(x) = 0\}$ and so (H2) is formulated for such a set.

Definition 3.1. Fix $p > 1$, $M \in L^p[0,T]$, and a neighborhood I of S. A set H is defined as follows. Let $\{a_\eta \mid \eta > 0\}$ be a 1-parameter family of $\mathbf{R}^m$-valued functions whose components all lie in $L^p[0,T]$. Suppose that

$$|a_\eta(t)| \leq M(t) \quad (\eta > 0) \tag{4}$$

for a.a. $t \in [0,T]$, and that

$$\sup\Big\{ \Big| \int_0^t a_\eta(s)\, ds \Big| : 0 \leq t \leq T \Big\} \to 0 \quad \text{as } \eta \to 0^+ \tag{5}$$

Suppose further that $u^*(t,x,a_\eta(t))$ is defined for a.a. $t \in [0,T]$ and all $x \in I$. The family $\{a_\eta\}$ belongs to H if (and only if) all these conditions hold.

Definition 3.2. We say the system (1)-(2) with controls $u = u(t,x) \in U$ obeys the approximability property if (H2) is valid and if the following conditions hold.

First, there exist $p > 1$ and $M \in L^p[0,T]$ such that the set H defined above (with I given by (H2)) is non-empty.

Second, if $\{a_\eta\} \subset H$, if $\{x_\eta \mid \eta > 0\}$ is a family of a.e. solutions in $[0,T]$ of

$$\dot{x} = f(t,x,u^*(t,x,a_\eta(t)))$$

such that $s(x_\eta(0)) \to 0$ as $\eta \to 0$, if y is an a.e. solution in $[0,T]$ of $\dot{x} = f(t,x,u^*(t,x,0))$ such that $s(y(0)) = 0$, and if $x_\eta(0) \to y(0)$, then $x_\eta(t) \to y(t)$ uniformly in $[0,T]$ as $\eta \to 0$.

We now propose a different definition of approximability which is based on the dynamical feedback control law $u(t,\epsilon)$ and on the equivalent control $u^*(t,x)$.

In order to formulate it, let H_1 be the set of all one-parameter families $\{a_\eta \mid \eta > 0\}$ of $\mathbf{R}^n$-valued functions $a_\eta \in L^1[0,T]$ which satisfy (4) and (5) for some $M \in L^1[0,T]$. Once again I is given by (H2), but now (H2) is formulated in terms of $s = s(t,x)$.

Consider the system

$$\begin{aligned} \dot{x} &= f(t,x,u) \\ \epsilon\dot{u} &= g(t,x,u) + a_\eta(t)\,. \end{aligned} \tag{6}$$

For each $\epsilon > 0$, $\eta > 0$, let $(x(t,\epsilon,\eta), u(t,\epsilon,\eta))$ be a solution of (6). By our assumption on $\{a_\eta\}$, we have for each $\epsilon > 0$:

$$\lim_{\eta \to 0}(x(t,\epsilon,\eta), u(t,\epsilon,\eta)) = (x(t,\epsilon,0), u(t,\epsilon,0)) \tag{7}$$

uniformly in $[0,T]$ where $(x(t,\epsilon,0), u(t,\epsilon,0))$ is a solution of (3). Moreover under the assumptions of Theorem 2.4, we get

$$\lim_{\epsilon \to 0}(x(t,\epsilon,0), u(t,\epsilon,0)) = (x_0(t), u_0(t)) \tag{8}$$

uniformly in $[t_1, T]$ for each $0 < t_1 < T$. We note that, in general, the controls $u(t,0,\eta)$ do not converge to $u_0(t)$ as $\eta \to 0$. Under suitable assumptions and for particular forms of the nonlinear term f, this convergence was proved in [3].

The above discussion indicates that dynamical feedback controls are necessary in order to obtain the good behaviour described in (7) and (8) in the presence of perturbations like $\{a_\eta\} \subset H_1$. Indeed (7) implies that the approximability property as given in [3] concerns system (3) together with the condition $s(t,x) = 0$, rather than equations (1)-(2). However, properties (7)-(8) allow us to formulate an approximability property for equations (1)-(2), as we see in the following definition.

