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Optimal Control Problems Via a Direct Method (*) (**).

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Summary. – *In this paper we deal with the minimization problem of a cost functional associated to a nonlinear boundary value control problem of a general form, defined in the fixed time interval* $[0, 1]$. *Specifically, we first give conditions which ensure that the nonlinear boundary value control problem is solvable and we study the structure of the relative solution set. Then, based on the properties of this set, we establish conditions ensuring both the existence of quasisolutions and that of solutions of the minimization problem under consideration. Such conditions will depend also on the choice of the control space* $L^r([0,1], \boldsymbol{R}^m)$ *where* $1 \leq r \leq +\infty$.

1. – Introduction.

In this paper we consider the nonlinear control process described by the differential system

$$\dot{x}(t) = f(t, x(t), u(t)) \text{ for almost all (a.a.) } \quad t \in [0, 1], \tag{1}$$

where the function $f\colon [0,1] \times \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}^n$ satisfies the following Carathédory conditions

(f_1) for any $(x, u) \in \boldsymbol{R}^n \times \boldsymbol{R}^m$, the map $f(\cdot, x, u)$ is (Lebesgue) measurable on $[0, 1]$; for a.a. $t \in [0, 1]$, the map $f(t, \cdot, \cdot)$ is continuous on $\boldsymbol{R}^n \times \boldsymbol{R}^m$;

(f_2) there exist $\alpha, \beta \in L^1([0, 1], \boldsymbol{R}_+)$ and $\gamma \in L^{r'}([0, 1], \boldsymbol{R}_+)$, where $1 \leq r' \leq +\infty$, such that

$$|f(t, x, u)| \leq \alpha(t) + \beta(t)|x| + \gamma(t)|u|$$

for all $x \in \boldsymbol{R}^n$, $u \in \boldsymbol{R}^m$ and a.a. $t \in [0, 1]$.

The function $t \to u(t)$, describing the control law, is assumed to belong to $L^r([0, 1], \boldsymbol{R}^m)$, where $1/r + 1/r' = 1$, and the function $t \to x(t)$, describing the state of (1), is in $AC([0, 1], \boldsymbol{R}^n)$, the Banach space of absolutely continuous functions $x\colon [0, 1] \to \boldsymbol{R}^n$.

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In [3] we considered a general notion of controllability, the so-called l-controllability, for the non linear control process (1). For the sake of completness we recall below the basic definitions. Let l: $AC([0,1], \boldsymbol{R}^n) \times L^r([0,1], \boldsymbol{R}^m) \to \boldsymbol{R}^k$, with $k > n$, be a (not necessarily linear) continuous «boundary» operator sending bounded sets into bounded sets. For a given $w \in \boldsymbol{R}^k$ we will say that the system (1) is *l-controllable* at w if the nonlinear boundary value control problem

$$(2) \qquad \begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \text{for a.a. } t \in [0,1] \\ l(x,u) = w, \end{cases}$$

is solvable for some $u \in L^r([0,1], \boldsymbol{R}^m)$. The system (1) is said to be *locally* l-controllable at $w_0 \in \boldsymbol{R}^k$ if there exists a neighborhood $N(w_0)$ of w_0 such that, for any $w \in N(w_0)$, problem (2) is solvable for some $u \in L^r([0,1], \boldsymbol{R}^m)$.

Finally, (1) is globally l-controllable if it is l-controllable for any $w \in \boldsymbol{R}^k$.

In [3] it is shown that one can obtain local and global l-controllability by means of controls belonging only to a subspace of $L^\infty([0,1], \boldsymbol{R}^m)$ of dimension $d = k - n$.

Let $D \subset \boldsymbol{R}^k$ be a nonempty compact set and let U_M be the closed ball of $L^r([0,1], \boldsymbol{R}^m)$, with $1 \leq r \leq +\infty$, centered at the origin and of radius $M > 0$. Let

$$S^D = \{(x,u) \in AC([0,1], \boldsymbol{R}^n) \times U_M : (x,u) \text{ is a solution of (2) for some } w \in D\}$$

and let $U_a^D = \{u \in U_M : (x,u) \in S^D\}$ be the set of admissible controls. Define the cost functional C: $AC([0,1], \boldsymbol{R}^n) \times L^r([0,1], \boldsymbol{R}^m) \to \boldsymbol{R}$ by

$$C(x,u) = \int_0^1 f_0(t, x(t), u(t))\, dt + g_0(x(1)) + h_0(x(0)),$$

where the function f_0: $[0,1] \times \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}$ satisfies assumptions (f_1)-(f_2) and g_0, h_0: $\boldsymbol{R}^n \to \boldsymbol{R}$ are continuous functions.

