CONTROLLABILITY PROBLEMS VIA SET-VALUED MAPS

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ABSTRACT

In this paper we give two examples of application to control theory of some results on the solvability of set-valued systems of the form

$$\begin{cases} 0 \in F(x, y) \\ 0 \in G(x, y) \end{cases} \tag{1}$$

where F and G are upper semicontinuous multivalued maps defined on Banach spaces. More precisely, in Problem 1 we give conditions for the solvability of two point boundary value problems for a multivalued control system. In Problem 2 we consider nonlinear control problems with dynamical feedback controls.

Key Words: Nonlinear control problems, set-valued differential equations.

1. PRELIMINARY RESULTS

We will denote a multivalued map M from X to Y with the symbol $M : X -\!\circ\ Y$. In the sequel, deg will denote the Leray- Schauder topological degree for single-valued compact fields. We will denote by Deg_{CL} the degree for multivalued compact vector fields with convex values, defined by Cellina and Lasota in [1]. By a r-neighborhood of a subset Ω of a metric space X we mean the set $B(\Omega,r) = \{y \in X : \exists x \in \Omega \text{ such that } d(x,y) < r\}$.

Definition 1.1 . Let X be a metric space. An upper semicontinuous set valued map $M : X -\!\circ\ X$ is *admissible* if there are maps $G_i : Y_i \to Y_{i+1}$, $i = 0, 1, \dots, n$ (Y_i metric spaces, $Y_0 = Y_{n+1} = X$) satisfying

i) $F = G_n \circ \cdots \circ G_0$;

ii) G_i is upper semicontinuous with acyclic, compact values for each $i=0,1, \dots, n$.

Each sequence $G_0, \cdots, G_n$ is called an admissible sequence for M. □

Definition 1.2 . Let X be a Banach space and let $\overline{B}(0,r)$ be a closed ball in X of radius r. We will say that the upper semicontinuous map $M : \overline{B}(0,r) -\!\circ\ X$ verifies the *Borsuk - Ulam*

(B.U.) property on ∂B if for all $x \in \partial B(0,r)$, $M(x)$ and $M(-x)$ are strictly separated by a hyperplane, i.e. for all $x \in \partial B(0,r)$ there exists a continuous functional $x^* \in X^*$, the dual space of X, such that $x^*(y) > 0$ for all $y \in M(x)$ and $x^*(y) < 0$ for all $y \in M(-x)$. □

Definition 1.3 . Let $M : \overline{B}(0, r) \subset X -\!\circ\, X$ be an upper semicontinuous set valued map. The map M *satisfies the boundary condition "P"* if $x \in \partial B(0,r)$ and $\lambda x \in M(x)$ implies $\lambda \leq 1$. □

Definition 1.4 . Let X and Y be metric spaces. Let $U \subset X \times Y$ be open and locally bounded over X. We shall say that $F : \overline{U} -\!\circ\, Y$ is a *parametrized compact vector field* if $F(x,y) = y - \hat{F}(x,y)$ with $\hat{F}$ upper semicontinuous and $\hat{F}(D)$ relatively compact in Y for any bounded set D of $\overline{U}$. We shall denote by $S^F = \{(x,y) \in \overline{U} : y \in \hat{F}(x,y)\}$; $S(x) = \{y \in Y : y \in \hat{F}(x,y)\}$; $S_x^F = S^F \cap (\{x\} \times Y)$; $\mathfrak{D}^F = \{x \in X : S_x^F \cap \partial U = \emptyset\}$. □

We want to investigate now the existence of solutions for the system

$$\begin{cases} y \in \hat{F}(x,y) \\ x = \hat{g}(x,y) \end{cases} \quad \text{or} \quad \begin{cases} 0 \in y - \hat{F}(x,y) = F(x,y) \\ 0 = x - \hat{g}(x,y) = g(x,y) \end{cases} \tag{2}$$

Theorem 1.5 [2]. Let X, Y be Banach spaces, let $U \subset X \times Y$ be an open, locally bounded set over X. Let $\hat{F} : \overline{U} -\!\circ\, Y$ be an upper semicontinuous compact map with convex values and $\hat{g}: \overline{U} \to X$ be a continuous and compact map. Suppose that there exists $r > 0$ such that $\overline{B}(0,r) \subset \mathfrak{D}^F$, and $\mathrm{Deg}_{CL}(F(0,\cdot),U(0),0) \neq 0$. Let $T: \overline{B}(0,r) -\!\circ\, X$ be defined by $T(x) = x - \hat{T}(x)$, where $\hat{T}(x) = \hat{g}(x, S(x))$ and $S(x)$. Let us suppose that for any $x \in \partial B$, the sets $T(x)$ and $T(-x)$ are strictly separated by an hyperplane. Then, there exists $x \in B$ such that $0 \in T(x)$ and hence, system (2) has a solution. □

In the next theorem we give an existence result for system (2) under the assumption that F and G are both multivalued maps and G is admissible. However, in this case we have to assume that the set $S(x)$ is acyclic for every $x \in \mathfrak{D}^F$. Notice that this assumption holds in many cases (see e.g. [4] and [5]).

Theorem 1.6 [2]. Let X, Y be Banach spaces, let $U \subset X \times Y$ be an open, locally bounded set over X. Let $\hat{F} : \overline{U} -\!\circ\, Y$ be an upper semicontinuous, uniformly quasibounded with respect to x, compact map Let us suppose that there exists $r > 0$ such that $\overline{B}(0,r) \subset \mathfrak{D}^F$, and for any $x \in \overline{B}(0,r)$ the set $S(x)$ is non empty and acyclic. Let $\hat{G} : \overline{U} -\!\circ\, X$ a compact, admissible map. Let $\hat{T} : \overline{B}(0,r) -\!\circ\, X$ be the map defined by $x -\!\circ\, \hat{T}(x) = \hat{G}(x, S(x))$. Suppose that the map $\hat{T}$ satisfies property "P" of Definition 1.3, then the system (1) has a solution. □

2. APPLICATIONS.

Problem 1 .

Consider the nonlinear multivalued control system,

$$\dot{y} \in \Phi(t, y, u),\ t \in [0,1],\ u \in \mathbf{R}^m,\ y \in \mathbf{R}^n. \tag{E_1}$$

The map $\Phi : [0,1] \times \mathbf{R}^n \times \mathbf{R}^m \multimap \mathbf{R}^n$ satisfies the following hypotheses.

(Φ_1) Φ is t-measurable, (p, q)-upper semicontinuous and $\Phi(t, p, q)$ is a compact, convex set for $(t, p, q) \in [0,1] \times \mathbf{R}^n \times \mathbf{R}^m$;

(Φ_2) for each $\rho > 0$ there exists $\gamma_\rho \in L^1((0,1), \mathbf{R}^+)$ such that for a.a. $t \in [0,1]$ and every $(p, q) \in \mathbf{R}^n \times \mathbf{R}^m$ with $|p| + |q| \leq \rho$ one has $|\Phi(t, p, q)| \leq \gamma_\rho(t)$.

The controls $u = u(t)$ are taken in a given n-dimensional subspace $\mathcal{U}$ of $L^\infty((0,1), \mathbf{R}^m)$. In [6] and [7] one can find examples of possible choices of $\mathcal{U}$.

We consider the boundary conditions $\begin{cases} y(0) = 0 \\ y(1) = y_1, \end{cases}$ where y_1 is a given vector in $\mathbf{R}^n$. (E_2)

Formulation of the problem : we want to prove the existence of a control law $u \in \mathcal{U}$ for which the multivalued nonlinear boundary value control problem (E_1) - (E_2) is solvable.

Let $X = \mathbf{R}^n$ and $Y = AC([0,1], \mathbf{R}^n)$. Let $g : Y \longrightarrow \mathbf{R}^n$ be the map defined by $g(y) = y(1) - y_1$. Clearly g is compact and continuous. Assume the following conditions

(i) let $J : \mathbf{R}^n \longrightarrow \mathcal{U}$ be an isomorphism. Let B be a bounded open set in $\mathbf{R}^n$. For every $b \in \overline{B}$ the solution set $S(b) = \{y \in Y : \dot{y} \in \Phi(t, y(t), J(b)(t)),\ y(0) = 0\}$ is bounded in $C([0,1], \mathbf{R}^n) = (C)^n$, and so, in virtue of ($\Phi_2$), it is bounded in Y by some constant $M = M(b)$;

(ii) The Cauchy problem $\begin{cases} \dot{y} \in \Phi(t, y, 0) \\ y(0) = 0 \end{cases}$ has the Cellina-Lasota topological degree different from 0 ;

(iii) Consider the map $T : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ defined by $T(b) = g(S(b))$.

There exists $r_1 = r_1(y_1) > 0$ such that for any $b \in \mathbf{R}^n$ with $|b| = r_1$ and $0 \notin T(b)$, we have that $\inf\limits_{z \in T(b)} \langle b, z \rangle > 0$ and $\sup\limits_{z \in T(-b)} \langle b, z \rangle < 0$.

Under assumptions (Φ_1) - (Φ_2), for any $b \in X$, the Cauchy problem $\begin{cases} \dot{y} \in \Phi(t, y, J(b)) \\ y(0) = 0 \end{cases}$ can be rewritten in the integral form $y \in \hat{\Phi}(b, y)$ where $\hat{\Phi} : X \times Y \longrightarrow Y$ is defined by

$$\hat{\Phi}(b, y)(t) = \int_0^t \Phi(s, y(s), J(b)(s))\, ds, \qquad \text{for any } t \in [0,1].$$

The integral is intended in the Aumann sense. It is easy to see that, in virtue of (Φ_1) - (Φ_2), $\hat{\Phi}(b, \cdot) : Y \multimap Y$ is a u.s.c. compact operator with convex values for any $b \in X$.

Proposition 2.1 . Under assumptions (i) - (iii) and (Φ_1) - (Φ_2) there is a control $u \in \mathcal{U}$ such that system (E_1) - (E_2) is solvable.

Proof : We show that all the conditions of Theorem 1.5 are fulfilled. First of all, assumption (i) implies that for every $\overline{B}(0, r) \subset \mathbf{R}^n$ the set $(S(\overline{B}(0, r)))$ is bounded and so $\mathfrak{D}^F = \mathbf{R}^n$. To prove this, given $r > 0$, let us consider the sequence $\{y_n\}_{n \in \mathbf{N}} \subset (S(\overline{B}(0, r)))$. Let $\{b_n\}_{n \in \mathbf{N}} \subset \overline{B}(0, r)$ be a sequence such that $\begin{cases} \dot{y}_n(t) \in \Phi(t, y_n(t), J(b_n)(t)) \text{ for a.a. } t \in [0,1] \\ y_n(0) = 0 \end{cases}$.

Passing to a subsequence, if necessary, we have that $b_n \to b_0$, $b_0 \in \overline{B}(0,r)$. For every $n \in \mathbf{N}$, define $\Phi_n : [0,1] \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ as follows $\Phi_n(t,p) = \Phi(t,p,J(b_n)(t))$. Using (Φ_1) - (Φ_2) and adapting the proof of Theorem 2.4 in [6] we can prove that the sequence $\{y_n\}_{n \in \mathbf{N}} \subset (S(\overline{B}(0,r))$ has a subsequence converging uniformly in $[0,1]$ to a function y_0 which is a solution of the following problem

$$\begin{cases} \dot{y}(t) \in \Phi(t,y(t),J(b_0)(t)) & \text{for a.a. } t \in [0,1] \\ y(0) = 0 \end{cases} .$$

Thus S has closed graph and $S(\overline{B}(0,r))$ is a compact set in $(C)^n$. This implies, by (Φ_2), that $S(\overline{B}(0,r))$ is bounded in Y, say by a constant $C = C(r) > 0$, and so the map S is u.s.c. . If we set $U = \{(b,Y) \in X \times Y \ : \ b \in \overline{B}(0,r_1),\ \| y \|_Y \le C(r_1)\}$, then assumption (ii) implies that $\mathrm{Deg}_{CL}(F(0,\cdot),U(0),0) \neq 0$, where $F = I - \hat{\Phi}$. Finally, assumption (iii) ensures that there exists a ball $\overline{B}(0,r_1) \subset X$ such that for any $b \in \partial B(0,r_1)$ for which $0 \notin T(b)$ we have that the sets $T(b)$ and $T(-b)$ are strictly separated by an hyperplane. □

Note 2.2 . We conclude with some remarks concerning conditions (i) - (iii) of Proposition 2.1 .

Condition (i) is verified if, for every $\rho > 0$, there exists a function $\omega_\rho : [0,1] \times \mathbf{R} \to \mathbf{R}$ satisfying the following conditions:

a) $| \Phi(t, p, q)| \le \omega_\rho(t, |p|)$ for all $|q| \le \rho$, all $p \in \mathbf{R}^n$ and a.a. $t \in [0,1]$;

b) $\omega_\rho(t, \zeta)$ is measurable in t for every ζ and continuous in ζ for a.a. $t \in [0,1]$;

c) $\forall K > 0$ there exist γ_K such that $|\omega_\rho(t, \zeta)| \le \gamma_K(t)$ for $|\zeta| \le K$ and for a.a. $t \in [0,1]$;

d) $\omega_\rho(t, \zeta)$ is non decreasing with respect to the second variable;

e) the maximal solution of the Cauchy problem $\begin{cases} \dot{\eta}(t) = \omega_\rho(t,\eta(t)) \\ \eta(t_0) = y_0 \end{cases}$ a.e. in $[0,1]$ is defined in all of $[0,1]$.

The existence of such function ω_ρ is assured, for instance, if for any $\rho > 0$ there exist two functions $\alpha_\rho, \beta_\rho \in L^1$ such that $|\Phi(t, p, q)| \le \alpha_\rho(t)|p| + \beta_\rho(t)$ for all $|q| \le \rho$, all $p \in \mathbf{R}^n$ and a.a. $t \in [0,1]$. Furthermore, the topological degree theory for compact vector fields with convex values (see [7]) provides several conditions ensuring that (ii) is verified. Finally, in order to guarantee (iii) assume the multivalued map Φ satisfies the following condition

(f) there exists $\delta > 0$ and a function $\alpha \in L^1((0,1),\mathbf{R})$ with $\int_0^1 \alpha(t)\,dt > 0$ such that

$$\liminf_{|b| \to \infty} \inf \left\{ \frac{<b,w>}{|b|^{1+\delta}} : w \in \Phi(t,B(0,R),J(b)(t)) \right\} \ge \alpha(t) \quad \forall\, R > 0 \text{ and for a.a. } t \in [0,1].$$

Under condition (f), it is not hard to see that for any vector $y_1 \in \mathbf{R}^n$ there exists $r_1 > 0$ such that for any $b \in \mathbf{R}^n$, $|b| = r_1$, we have

$$\inf_{w \in \Phi(t,p,J(b)(t))} \frac{<b,w-y_1>}{|b|^{1+\delta}} \ge \frac{\alpha(t)}{2}, \text{ for a.a. } t \in [0,1] \text{ and } |p| \le R_1,$$

where $R_1 > 0$ is such that $\| y \|_Y \le R_1$ for any $y \in S(\overline{B}(0,r_1))$. Substituting b with $-b$ we also get

$$\sup_{w \in \Phi(t,p,J(-b)(t))} \frac{<b,w-y_1>}{|b|^{1+\delta}} \le -\frac{\alpha(t)}{2}, \text{ for a.a. } t \in [0,1] \text{ and } |p| \le R_1.$$

In conclusion, by (f), we have that (E_1) - (E_2) is solvable for any $y_1 \in \mathbf{R}^n$. Therefore it is possible to reach any point of $\mathbf{R}^n$ from the origin along the trajectories of (E_1).

Problem 2 .

Let f and g Caratheodory maps. We consider the nonlinear boundary value control problem

$$\begin{cases} \dot{y} = f(t,y,u) \\ y(0) = y_0,\ y(1) \in B \end{cases} \qquad y_0 \text{ assigned , B compact, contractible set of } \mathbf{R}^n. \qquad (P_1)$$

The control u satisfies the differential system $\begin{cases} \dot{u} = g(t,y,u) \\ u(0) = u_0 \end{cases}$, $t \in [0,1]$, $u \in \mathbf{R}^n$, $y \in \mathbf{R}^n$. $\qquad (P_2)$

<u>Formulation of the Problem</u> : we want to give conditions under which there exists a pair of functions $(y,u) \in Y \times Y$ satisfying (P_1) - (P_2), where Y denotes the Banach space $AC([0,1],\mathbf{R}^n)$.

Assume the following conditions on the dynamics g.

(g_1) Let $\alpha, \beta \in L^1((0,1),\mathbb{R}^+)$ such that $|g(t,p,q)| \le \alpha(t)|q| + \beta(t)$, $p \in \mathbb{R}^n$, $q \in \mathbb{R}^n$;

(g_2) the Cauchy problem $\begin{cases} \dot{u} = g(t,0,u) \\ u(0) = u_0 \end{cases}$ has the topological degree different from zero;

(g_3) for any $i = 1,2,\dots,n$ we have that $u_0^i \to \pm\infty$ implies $u^i(y,u_0)(t) \to \pm\infty$ for any $y \in Y$ and any $t \in [0,1]$. Here $(u^i(y,u_0))_{i=1}^n = u(y,u_0)$ which represents the solution to (P_2) corresponding to $(y,u_0) \in Y \times \mathbf{R}^n$.

Under assumption (g_1) and using the continuous dependence on the parameters $(y,u_0) \in Y \times \mathbb{R}^n$ of the solutions u to problem (P_2), (see [6]), it can be shown that for any $r > 0$, the set $S(\overline{B}(0,r)) = \{u \in U : \dot{u} = g(t,y,u),\ u(0) = u_0\}$ is bounded in $C([0,1],\mathbb{R}^n) = (C)^n$ and thus bounded in Y. Here $B(0,r) = \left\{x = (y,u_0) \in Y \times \mathbf{R}^n : \|x\| = \max\{\|y\|_Y, |u_0|\} < r\right\}$. Moreover, conditions (g_1)-(g_2) ensure that the upper semicontinuous map $S : \overline{B}(0,r) \multimap (C)^n$ defined by $S(y) = \{u : \dot{u} = g(t,y,u),\ u(0) = u_0\}$ has nonempty, compact, acyclic values.

Finally, assume the following conditions on f.

(f_1) for any $\rho > 0$, there exists $\gamma_\rho \in L^1((0,1),\mathbb{R}^+)$ such that for a.a. $t \in [0,1]$ and any $(p,q) \in \mathbf{R}^n \times \mathbb{R}^n$ with $|p| + |q| \le \rho$ one has $|f(t,p,q)| \le \gamma_\rho(t)$;

(f_2) for any $\eta > 0$ there exist $\zeta > 0$ and a function $\alpha_\eta \in L^1((0,1),\mathbb{R}^+)$ with $\alpha_\eta(t) > 0$ for a.a. $t \in [0,1]$ such that for any $|p| \ge \zeta$ we have that $\dfrac{<p, f(t,p,q)>}{|p|^{1+\delta}} \ge \alpha_\eta(t)$ for some $\delta > 0$, for any $|q| < \eta$ and for a.a. $t \in [0,1]$;

(f_3) for any $i = 1,2,\dots,n$, $\lim\limits_{q_i \to \pm\infty} f_i(t,p,q) = \pm\infty$ for any $p \in \mathbb{R}^n$ and for a.a. $t \in [0,1]$.

Let $\hat{F} : Y \times \mathbb{R}^n \times Y \to Y$ and $\hat{G} : Y \times \mathbb{R}^n \times Y \multimap \mathbb{R}^n$ be the operators defined respectively by

$$\hat{F}(y,u_0,u)(t) = z(t) = y(0) + \int_0^t f(t,y(t),u(t))\,dt\ , \quad \text{for any } t \in [0,1]$$

$$\hat{G}(y,u_0,u) = u_0 + B - y_0 - \int_0^1 f(t,y(t),u(t))\,dt$$

$\hat{F}$ is a continuous and compact map. $\hat{G}$ is an admissible map, in fact $\hat{G}$ can be considered as the composition of the following applications

$V : Y \times \mathbf{R}^n \times Y \longrightarrow \mathbb{R}^n$ defined by $V(y, u_0, u) = u_0 - \int_0^1 f(t,y(t),u(t))\,dt$;

$W_1 : \mathbf{R}^n \multimap \mathbf{R}^n \times \mathbf{R}^n$ defined by $W_1(v) = (v, B)$;

$W_2 : \mathbf{R}^n \times \mathbf{R}^n \multimap \mathbf{R}^n$ defined by $W_2(v,b) = v + b - y_0$;
V is continuous, W_1, W_2 are u.s.c. with compact, contractible values, since the Cartesian product of contractible sets is contractible, then $\hat{G} = (W_2 \circ W_1 \circ V)$ is an admissible map.

Proposition 2.3 . Assume (f_1) - (f_3) and (g_1) - (g_3). Then there exists a solution to (P_1) - (P_2).

Proof : Consider the map $\hat{T} : Y \times \mathbf{R}^n \multimap Y \times \mathbf{R}^n$ defined by $\hat{T}(x) = \hat{H}(x, S(x))$, where $x = (y, u_0)$ and $\hat{H} = (\hat{F}, \hat{G})$. $\hat{T}$ is admissible since it is the composition of the admissible map $\hat{H}$ with the u.s.c. compact, acyclic-valued map S. Furthermore it can be proved that $\hat{T}$ satisfies property "P" on some ball B(0,r) and so by Theorem 1.6 there is a solution to (P_1)-(P_2). □

Observe that, by slight modifications we can assume in the previous example that f, g are t-measurable, (p,q)-upper semicontinuous, compact, convex valued maps.
Controllability problems of this kind are also treated in [3] by a different approach.

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