Definition 3.3. We say that the system (1)-(2) fulfills the approximability property if and only if the following conditions hold:

(i) the hypothesis (H2) is valid;

(ii) there exists $M \in L^1[0,T]$ such that the set H_1 is not empty and such that

(iii) if $\{a_\eta\} \subset H_1$, if $\epsilon > 0$, $\eta > 0$, if $(x(t,\epsilon,\eta), u(t,\epsilon,\eta))$ is a solution of (6) such that $s(0,x(0,\epsilon,\eta)) \to 0$ as $\eta \to 0$, and if $y(t)$ is the solution in $[0,T]$ of the system

$$\begin{aligned} \dot{x} &= f(t,x,u) \\ 0 &= g(t,x,u) \end{aligned}$$

satisfying $s(0,y(0)) = 0$, then the condition $\lim_{\eta \to 0} x(0,\epsilon,\eta) = y(0)$ implies that $\lim_{\epsilon \to 0} \lim_{\eta \to 0} x(t,\epsilon,\eta) = y(t)$ uniformly in $[0,T]$.

Reviewing previous considerations, we see that the system (1)-(2) fulfills the approximability property.

Note that, in general, $\lim_{\eta \to 0} \lim_{\epsilon \to 0} x(t,\epsilon,\eta)$ does not exist.

§4. Applications

We first consider a tracking problem. That is, we want to control the state x of system (1) with given initial condition in such a way it is asymptotically equal to a given state of a reference model up to a preassigned error.

More precisely, consider a reference control model

$$\begin{aligned} \dot{y} &= \varphi(t,y,v), \qquad y \in \mathbb{R}^n,\ v \in V \subset \mathbb{R}^p \\ y(0) &= y_0, \qquad |y_0| \leq M \quad \text{for some } M > 0\,, \end{aligned} \tag{9}$$

where $t \in [0,\infty)$ and V is a specified set. We pose the following problem.

(P1) Let $\alpha > 0$, $\delta > 0$, and $x_0 \in \mathbb{R}^n$ be given together with a fixed state-control (y,v) for the reference model (9). It is required to design a dynamical feedback control law (equivalently, define a function s) in such a way that, if $\epsilon > 0$ is sufficiently small, then the solution $x(t,\epsilon)$ of system (1) for which $x(0,\epsilon) = x_0$ satisfies the inequality

$$|y(t) - x(t,\epsilon)| \leq \delta + A\, e^{-\alpha t} \qquad (0 \leq t < \infty) \tag{10}$$

for some constant $A = A(x_0) > 0$.
We can solve (P1) as follows.
Define a function $s: [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$s(t,x) = y(t) - x - e^{Ct}(y_0 - x_0) \tag{11}$$

where C is a specified $n \times n$ matrix whose logarithmic norm $\mu(C)$ satisfies $\mu(C) = -\alpha < 0$. Note that $s(0,x_0) = 0$, i.e. $(0,x_0) \in S$.

We have the following

Proposition 4.1. Let $s(t,x)$ be defined as in (11). Assume that all the hypotheses of Theorem 2.4 are satisfied. Let $u_0 \in \mathbb{R}^n$ be a point such that $(0,x_0,u_0)$ is in the domain of influence of $\bar{u}_0 = u^*(0,x_0)$. Then there exists $\epsilon_0 > 0$ such that, for each $0 < \epsilon \leq \epsilon_0$, the solution $(x(t,\epsilon),u(t,\epsilon))$ of system (3) with initial condition $(x(0,\epsilon),u(0,\epsilon)) = (x_0,u_0)$ has the property that $x(t,\epsilon)$ satisfies (10).

Proof. System (3) takes the form

$$\begin{aligned} \dot{x} &= f(t,x,u), \qquad x(0) = x_0 \\ \epsilon\dot{u} &= f(t,y(t),u(t)) - f(t,x,u) - C\, e^{Ct}(y_0 - x_0) \end{aligned} \tag{12}$$

where $\epsilon > 0$ and $t \in [0,\infty)$.
Set $\epsilon = 0$, and let $(x_0(t),u_0(t))$ be the solution of (12) with initial condition $(x_0(0),u_0(0)) = (x_0,u_0)$. Then by Remark 2.3, we have $s(t,x_0(t)) = 0$ for all $t \geq 0$ (because $\frac{d}{dt}s(t,x_0(t)) = 0$ and $s(0,x_0(0)) = 0$). Thus $x_0(t)$ is the solution of (1) with $x(0) = x_0$ which corresponds to the equivalent control $u^*(t,x_0(t))$ and which satisfies

$$y(t) - x_0(t) = e^{Ct}(y_0 - x_0). \tag{13}$$

On the other hand, Theorem 2.4 implies that for sufficiently small $\epsilon > 0$ we have

$$|x_0(t) - x(t,\epsilon)| \leq \delta \tag{14}$$

for all $t \in [0,\infty)$. Now (10) follows from (13) and (14). □

Next consider the following problem.

(P2) Let $x_0, x_1 \in \mathbb{R}^n$ and $\delta > 0$ be given. We are to design a dynamical feedback control law in such a way that, if $\epsilon > 0$ is sufficiently small, then the solution $x(t,\epsilon)$ of (1) with initial condition $x(0,\epsilon) = x_0$ satisfies

$$x(T,\epsilon) \in B(x_1,\delta) \tag{15}$$

for a given $T < \infty$.

To solve (P2), let us define a function $s: [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$ as follows:

$$s(t,x) = y(t) - Tx + (T - t)x_0 + tx_1 \tag{16}$$

where $y(t)$ is any smooth function satisfying the boundary condition $y(0) = y(T) = 0$. Note that $s(0,x_0) = s(T,x_1) = 0$.
We have the following

Proposition 4.2. Let $s = s(t,x)$ be defined as in (16). Assume that all the assumptions of Theorem 2.4 are satisfied. Choose $u_0 \in \mathbb{R}^n$ such that the point $(0,x_0,u_0)$ is in the domain of influence of $\bar{u}_0 = u^*(0,x_0)$. Then there exists $\epsilon_0 > 0$ such that, for any $0 < \epsilon \leq \epsilon_0$, the solution $(x(t,\epsilon),u(t,\epsilon))$ of system (3) with $(x(0,\epsilon),u(0,\epsilon)) = (x_0,u_0)$ has the property that $x(t,\epsilon)$ satisfies (15).

Proof. System (3) takes the form

$$\left\{ \begin{aligned} \dot{x} &= f(t,x,u), \qquad x(0) = x_0 \\ \epsilon\dot{u} &= \dot{y}(t) - T\, f(t,x,u) + (T - t)x_0 + tx_1 \end{aligned} \right. \tag{17}$$

with $\epsilon > 0$, $t \in [0,T]$.

Set $\epsilon = 0$, and let $(x_0(t),u_0(t))$ be the solution of (17) satisfying $(x_0(0),u_0(0)) = (x_0,u_0)$. Then by Remark 2.3, we have $s(t,x_0(t)) = 0$ for all $t \in [0,T]$ (because $\frac{d}{dt}s(t,x_0(t)) = 0$ for all $0 \leq t \leq T$ and $s(0,x_0(0)) = 0$). Thus $x_0(t)$ is the solution of (1) with $x(0) = x_0$ which corresponds to the equivalent control u^* and which satisfies

$$x_0(T) = x_1 \tag{18}.$$

On the other hand, Theorem 2.4 implies that, for sufficiently small $\epsilon > 0$, we have

$$|x_0(T) - x(T,\epsilon)| \leq \delta \tag{19}.$$

Combining (18) and (19), we see that $x(T) \in B(x_1,\delta)$, as required. □

Remark 4.3. Consider the following time-invariant linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \tag{20}$$

where $x = (x_1, x_2) \in \mathbf{R}^k \times \mathbf{R}^m = \mathbf{R}^n$, $u \in \mathbf{R}^m$.
Note that there is no loss of generality in considering the previous form for a linear system model $\dot{x} = Ax + Bu$, since it can be always rewritten as in (20), provided that B is of full rank m. We assume that (20) is completely controllable.
For system (20), by using high-gain feedback controls in ([6], § 3.6) and [15], classical control problems have been solved by means of a singular perturbation analysis. We would like to point out that the method presented in this paper is different.

since the resulting controls, in general, are not high-gain controls and the fast dynamic is only that concerning the control law. However, under the same assumptions, related control problems for (20), as well as, control problems of inertial systems can be successfully tackle by means of the proposed approach. The work concerning these topics is still in progress.

§5. Conclusions

We have defined a dynamical feedback controller for the non linear control problem (1)-(2). The dynamical feedback control can, under appropriate assumptions, be designed in such a way that the corresponding state of (1) lies as close as one wants to the sliding manifold (2). Moreover, chattering does not occur. By means of this controller we have also defined an appropriate approximability property for our control problem, and we have seen that it is valid under our assumptions. Finally, we have applied the dynamical feedback control method to solve a general tracking problem and a general controllability problem.

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