In this paper we consider the following

Optimization Problem (OP). Establish the existence of a couple $(x^*, u^*) \in S^D$ such that

$$C(x^*, u^*) = \inf\{C(x,u) : (x,u) \in S^D\}.$$

The paper is organized as follows. In Section 2 we shall give sufficient conditions (Theorem 1 and Theorem 2) for the nonlinear boundary value control problem (2) to be solvable using as control spaces p-dimensional subspaces of $C([0,1], \boldsymbol{R}^m)$, where $p \geq d = k - n$. More precisely, we will fix a Schauder basis $\{v_1, v_2, \ldots, v_n, \ldots\}$ of $C([0,1], \boldsymbol{R}^m)$ and we will give conditions which ensure that system (2) is solvable, with $D = \{0\}$, in the control space $U_d = \operatorname{span}\{v_1, v_2, \ldots, v_d\}$ and, for any positive integer $p > d$, we will study the structure of the solution set S_p of (2) when the controls are in a (suitable) ball of any space $U_p \supset U_d$. As a consequence we will have that $S \supset \overline{\bigcup_{p \geq d} S_p} \neq \emptyset$, where the closure is in $C([0,1], \boldsymbol{R}^n) \times L^r([0,1], \boldsymbol{R}^m)$, $1 \leq r \leq +\infty$.

In Section 3 we shall establish in Theorem 3 the existence in S of solutions of a regularized minimization problem which are quasisolutions to our problem. Conditions to ensure that the sequence $\{(x_p^*, u_p^*)\}_{p \geq d}$, such that $C(x_p^*, u_p^*) = \inf_{(x,u) \in S_p} C(x, u)$, is a minimizing sequence for C in S, are also given.

In Section 4, under the usual convexity assumptions, we shall give a result (Theorem 4) concerning the existence of solutions to (OP) when the controls are in $L^\infty([0,1], \boldsymbol{R}^m)$. Moreover, some particular cases, when the controls are in $L^r([0,1], \boldsymbol{R}^m)$ with $r \neq +\infty$, are also considered. The case when D is any nonempty compact set in $\boldsymbol{R}^k$ is also treated.

Finally, in Section 5 we give some examples, illustrating our results.

In what follows $(AC)^s$, $(C)^s$ and $(L^r)^s$, $s \in \boldsymbol{N}$, $1 \leq r \leq +\infty$, will denote the Banach spaces $AC([0,1], \boldsymbol{R}^s)$, $C([0,1], \boldsymbol{R}^s)$ and $L^r([0,1], \boldsymbol{R}^s)$ equipped with the usual norms (we will omit the superscript if $s = 1$). Finally, $|\cdot|$ and $|\cdot|_X$ will denote respectively the Euclidean norm in any finite dimensional space and the norm in any infinite dimensional Banach space X, while $\langle \cdot, \cdot \rangle$ will denote the inner product in $\boldsymbol{R}^s$ corresponding to the Euclidean norm.

2. – Existence of solutions to the nonlinear boundary value control problem (2).

Let $\{v_1, v_2, \ldots, v_n, \ldots\}$ be a Schauder basis for $(C)^m$. For any integer $p \geq d$, with $d = k - n$, let $U_p = \text{span}\{v_1, v_2, \ldots, v_d, \ldots, v_p\}$ and define the algebraic isomorphism $J_p \colon \boldsymbol{R}^p \to U_p$ as follows

$$J_p(b) = \sum_{i=1}^{p} b_i v_i \qquad \text{for any } b = (b_i)_{i=1}^p \in \boldsymbol{R}^p .$$

It is well known that on the space U_p the norm defined by $\|u\| = \left(\sum_{i=1}^{p} b_i^2 \right)^{1/2}$ is equivalent to the $(C)^m$ norm, thus $\boldsymbol{R}^p$ and U_p are topologically isomorphic, so that any bounded set in U_p is relatively compact. (i.e. J is a homeomorphism).

Finally, let $\pi_p \colon (AC)^n \times U_p \to U_p$ be the projection defined, for any $(x, u) = (x, J_p(b)) \in (AC)^n \times U_p$, by

$$\pi_p(x, u) = \begin{cases} \sum_{i=d+1}^{p} b_i v_i & \text{if } p > d, \\ 0 & \text{if } p = d. \end{cases}$$

Therefore, for $(x, u) \in (AC)^n \times U_p$, one has

$$(x, u) = (I - \pi_p)(x, u) + \pi_p(x, u) = (x, u_d) + (0, u_\alpha),$$

where $(x, u_d) \in (AC)^n \times U_d$ and $(0, u_\alpha) \in \{0\} \times U_\alpha$, with $U_\alpha = \text{span}\{0, 0, \ldots, 0, u_{d+1}, \ldots, u_p\}$ and $\alpha = p - d$. (In the sequel we will identify $(0, u_\alpha)$ with u_α).

Observe that, since any Schauder basis of $(C)^m$ is also a Schauder basis of

$(L^r)^m$ whenever $1 \leq r < +\infty$, $\{v_1, v_2, \ldots, v_n, \ldots\}$ is a common basis for all the space $(L^r)^m$, with $1 \leq r < +\infty$.

We will assume from now on that the basis $\{v_1, v_2, \ldots, v_n, \ldots\}$ of $(L^r)^m$ is generated as follows

$$v_n = z_i e_j\,,$$

where $n = (i-1)m + j$, $\{e_j\}_{j=1}^m$ is the standard basis of $\boldsymbol{R}^m$ and $\{z_i\}_{i \in \boldsymbol{N}}$ is the Schauder basis of $C([0,1], \boldsymbol{R})$ defined by

$$z_1(t) \equiv 1, \quad z_2(t) = t, \ldots, z_i(t) = \begin{cases} 0 \text{ for } t \notin \left(\dfrac{2k-2}{2^{l+1}}, \dfrac{2k}{2^{l+1}}\right), \\ 1 \text{ for } t = \dfrac{2k-1}{2^{l+1}}, \\ \text{linear in } \left[\dfrac{2k-2}{2^{l+1}}, \dfrac{2k-1}{2^{l+1}}\right] \text{ and } \left[\dfrac{2k-1}{2^{l+1}}, \dfrac{2k}{2^{l+1}}\right], \end{cases}$$

with $i = 2^l + k + 1$, $k = 1, 2, \ldots, 2^l$, $l = 0, 1, 2, \ldots$ (see [6]).

Assume now, for simplicity, that $D = \{0\}$ and omit D in the notation. Furthermore, we shall consider the vectors in $\boldsymbol{R}^n$ as the constant functions of $(AC)^n$.

We can prove the following result.

THEOREM 1. – Assume that

(i) the set

$$R_d = \{(x, u) \in (AC)^n \times U_d : (x, u) \text{ is a solution of } \dot{x}(t) = \lambda f(t, x(t), u(t))$$
$$\text{for a.a. } t \in [0, 1] \text{ and } l(x, u) = 0 \text{ for some } \lambda \in [0, 1]\}$$

is bounded in $(C)^n \times (L^r)^m$ (i.e., there exist constants $L, M > 0$ such that for any $(x, u) \in R_d$ one has $\max\limits_{t \in [0,1]} |x(t)| < L$ and $|u|_{(L^r)^m} < M$);

(ii) $\deg(l_d^0, \Omega_d^0, 0) \neq 0$, where $l_d^0 = l|_{\boldsymbol{R}^n \times U_d}$ and Ω_d^0 is any open bounded subset of $\boldsymbol{R}^n \times U_d$ containing the set $(l_d^0)^{-1}(0)$;

(iii) let $U_p^M = (U_d \times U_x) \cap U_{M'}$, for any $p > d$ and for any $u_x \in \pi_p((AC)^n \times U_p^M)$ the x-components and the u_d-components of the set

$$S_p = \{(x, u) \in (AC)^n \times U_p : (x, u) \text{ is a solution of } \dot{x}(t) = f(t, x(t), u(t))$$
$$\text{for a.a. } t \in [0, 1],\ l(x, u) = 0 \text{ and } \pi_p(x, u) = u_x\}$$

are bounded in $(C)^n$ and $(L^r)^m$ by L and M respectively.

Then for any $p \geq d$ and any $u_x \in \pi_p((AC)^n \times U_p^M)$ the nonlinear boundary value control problem

$$(3) \qquad \begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \text{for a.a. } t \in [0, 1], \\ l(x, u) = 0, \end{cases}$$

has a solution $(x, u) \in (AC)^n \times U_p^M$ with $\pi_p(x, u) = u_x$.

PROOF. – We consider at first the case $p = d$. Let $L_d\colon (AC)^n \times U_d \to (L^1)^n$ be the linear operator defined by

$$L_d(x, u)(t) = \dot{x}(t) \qquad \text{for a.a. } t \in [0, 1].$$

Clearly $\ker L_d = \boldsymbol{R}^n \times U_d$ and $\operatorname{Im} L_d = (L^1)^n$.

Let $\Lambda_d\colon (AC)^n \times U_d \to (L^1)^n \times \boldsymbol{R}^k$ be the operator defined by

$$\Lambda_d(x, u) = (L_d(x, u),\ l_d(x, u))$$

with $l_d = l|_{(AC)^n \times U_d}$.

We show now that $\deg(\Lambda_d, \Omega_d, 0) \neq 0$, where Ω_d is the open bounded set defined by $\Omega_d = \{(x, u) \in (AC)^n \times U_d\colon |x|_{(C)^n} < L,\ |u|_{(L^r)^m} < M\}$. Clearly $R_d \subset \Omega_d$. Furthermore, let $c = (a, b) \in \boldsymbol{R}^n \times \boldsymbol{R}^d$ and let $\theta\colon (L^1)^n \times \boldsymbol{R}^k \to (AC)^n$ be the operator defined by

$$\theta(y, c) = (x, J_d(b)),$$

where $x \in (AC)^n$ is given by $x(t) = a + \int\limits_0^t y(s)\, ds$ for any $t \in [0, 1]$. It is immediate that θ is an isomorphism. Consider now

$$(\Lambda_d \circ \theta)(y, c) = (y, c) - (0, c - l_d(x, J(b))).$$

Since $|\deg(\Lambda_d, \Omega_d, 0)| = |\deg(\Lambda_d \circ \theta, \widetilde{\Omega}_d, 0)|$, with $\widetilde{\Omega}_d = \theta^{-1}(\Omega_d)$, the reduction property of the topological degree implies that

$$\deg(\Lambda_d \circ \theta, \widetilde{\Omega}_d, 0) = \deg(\Lambda_d \circ \theta, \widetilde{\Omega}_d^0, 0),$$

where $\widetilde{\Omega}_d^0 = \widetilde{\Omega}_d \cap (\{0\} \times \boldsymbol{R}^k)$. On the other hand $\Lambda_d \circ \theta|_{\{0\} \times \boldsymbol{R}^k} = l_d^0(a, J(b)) = \widetilde{l}_d^0(a, b)$ with $\widetilde{l}_d^0\colon \boldsymbol{R}^n \times \boldsymbol{R}^d \to \boldsymbol{R}^k$ given by $\widetilde{l}_d^0 = l_d^0 \circ (I \times J)$, ($I$ is the identity in $\boldsymbol{R}^n$).

Therefore

$$\deg(\Lambda_d \circ \theta, \widetilde{\Omega}_d^0, 0) = \deg(\widetilde{l}_d^0, \widetilde{\Omega}_d^0, 0)$$

and the last topological degree is nonzero by assumption (ii). Let $\Phi_d\colon (AC)^n \times U_d \to (L^1)^n$ be the Nemitskii operator generated by f, that is

$$\Phi_d(x, u)(t) = f(t, x(t), u(t)) \qquad \text{for a.a. } t \in [0, 1].$$

Using (f_1)-(f_2), the Lebesgue dominated convergence theorem, the compact imbedding of $(AC)^n$ into $(L^1)^n$ and the fact that U_d is a finite dimensional subspace of $(C)^m$, one can easily see that Φ_d is a continuous and compact operator (i.e. it sends bounded sets of $(AC)^n \times U_d$ into relatively compact sets of $(L^1)^n$).

Finally, let $\psi_d\colon (AC)^n \times U_d \to (L^1)^n \times \boldsymbol{R}^k$ be the operator defined by

$$\psi_d(x, u) = (\Phi_d(x, u), 0).$$

Obviously ψ_d has the same properties as Φ_d. If we consider the homotopy $H_d\colon [0, 1] \times (AC)^n \times U_d \to (L^1)^n \times \boldsymbol{R}^k$ defined by

$$H_d(\lambda, x, u) = \Lambda_d(x, u) - \lambda\psi_d(x, u), \qquad \lambda \in [0, 1],$$

assumption (i) implies that it is admissible in $[0,1]\times\Omega_d$ (i.e. $H_d(\lambda, x, u)\neq 0$ for all $(x,u)\in\partial\Omega_d$ and any $\lambda\in[0,1]$) and so by the homotopy invariance property of the topological degree one has $\deg(\Lambda_d-\psi_d, \Omega_d, 0)\neq 0$. The solution property implies that (3) has a solution in Ω_d. This proves Theorem 1 in the case $p=d$.

Suppose now that $p>d$. By assumption (iii) for any $u_\alpha\in\pi_p((AC)^n\times U_p^M)$ the homotopy

$$\begin{cases}\dot{x}(t)=f(t,x(t),u(t)) & \text{for a.a. } t\in[0,1],\\ l(x,u)=0, & \\ \pi_p(x,u)=\mu u_\alpha & \mu\in[0,1],\end{cases}$$

is admissible in $[0,1]\times\Omega_{p'}$, where $\Omega_p=\{(x,u_d,u_\alpha)\in(AC)^n\times U_d\times U_\alpha\colon |x|_{(C)^n}<L,$ $|u_d|_{(L^r)^m}<M$ and $|u_\alpha|_{(L^r)^m}\leqslant M\}$.

Therefore for any $u_\alpha\in\pi_p((AC)^n\times U_p^M)$ it follows that

$$\deg((\Lambda_p-\psi_p, \pi_p), \Omega_p, (0,u_\alpha))=\deg(\Lambda_d-\psi_d, \Omega_d, 0)\neq 0,$$

where $\Lambda_p, \psi_p\colon (AC)^n\times U_p\to(L^1)^n\times\boldsymbol{R}^k$ are defined in the same way as Λ_d, ψ_d.

The solution property of the topological degree guarantees the existence of a solution $(x,u)\in\Omega_p$ of the problem

$$\begin{cases}\dot{x}(t)=f(t,x(t),u(t)) & \text{for a.a } t\in[0,1],\\ l(x,u)=0, & \\ \pi_p(x,u)=u_\alpha, & \end{cases}$$

for any $u_\alpha\in\pi_p((AC)^n\times U_p^M)$. This concludes the proof. ∎

REMARK 1. – Observe that, by assumption (f_2), the boundedness of the set S_p, with $p\geqslant d$, in the space $(C)^n\times(L^r)^m$ implies that the $x=x(t)$ are bounded in $(AC)^n$.

Since the subsequent results are based on Theorem 1 we shall give some useful conditions which guarantee that Theorem 1 applies.

To this aim, let A, B be bounded open subset of $\boldsymbol{R}^n$ and $\boldsymbol{R}^d$, respectively. Then we can state the following result.

THEOREM 2. – Assume that $(f_1)-(f_2)$ hold for the nonlinearity f in system (1). Suppose further that for every pair $(x,u)\in R_d$ one has $x(t)\in A$ for some $t\in[0,1]$ and $J^{-1}(u)\in B$. Then assumption (i) in Theorem 1 holds.

The proof can be found in [3, Theorem 3]. It is essentially based on the continuous dependence property of the solutions on the parameters as they vary in $A\times B$ and $\lambda\in[0,1]$ when one considers the homotopy $\dot{x}(t)-\lambda f(t,x(t),u(t))=0$, for a.a. $t\in[0,1]$, (see [1, Theorem 2.4]). Note that assumption (i) in Theorem 1 holds if for any $(x,u)\in R_d$ one has $\min\limits_{t\in[0,1]}|x(t)|+|J^{-1}(u)|\leqslant K_1$ for some constant $K_1>0$. In particular,

Theorem 2 applies if for any $a \in A$, $b \in B$ and $\lambda \in [0, 1]$ the above homotopy has a unique solution x defined on $[0, 1]$.
Let us mention that, in general, assumption (iii) in Theorem 1 is satisfied under the same conditions which ensure that assumption (i) is satisfied. Furthermore, notice that assumptions (i) and (iii) imply that S_p is a compact set in $(C)^n \times U_p^M$ for any $p \geq d$ (see (α) of Lemma 1 below).

Finally, concerning assumption (ii) in Theorem 1 we refer to [3], where we presented some methods to compute the topological degree of the boundary operator and some examples where a priori bounds have been obtained.

3. – **Quasisolutions to the optimization problem** (OP).

In the previous Section we gave conditions to ensure that the nonlinear boundary value control problem (2) admits solutions. Theorem 1 describes the structure of the (nonempty) solution set S_p, $p \geq d$, when the controls are taken in the finite dimensional space U_p. As a consequence the solution set $S \subset (AC)^n \times (L^r)^m$ is nonempty since $\bigcup_{p \geq d} S_p \subset S$. On the other hand the assumptions of Theorem 1 and $(f_1) - (f_2)$, are in general not sufficient to guarantee the existence of a solution of our optimization problem. Indeed, the u-component of any minimizing sequence $\{(x_q, u_q)\}_{q \in N} \subset S$ is not compact in $(L^r)^m$, $1 \leq r \leq +\infty$; compactness together with the closure of S in the $(C)^n \times (L^r)^m$ topology (see Lemma 2 below) would imply the existence of a minimum for C. In fact, it is not hard to show that under our assumptions concerning the functions f_0, g_0 and h_0 which define the cost functional C, we have the following

LEMMA 1. – Let $1 \leq r \leq +\infty$, the following facts hold.

(α) C is continuous in the space $(C)^n \times U_p$, $p \geq d$, equipped with the $(C)^n \times (L^r)^m$ topology. Indeed, it is continuous in the $(C)^n \times (C)^m$ topology by virtue of our choice of the Schauder basis of $(L^r)^m$, $1 \leq r < +\infty$, as a Schauder basis of $(C)^m$, and the fact that U_p and $\boldsymbol{R}^p$ are homeomorphic for any norm in U_p.

(β) For any sequence $\{(x_n, u_n)\}$ such that $x_n \to x_0$ in the $(C)^n$ norm, $u_n(t) \to u_0(t)$ for a.a. $t \in [0, 1]$ and $|u_n|_{(L^r)^m} \leq M$ for any $n \in N$, one has

$$\lim_{n \to \infty} C(x_n, u_n) = C(x_0, u_0).$$

(γ)

$$I := \inf_{(x, u) \in S} C(x, u) > -\infty$$

since S is a bounded, nonempty set in $(C)^n \times (L^r)^m$.

Furthermore, if we restrict ourselves to a reflexive control space, i.e. $(L^r)^m$ with $1 < r < +\infty$, and to weak convergence for the controls u then, in general, S is not closed in the $(C)^n \times w-(L^r)^m$ topology, where $w-(L^r)^m$ denotes the weak topology in $(L^r)^m$.

Under our assumptions we will prove the existence of approximate solutions, or quasisolutions, to our optimization problem. Specifically, we consider the following regularized optimization problem (RP).

Given $\delta > 0$, establish the existence of a pair $(x_\delta, u_\delta) \in S$ such that

$$\text{(RP)} \qquad C(x_\delta, u_\delta) = \inf \{C(x,u) + \delta d((x,u),(x_\delta,u_\delta)) \colon (x,u) \in S\},$$

where $d((x,u),(x_\delta,u_\delta)) := |x - x_\delta|_{(C)^n} + |u - u_\delta|_{(L^r)^m}$.

Clearly any solution of the original problem (OP) is a solution of the regularized one. We give the following

DEFINITION. – A solution (x_δ, u_δ) of the regularized optimization problem (RP) is called a δ-quasisolution of (OP).

In the sequel we need the following

LEMMA 2. – The solution set S is closed in the $(C)^n \times (L^r)^m$ topology, where $1 \leq r \leq +\infty$.

The proof is a direct consequence of Theorem 2.4 in [1].

The following result guarantees the existence of quasisolutions of (OP) with certain useful properties.

THEOREM 3. – Assume hypotheses (i)-(ii) and (iii) of Theorem 1. For given ε, $\lambda > 0$ let (x_0, u_0) be such that

$$C(x_0, u_0) \leq I + \varepsilon .$$

Then there exists a pair $(\hat{x}, \hat{u})$ with the following properties

$$C(\hat{x}, \hat{u}) \leq C(x_0, u_0), \tag{1}$$

$$d((\hat{x}, \hat{u}), (x_0, u_0)) \leq \lambda, \tag{2}$$

$$C(x,u) > C(\hat{x}, \hat{u}) - \varepsilon \frac{d((x,u),(\hat{x},\hat{u}))}{\lambda} \quad \text{for all } (x,u) \in S, \ (x,u) \neq (\hat{x}, \hat{u}). \tag{3}$$

PROOF. – Under assumptions (i), (ii), (iii) and Lemma 2 the solution set S is nonempty and closed in the Banach space $(C)^n \times (L^r)^m$. This implies in particular the existence of a pair $(x_0, u_0) \in S$ with the required property. Furthermore, the functional $C \colon S \to \boldsymbol{R}$ is continuous and bounded (see Lemma 1). Therefore we can apply Proposition 38.22 of [7] to conclude the proof. ■

As a particular case let $1 \leq r < +\infty$. Let $\pi \colon (AC)^n \times (L^r)^m \to (L^r)^m$ be the projection defined by

$$\pi(x, u) = \sum_{i=d+1}^{\infty} b_i v_i \,,$$

and set $\tilde{U} = \pi((AC)^n \times (L^r)^m)$. Recall that $\{v_i\}_{i=1}^{\infty}$ is a Schauder basis of $(L^r)^m$, $1 \leq r < +\infty$.

Consider the following condition

(A) for any $\tilde{u} \in \tilde{U}$ there is at most one solution $(x, u) \in (AC)^n \times (L^r)^m$ of the following problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \text{for a.a. } t \in [0, 1]\,, \\ l(x, u) = w\,, \pi(x, u) = \tilde{u}\,, \end{cases}$$

whenever $w \in \boldsymbol{R}^k$.

In the results below condition (A) is considered only for $w = 0$, since $D = \{0\}$.

LEMMA 2. – Under the assumptions of Theorem 1 and (A)

$$S = \overline{\bigcup_{p=d}^{+\infty} S_p}$$

where the closure is considered in the space $(C)^n \times (L^r)^m$.

PROOF. – By Lemma 2, $\overline{\bigcup_{p=d}^{+\infty} S_p} \subset S$. Conversely, let $(x_0, u_0) \in S$ and $\tilde{u}_0 = \pi(x_0, u_0)$. Then $u_0 = u_d^0 + \tilde{u}_0$, for some $u_d^0 \in U_d$. Since $\tilde{u}_0 \in \tilde{U} = \overline{\bigcup_{\alpha=1}^{+\infty} U_\alpha}$, with $\alpha = p - d$, there exists a sequence $\{\tilde{u}_s\}_{s \in \boldsymbol{N}} \subset \bigcup_{p \geq d} \pi((AC)^n \times U_p^M) \subset \bigcup_{\alpha=1}^{+\infty} U_\alpha$ such that $\tilde{u}_s \to \tilde{u}_0$ in the $(L^r)^m$ norm. Theorem 1 and assumption (A) ensure that, for any $s \in \boldsymbol{N}$, there exists exactly one solution $(x_s, u_s) \in S_p$ for some $p = p(s) \geq d$ with $\pi_p(x_s, u_s) = \tilde{u}_s$. Hence $\{(x_s, u_s)\}_{s \in \boldsymbol{N}} \subset \bigcup_{p=d}^{+\infty} S_p$, but $u_s = u_d^s + \tilde{u}_s$ where $\{u_d^s\}$ is a bounded sequence in the d-dimensional space U_d and $\tilde{u}_s \to \tilde{u}_0$ in the $(L^r)^m$ norm. Thus, passing to a subsequence if necessary, $u_s \to u_1$ in the $(L^r)^m$ norm and so, passing to a subsequence again, we have also that $u_s \to u_1$ a.e. in $[0, 1]$. On the other hand $|x_s|_{(C)^n} < L$ for any $s \in \boldsymbol{N}$, therefore by the continuous dependence property of the solutions x_s on the parameters u_s and on the initial conditions $x_s(0)$ (see [1, Theorem 2.4]) we conclude that, considering possibly a subsequence, $x_s \to x_1$ in the $(C)^n$ norm. Hence $(x_s, u_s) \to (x_1, u_1)$ in $(C)^n \times (L^r)^m$ and by the continuity of l we have that $(x_1, u_1) \in S$ with $\pi(x_1, u_1) = \tilde{u}_0$. Now assumption (A) guarantees that $(x_1, u_1) = (x_0, u_0)$.

This completes the proof. ■

REMARK 2. – It is easy to see that assumption (A) is satisfied if the following conditions hold

(a_1) – For any $u \in U_M$ and for any $a \in \boldsymbol{R}^n$ there exists an unique solution $x \in (AC)^n$ of the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \text{a.a. } t \in [0,1], \\ x(t_0) = a, \end{cases}$$

for some $t_0 \in [0,1]$.

(a_2) – If $(x_1, u_1), (x_2, u_2) \in (AC)^n \times U_M$ are such that $u_i = u_d^i + \tilde{u} \in U_d + \tilde{U}$, with $i = 1, 2$, and either $x_1(t) \neq x_2(t)$ for any $t \in [0,1]$, or $u_d^1(t) \neq u_d^2(t)$ for a.a. $t \in E \subseteq [0,1]$ with meas $(E) > 0$, then

$$l(x_1, u_1) \neq l(x_2, u_2).$$

REMARK 3. – In Lemma 2 we showed that any pair $(x, u) \in S$ is the limit in the $(C)^n \times (L^r)^m$ norm of a sequence $\{(x_s, u_s)\} \subset \bigcup_{p \geq d} S_p$. In fact we proved more. Condition (A) and the local surjectivity of the map $((\Lambda_p - \psi_p), \pi_p)$ at the point $(0, u_\alpha)$ (which is a consequence of the fact that $\deg(((\Lambda_p - \psi_p), \pi_p), \Omega_p, (0, u_\alpha)) \neq 0$ for any $p \geq d$) imply that for any $s \in \boldsymbol{N}$ there exist an integer $p = p(s) \geq d$ and $\varepsilon = \varepsilon(s) > 0$ such that (for $p > d$) the problem

$$(P) \qquad (\Lambda_p - \psi_p)(x, u) = (\varphi, w),$$

has exactly one solution $(y, v) \in (AC)^n \times (L^r)^m$ with $\pi_p(y, v) = v_\alpha$ for any $(\varphi, w) \in (L^1)^n \times \boldsymbol{R}^k$ and any $v_\alpha \in U_\alpha$ for which

$$|\varphi|_{(L^1)^n} + |w| + |v_\alpha - \pi_p(x_s, u_s)|_{(L^r)^m} < \varepsilon.$$

If $p = d$ then $U_0 = \{0\}$, $\pi_p(x_s, u_s) = 0$ and problem (P) has exactly one solution $(y, v) \in (AC)^n \times (L^r)^m$ for any $(\varphi, w) \in (L^1)^n \times \boldsymbol{R}^k$ with $|\varphi|_{(L^1)^n} + |w| < \varepsilon$.

Moreover, $(y, v) \to (x_s, u_s)$ in the $(C)^n \times (L^r)^m$ norm as $\varepsilon \to 0$. Therefore S consists of stable solutions (in the sense defined above) and their limits.

Assumption (A) is restrictive but, on the other hand, if we remove it we cannot represent the whole solution set S as the «limit» of the solution sets S_p, $p \geq d$.

The above observation suggests that if (A) is not satisfied, one should consider the subset of S consisting of all the stable solutions of (3) belonging to S_p, whenever $p \geq d$, and their limits in the $(C)^n \times (L^r)^m$ norm. This is, in our opinion, the most interesting set on which to minimize C.

To this end, for any $\varepsilon > 0$ and any $p > d$, let us define

$S_p^\varepsilon = \{(x, u) \in S_p$: the problem (P) has a solution $(y, v) \in (AC)^n \times U_p$ with $\pi_p(y, v) = v_\alpha$ for any $(\varphi, w) \in (L^1)^n \times \boldsymbol{R}^k$ and any $v_\alpha \in U_a$ such that $|\varphi|_{(L^1)^n} + |w| + |v_\alpha - \pi_p(x, u)|_{(L^r)^m} < \varepsilon\}$;

and, if $p = d$, let

$S_d^\varepsilon = \{(x, u) \in S_d$: the problem (P) has a solution $(y, v) \in (AC)^n \times U_d$ for any $(\varphi, w) \in (L^1)^n \times \boldsymbol{R}^k$ such that $|\varphi|_{(L^1)^n} + |w| < \varepsilon\}$.

Let

$$\mathcal{S}_p = \overline{\bigcup_{\varepsilon > 0} S_p^\varepsilon},$$

and

$$\mathcal{S} = \overline{\bigcup_{p \geq d} \mathcal{S}_p}.$$

Now, consider the optimization problem (OP) on the set $\mathcal{S}$ and state the next Lemma 3 also in this context, since $\mathcal{S}_p$ is a compact set of $(C)^n \times (L^r)^m$ for any $p \geq d$.

We can prove now the following result, under the assumptions of Theorem 1 and condition (A).

LEMMA 3. – The sequence $\{(x_p^*, u_p^*)\}_{p=d}^{+\infty} \subset S$ such that $(x_p^*, u_p^*) \in S_p$ and $C(x_p^*, u_p^*) = \inf_{(x, u) \in S_p} C(x, u)$, for any $p \geq d$, is a minimizing sequence.

PROOF. – Since S_p is a compact set in $(C)^n \times (L^r)^m$ for any $p \geq d$ and $S_{p_1} \subset S_{p_2}$ whenever $p_1 < p_2$, Lemma 1-(α) implies

$$C(x_{p_1}^*, u_{p_1}^*) = \inf_{(x, u) \in S_{p_1}} C(x, u) \geq \inf_{(x, u) \in S_{p_2}} C(x, u) = C(x_{p_2}^*, u_{p_2}^*),$$

for some $(x_{p_1}^*, u_{p_1}^*) \in S_{p_1}$ and $(x_{p_2}^*, u_{p_2}^*) \in S_{p_2}$.

For $p \geq d$ set $I_p = \inf_{(x, u) \in S_p} C(x, u)$ $(= C(x_p^*, u_p^*)$ for some $(x_p^*, u_p^*) \in S_p)$. We want to prove that $\inf_{p \in \boldsymbol{N}} I_p = \lim_{p \to +\infty} I_p = I$. Obviously $\lim_{p \to +\infty} I_p \geq I$.

Assume that $\lim_{p \to +\infty} I_p = I_0 > I$ and let $\beta = I_0 - I > 0$.

By Lemma 2, for any $q \in \boldsymbol{N}$, there exists $(x_q, u_q) \in \bigcup_{p \geq d} S_p$ such that

$$I \leq C(x_q, u_q) < I + \frac{1}{q}.$$

For $\dfrac{1}{q} < \beta$ we get

$$(o) \qquad 0 \leq C(x_q, u_q) - I < \beta, \qquad (x_q, u_q) \in S_p.$$

But

$$C(x_q, u_q) \geq C(x_p^*, u_p^*) \geq I_0 > I,$$

with $p = p(q)$, and so

$$C(x_q, u_q) - I \geq \beta,$$

for sufficiently large q, contradicting (o). ■

In conclusion, we can apply now Theorem 3 with $\lambda = \sqrt{\varepsilon}$, $\varepsilon = 1/p$, and $(x_p^*, u_p^*) \in S$ in order to derive for any $p \geq d$ the existence of an $\sqrt{\varepsilon}$-quasisolution of (OP) $(\hat{x}_p, \hat{u}_p)$ such that $d((x_p^*, u_p^*), (\hat{x}_p, \hat{u}_p)) < 1/\sqrt{p}$.

4. – Existence of solutions to the optimization problem (OP).

Under the usual convexity assumption on the multivalued vector field associated to the dynamics f, boundary condition l and the cost function C we shall give a result on the existence of solutions when the controls are in $U_M \subset (L^\infty)^m$. This kind of assumption together with the boundedness of the controls and (f_1)-(f_2) guarantees that the x-component of the set S is closed in the $(C)^m$ norm.

In order to formulate the result (Theorem 4) in a more convenient way we specialize the nonlinear boundary operator $l \colon (AC)^n \times (L^\infty)^m \to \boldsymbol{R}^k$ as follows

$$l(x, u) = \int_0^1 k(t, x(t), u(t))\, dt + m(x(1)) + n(x(0)),$$

where $k \colon [0,1] \times \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}^k$ satisfies conditions (f_1)-(f_2) and $m, n \colon \boldsymbol{R}^n \to \boldsymbol{R}^k$ are continuous functions.

Let $F(t, x, u) = (f(t, x, u), k(t, x, u), f_0(t, x, u))$ for any $(x, u) \in \boldsymbol{R}^n \times \boldsymbol{R}^m$ and for a.a. $t \in [0,1]$ and let $B(0, M) = B_M$ be the ball in $\boldsymbol{R}^m$ centered at the origin with radius $M > 0$.

We have the following result.

THEOREM 4. – Assume that

(a) the solution set S of (3) is bounded and nonempty in $(C)^n \times (L^\infty)^m$;

(b) the set $F(t, x, B_M)$ is a nonempty convex subset of $\boldsymbol{R}^{n+k+1}$ for any $x \in \boldsymbol{R}^n$ and for a.a. $t \in [0,1]$.

Then there exists a couple $(x^*, u^*) \in S$ such that

$$C(x^*, u^*) = I.$$

PROOF. – The proof follows closely similar proofs in [2] and [4]. Let $\{(x_q, u_q)\}_{q\in N} \subset S$ be a minimizing sequence and let

$$x_{0,q}(t) = \int_0^t f_0(s, x_q(s), u_q(s))\, ds$$

and

$$y_q(t) = \int_0^t k(s, x_q(s), u_q(s))\, ds .$$

Then $X_q(t) = (x_q(t),\ y_q(t),\ x_{0,q}(t))$ solves the following Cauchy problem for any $q \in N$

$$\begin{cases} \dot{X}_q(t) = F(t, x_q(t), u_q(t)) & \text{for a.a. } t\in[0,1], \\ x_q(0) = i_q , \\ y_q(0) = 0 , \\ x_{0,q}(0) = 0 . \end{cases}$$

Moreover $y_q(1) = -m(x_q(1)) - n(x_q(0))$ and by the boundedness of $\{(x_q, u_q)\}_{q\in N}$ in the space $(C)^n \times (L^\infty)^m$, (assumption (a)), and the assumptions on f, k, f_0 it follows that the sequence $\{X_q\}$ is compact in $(C)^n$. Thus, by passing to a subsequence if necessary, we get that

$$X_q \to X \qquad \text{as} \qquad q \to +\infty \qquad in \qquad (C)^n ,$$

for some $X = (x^*, y^*, x_0^*)$.

Now set

$$\mathcal{F}_q(t) = F(t, x_q(t), u_q(t)) ,$$

for a.a. $t \in [0, 1]$. We have

$$\lim_{q\to+\infty} \int_0^t \mathcal{F}_q(s)\, ds = \int_0^t \mathcal{F}_0(s)\, ds ,$$

for any $t \in [0, 1]$, with $\mathcal{F}_0 \in (L^1)^{n+k+1}$, i.e. $\{\mathcal{F}_q\}_{q\in N}$ converges weakly to $\mathcal{F}_0$ in $(L^1)^{n+k+1}$. