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Brief Paper

Continuous-time LQ Regulator Design by Polynomial Equations*

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Key Words—Optimal control; linear systems; multivariable systems; polynomials.

Abstract—The deterministic continuous-time LQ output regulation problem is solved by polynomial equations as an alternative to the usual Riccati equation approach. In particular, it is shown that Riccati-based and polynomial methods are fully conceptually equivalent in steady-state LQ regulation.

1. *Introduction*

THIS PAPER deals with the classic Linear Quadratic Output Regulation (LQOR) problem. Only the continuous-time case is considered here, the discrete-time case having been recently reported elsewhere (Mosca and Nistri, 1989).

A continuous-time, linear, time-invariant state-space representation of the plant to be output regulated is considered

$$\begin{cases} \dot{x}(t) = \Phi x(t) + Gu(t) \\ y(t) = Hx(t) \end{cases} \tag{1}$$

with $x(t) \in \mathscr{R}^n$, $u(t) \in \mathscr{R}^m$ and $y(t) \in \mathscr{R}^p$. The problem is to find, if it exists, an input variable $u(\cdot) \in L_2$ minimizing the quadratic-cost

$$J = \int_0^\infty (\|y(t)\|^2_{\Psi_y} + \|u(t)\|^2_{\Psi_u})\, dt. \tag{2}$$

for any initial state $x(0)$. In (2) $\Psi_y = \Psi_y' > 0$, $\Psi_u = \Psi_u' > 0$, $\|v(t)\|^2_\Psi := v'(t)\Psi v(t)$ and the prime denotes transpose. By Parseval's Lemma, the above cost can be expressed in terms of the Laplace transforms of $y(t)$ and $u(t)$

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} (\|y(s)\|^2_{\Psi_y} + \|u(s)\|^2_{\Psi_u})\, ds. \tag{3}$$

It is well known that under stabilizability and detectability assumptions on the triplet (Φ, G, H), problem (1)–(3) can be solved in state-feedback form by using the unique nonnegative definite solution of the relevant algebraic Riccati equation. Moreover, the resulting closed-loop system turns out to be asymptotically stable.

The aim of this paper is to provide a direct matrix-fraction approach to the problem. In this way, the solution is obtained by: first, solving a spectral factorization problem; and, next, finding the minimum-degree solution with respect to a "dummy" polynomial matrix of a pair of bilateral Diophantine equations. The specific "minimum-degree" property that identifies the required solution will be made precise in Section 2.

The problem was previously addressed by Kučera (1983) for scalar input plants and completely reachable pairs (Φ, G). Because of these rather restrictive assumptions, the solution can be obtained in terms of a single Diophantine equation (Kučera, 1983). In Section 2 it is shown that, in the general case, two bilateral Diophantine equations must be solved simultaneously. Another related contribution is Grimble (1987), where the polynomial solution to the LQ *stochastic* regulator with complete state information was given for the discrete-time case.

One of the reasons for considering a polynomial solution for the standard deterministic LQOR problem is to show that Riccati-based and polynomial methods are fully conceptually equivalent, as far as steady-state (semi-infinite horizon) results are concerned. In particular, stabilizability and/or detectability requirements in the Riccati equation approach are replaced by conditions on the stability of greatest common left and right divisors of polynomial matrices.

The reader is referred to Kučera (1979), whose notation is adopted hereafter as much as possible considering the differences between the discrete and continuous-time cases. For any real rational matrix $R(s)$, $R^*(s) := R'(-s)$. Further, for any polynomial matrix $P(s)$ in the indeterminate s the following notations or definitions are assumed hereafter: $\partial P(s)$ denotes the degree of $P(s)$; $P(s)$ is said to be *row-reduced* if the matrix of the coefficients of the highest power of s in each row of $P(s)$ has full row-rank; similarly, $P(s)$ is said to be *column-reduced* if the matrix of the coefficients of the highest power of s in each column of $P(s)$ has full column-rank; a square polynomial matrix $P(s) = P_0 s^n + P_1 s^{n-1} + \cdots + P_n$ is said to be *regular* if its leading matrix coefficient P_0 is nonsingular. Any regular polynomial matrix is both row-reduced and column-reduced. The opposite implication is in general false.

2. *Main results*

As is well known, the problem (1)–(3) only depends on a completely observable subsystem of (1) obtainable via Kalman's canonical decomposition. Thus, from the outset assume that

(A.1) (Φ, H) is a completely observable pair.

Hereafter all quantities are assumed to be Laplace transforms. If $y(s)$ denotes the output of (1) due to $x(0)$ and the input signal is denoted by $u(s)$,

$$y(s) = HA^{-1}(s)[x(0) + Bu(s)] \tag{4}$$

where $A(s)$ and B are the following polynomial matrices

$$A(s) := sI - \Phi \tag{5}$$

$$B := G. \tag{6}$$

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Hereafter, for the sake of simplicity, the argument s will be omitted unless it is required to avoid possible confusion. The quadratic cost (3) can be conveniently rewritten as

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} (y^*\Psi_y y + u^*\Psi_u u)\,ds. \tag{7}$$

Let us also rewrite (4) as follows

$$y(s) = HA^{-1}(s)x(0) + HB_2(s)A_2^{-1}(s)u(s) \tag{8}$$

where $B_2(s)A_2^{-1}(s)$ is a right-coprime (rc) column-reduced matrix-fraction description of $A^{-1}B$. Hence, $\partial_{ci}B_2 < \partial_{ci}A_2$, $i = 1, 2, \ldots, m$, where $\partial_{ci}B_2$ denotes the column-degree of the ith columns of B_2 (Kailath, 1980).

Substituting (8) in (7), obtain:

$$\begin{aligned} J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} & [u^*A_2^{-*}(A_2^*\Psi_u A_2 + B_2^*\Psi_x B_2)A_2^{-1}u \\ & + u^*A_2^{-*}B_2^*\Psi_x A^{-1}x(0) + x'(0)A^{-*}\Psi_x B_2 A_2^{-1}u \\ & + x'(0)A^{-*}\Psi_x A^{-1}x(0)]\,ds \end{aligned} \tag{9}$$

where $A_2^{-*} := (A_2^{-1})^*$ and $\Psi_x := H'\Psi_y H$. Let $E(s)$ be a Hurwitz polynomial matrix solving the following *spectral factorization problem*

$$E^*E = A_2^*\Psi_u A_2 + B_2^*\Psi_x B_2. \tag{10}$$

Since A_2 is column-reduced, $B_2A_2^{-1}$ strictly proper, $\Psi_u > 0$, from Kučera (1980) it follows that E is column-reduced. By using (10) and adding and subtracting $x'(0)A^{-*}\Psi_x B_2 E^{-1}E^{-*}\Psi_x A^{-1}x(0)$ in (9), obtain

$$J = J_1 + J_2 \tag{11}$$

with

$$\begin{aligned} J_1 &:= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} L^*L\,ds \\ J_2 &:= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} x'(0)A^{-*}(I - \Psi_x B_2 E^{-1}E^{-*}B_2^*)\Psi_x A^{-1}x(0)\,ds \\ L &:= E^{-*}B_2^*\Psi_x A^{-1}x(0) + EA_2^{-1}u. \end{aligned} \tag{12}$$

Note that J_2 does not depend on u. In order to simplify the above expression for J_1, let us consider the following bilateral Diophantine equation

$$E^*Y + ZA = B_2^*\Psi_x. \tag{13}$$

It is temporarily assumed that a solution (Y, Z) of (12) exists with $\partial_{ri}Z < \partial_{ri}E^*$, $i = 1, 2, \ldots, m$, where $\partial_{ri}Z$ denotes the row-degree of the ith row of Z. Under this assumption, (12) becomes

$$L = Yx(s) + (E - YB_2)A_2^{-1}u + E^{-*}Zx(0). \tag{14}$$

Further, let us temporarily assume that the following polynomial equation can be jointly solved along with (13)

$$XA_2 + YB_2 = E. \tag{15}$$

Substitution of (15) into (14) gives

$$L = (Yx + Xu) + E^{-*}Zx(0). \tag{16}$$

Consequently the cost index J_1 can be split into the following three components

$$J_1 = J_3 + J_4 + J_5 \tag{17}$$

$$J_3 := \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} x'(0)Z^*E^{-1}E^{-*}Zx(0)\,ds \tag{18}$$

$$\begin{aligned} J_4 := \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} & \{(Yx + Xu)^*E^{-*}Zx(0) \\ & + x'(0)Z^*E^{-1}(Yx + Xu)\}\,ds \end{aligned} \tag{19}$$

$$J_5 := \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} (Yx + Xu)^*(Yx + Xu)\,ds \tag{20}$$

where J_3 does not depend on u. Further, if X and Y are constant matrices, i.e. $\partial X = \partial Y = 0$ (Lemma 2 below verifies that this property holds true), by using the Residue Theorem, it is easy to prove that J_4 is identically zero. This follows from the fact that $\partial_{ci}Z^* < \partial_{ci}E$, and hence Z^*E^{-1} is a strictly proper stable transfer-matrix, and that $u(\cdot)$ and $x(\cdot)$ are in L_2. Thus, minimization of J amounts to minimizing J_5.

Finally, premultiplying both sides of (15) by E^* and taking into account (10), one gets

$$E^*X - ZB = A_2^*\Psi_u. \tag{21}$$

The following lemma summarizes the above discussion.

Lemma 1. Provided that

(i) (13) and (21) [or (13) and (16)] admit a solution (X, Y, Z) with $\partial_{ri}Z < \partial_{ri}E^*$, $i = 1, 2, \ldots, m$, $\partial X = \partial Y = 0$ and X nonsingular; and

(ii) $J_2 + J_3$ is bounded;

the solution of the LQOR problem is given by

$$u(s) = -X^{-1}Yx(s) \tag{22}$$

with X and Y specified in (i), and correspondingly,

$$J_{\min} = J_2 + J_3. \quad \square$$

A condition under which (i) of Lemma 1 is fulfilled is given by next lemma whose proof is given in the Appendix.

Lemma 2. Let the greatest common left divisors (GCLDs) of A and B in (5) and (6) be strictly Hurwitz. Then, there is a unique solution (X, Y, Z) of (13) and (21) [or (13) and (15)] such that

$$\partial_{ri}Z < \partial_{ri}E^*, \quad i = 1, 2, \ldots, m. \tag{23}$$

$\partial X = \partial Y = 0$, and X nonsingular. □

The unique solution (X, Y, Z) referred to in Lemma 2, will be called the *minimum row-degree* solution w.r.t. Z.

Boundedness of $J_2 + J_3$ is clearly guaranteed by the stability of the closed-loop system. This, in turn, if the plant has no unstable hidden modes, is fulfilled if and only if E is strictly Hurwitz. In fact, $\det E$ is proportional to the characteristic polynomial of the closed loop system made up by the plant $B_2A_2^{-1}$ together with the control law (22). The next lemma, whose proof is reported in the Appendix, gives a sufficient condition for E to be strictly Hurwitz.

Lemma 3. If (A.1) holds then the spectral factor E is strictly Hurwitz. □

The previous lemmas show that the LQ output regulation problem can be solved provided that (Φ, H) is a completely observable pair and the GCLD's of A and B are stable. This is the same as assuming that the given plant (Φ, G, H), which in general need not be completely observable, has all its observable-unreachable eigenvalues stable. The results are summarized in the following theorem.

Theorem 1. Consider the LQ output regulation problem (1)–(3) for the plant $\Sigma = (\Phi, G, H)$. Then,

1. The problem is solvable if and only if the GCLDs of A_0 and B_0 are stable, where $A_0 := sI - \Phi_0$ and $B_0 := G_0$ and $\Sigma_0 = (\Phi_0, G_0, H_0)$ is a completely observable subsystem of Σ obtainable via Kalman's canonical observability decomposition of Σ.
2. Provided that the solvability condition is fulfilled, the optimal input signal is given by (22), where X and Y are obtained by first solving the spectral factorization problem (10) and next finding the *minimum row-degree solution* w.r.t. Z of the pair of bilateral Diophantine equations (13) and (21) [or (13) and (15)], viz. $\partial_{ri}Z < \partial_{ri}E^*$.
3. The overall closed-loop system is asymptotically stable if and only if the plant Σ has no unstable hidden modes. □

It is interesting to explore the connection between the polynomial solution (X, Y, Z) given in Theorem 1 (part 2) with the classic Riccati-based solution of the LQOR problem. This is considered in Theorem 2, whose assertion can be easily proved, by showing that the triplet (25) fulfills (13) and (21) [or (13) and (15)] and satisfies the row-degree inequalities $\partial_{ri}Z < \partial_{ri}E^*$, $i = 1, 2, \ldots, m$.

Theorem 2. Let the plant $\Sigma = (\Phi, G, H)$ be stabilizable and detectable. Also let P denote the unique symmetric non-negative definite solution of the matrix algebraic Riccati equation

$$P\Phi + \Phi' P - PG'\Psi_u^{-1}GP + \Psi_x = 0. \quad (24)$$

Then, the minimum row-degree solution w.r.t. Z of the Diophantine equations (13) and (21) [or (13) and (15)] is given by

$$X'X = \Psi_u \quad X'Y = G'P \quad Z = B_2^* P. \quad \square \quad (25)$$

3. *Discussion and examples*

The polynomial solution to the LQOR problem given in Mosca and Nistri (1989) looks like a direct translation into the discrete-time context of the one covered in the present paper. In fact, in both cases the solution—if it exists—is given in terms of a spectral factorization problem and a pair of bilateral Diophantine equations. Nevertheless, the degree constraints for the discrete-time case turn out to be different from (23). The discrete-time case degree constraints, in the present continuous-time context, would translate as follows

$$\partial Z < \partial E^*. \quad (26)$$

This condition is weaker than the row-degree inequality (23). If E is *regular*—a property that is fulfilled in the standard discrete-time case discussed in Mosca and Nistri (1989)—(23) is equivalent to (26). The next example shows that in the continuous-time case, where E need not be regular (26) has to be replaced by (23) in order to get the desired constant solution (X, Y) of (13) and (21).

Example 1. Let

$$\Phi = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}; \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};$$

$$\Psi_x = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad \Psi_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A possible choice for A_2 and B_2 is

$$A_2 = \begin{bmatrix} s+1 & 0 \\ 0 & 1 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $B_2^*\Psi_x = 0_{2\cdot 2}$ and A_2 is Hurwitz, one can set $E = A_2$. Consequently, the solution in Theorem 1 (part 2) is as follows

$$X = I_2, \quad Z = Y = 0_{2\cdot 2}.$$

On the contrary, (26) does not yield either a unique solution of (13) and (21) or necessarily $\partial X = \partial Y = 0$. In fact, it can be checked that

$$X = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}; \quad Y = \begin{bmatrix} 0 & 0 \\ s+1 & s+\frac{1}{2} \end{bmatrix}; \quad Z = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$$

is one possible solution of (13) and (21) fulfilling (26). $\square$

There are in general, *two* bilateral Diophantine equations, viz. (13) and (21) [or, equivalently (13) and (15)], that must be solved with the row-degree constraints (23) in order to finding the LQ optimal feedback-gain matrix $F = X^{-1}Y$. In particular, the reader is referred to Mosca *et al.* (1990) and Hunt *et al.* (1987) where the need to solve two Diophantine equations is thoroughly investigated. Nevertheless an example is now presented to aid our understanding of this problem.

Example 2. Let

$$\Phi = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}; \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$$H = [1 \quad 1]; \quad \Psi_u = 1; \quad \Psi_y = 1.$$

Notice that (Φ, G) is not completely reachable, whereas (Φ, H) is completely observable. We find

$$A = \begin{bmatrix} s-1 & 0 \\ 0 & s+\frac{1}{2} \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

A GCLD of A and B is

$$\begin{bmatrix} 1 & 0 \\ 0 & s+\frac{1}{2} \end{bmatrix}$$

which is stable. Thus, according to Lemma 2 the problem is solvable, and, according to Theorem 1 (part 3), the resulting LQ optimal feedback stabilizes the plant being the only plant hidden eigenvalue $\lambda = -\frac{1}{2}$. We also find:

$$A_2 = s - 1; \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

and, via spectral factorization, $E = s + \sqrt{2}$. Further: $A_2^* = -s - 1$; $B_2^* = [1 \quad 0]$, $E^* = -s + \sqrt{2}$, which implies $\partial Z = 0$. (13) and (21) [or (13) and (15)], give

$$X = 1; \quad Y = \left[y_1 = \sqrt{2} + 1, \; y_2 = \frac{2(2\sqrt{2} - 1)}{7} \right];$$

$$Z = \left[z_1 = y_1, \; z_2 = \frac{2(2\sqrt{2} - 1)}{7} \right].$$

Hence,

$$F = -\left[\sqrt{2} + 1 \quad \frac{2(2\sqrt{2} - 1)}{7} \right]$$

and

$$\Phi + GF = \begin{bmatrix} -\sqrt{2} & -\frac{2(2\sqrt{2} - 1)}{7} \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

(15) alone yields $X = 1$, $y_1 = \sqrt{2} + 1$, $z_1 = \sqrt{2} + 1$. However, it does not provide any information on y_2 and z_2. $\square$

Conclusions

The deterministic continuous-time LQ output regulation problem has been solved via spectral factorization and a pair of bilateral Diophantine equations. Stabilizability and/or detectability requirements in the Riccati equation approach were replaced by conditions on the stability of greatest common left and right divisors of polynomial matrices.

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Appendix

Proof of Lemma 2. It is known (Kučera, 1979) that (13) and (21) are solvable provided that the GCLDs of A and B are strictly Hurwitz. All the solutions of (13) and (21) are given by:

$$X = X_0 - TB; \quad Y = Y_0 + TA; \quad Z = Z_0 - E^*T \qquad (27)$$

where (X_0, Y_0, Z_0) is a solution of (13) and (21) and T is any polynomial matrix of compatible dimensions. By the Division Theorem for Polynomial Matrices (Kailath, 1980) there exists a unique pair (Q, R) such that

$$Z_0 = E^*Q + R, \quad \partial_{ri}R < \partial_{ri}E^*.$$

Substituting this expression in (27), one obtains $Z = R + E^*(Q - T)$. Therefore the minimum row-degree solution of (13) and (21) w.r.t. Z is obtained by tacking $T = Q$ in (27):

$$X = X_0 - QB; \quad Y = Y_0 + QA; \quad Z = R. \qquad (28)$$

It remains to be shown that $\partial X = \partial Y = 0$. This can be concluded by first noting that from (10)

$$\partial_{ci}E = \partial_{ci}A_2. \qquad (29)$$

This can be seen by considering that since $\partial_{ci}B_2 < \partial_{ci}A_2$ and $\Psi_u > 0$,

$$2\partial_{ci}E = \partial(E^*E)_{ii} = \partial(A_2^*\Psi_uA_2)_{ii} = 2\partial_{ci}A_2$$

where $(E^*E)_{ii}$ denotes the ith diagonal entry of E^*E. From (13), it follows that

$$\partial_{ri}(E^*Y) \le \partial_{ri}E^*$$

which, in turn, by nonsingularity of E, implies that $\partial Y = 0$.

Similarly, from (21) it follows that

$$\partial_{ri}(E^*X) = \partial_{ri}E^*$$

and again, arguing as above, one finds that $\partial X = 0$. Finally, by using (23), (29) and the fact that $\Psi_u > 0$, from (21) obtain

$$\partial_{ri}(E^*X) = \partial_{ri}A_2^*.$$

Hence, since E^* and A_2^* are row-reduced, it follows that X is nonsingular.

In Mosca *et al.* (1990) it is shown that the solution of (13) and (21) coincide with that of (13) and (15).

Proof of Lemma 3. First, it will be shown that complete observability of (Φ, G) implies that HB_2 and A_2 are right coprime (rc). In order to prove this, we begin by noting that by PBH test (Kailath, 1980), complete observability of (Φ, H) is equivalent to H and A rc. This, in turn, implies that H and A_1 are rc, if $A^{-1}B = A_1^{-1}B_1$ with A_1 and B_1 left coprime (lc). In fact, $A = \Delta A_1$ and $B = \Delta B_1$ with Δ a GCLD of A and B. Hence, for some polynomial matrices U and V,

$$I = UA + VH = (U\Delta)A_1 + VH.$$

We finally show that if H and A_1 are rc, then HB_2 and A_2 are rc. In fact, consider the transfer-matrices

$$HA_1^{-1}B_1 = HB_2A_2^{-1}$$

for which $\partial \det A_1 = \partial \det A_2$. The expression on the LHS can be minimally realized in state-space form with a state of dimension equal to $\partial \det A_1$ since H and A_1 are rc. Thus, $HB_2A_2^{-1}$ must be also realized in minimal form having the same state dimension. Hence, we conclude that HB_2 are A_2 are rc.

Having proved that HB_2 and A_2 are rc, we next show that $\det E(j\omega) \neq 0$, $\forall \omega \in \mathcal{R}$ and hence that E is strictly Hurwitz. In order to prove this, we note that HB_2 and A_2 rc implies that, for $\varphi_y = \varphi_y' = \Psi_y^{1/2}$ and $\varphi_u = \varphi_u' = \Psi_u^{1/2}$, $\hat{B}_2 := \varphi_y HB_2$ and $\hat{A}_2 := \varphi_u A_2$, by nonsingularity of φ_y and φ_u, are rc. In fact for some polynomial matrices $\bar{U}$ and $\bar{V}$

$$I = \bar{U}A_2 + \bar{V}HB_2 = (\bar{U}\varphi_u^{-1})\hat{A}_2 + (\bar{V}\varphi_y^{-1})\hat{B}_2.$$

We now prove that $\det E(j\omega) \neq 0$, $\forall \omega \in \mathcal{R}$ by contradiction. Suppose there exists $u \in \mathcal{R}^m$, $u \neq 0$ and $\omega \in \mathcal{R}$ such that

$$0 = \|E(j\omega)u\|^2 = \|\hat{A}_2(j\omega)u\|^2 + \|\hat{B}_2(j\omega)u\|^2.$$

This implies that $A_2(j\omega)u = B_2(j\omega)u = 0$. But since $\hat{B}_2$ and $\hat{A}_2$ are rc, for some polynomial matrices $\bar{U}$ and $\bar{V}$, $(\bar{U}A_2 + \bar{V}B_2)u = Iu \neq 0$. This contradicts singularity of $E(j\omega)$. □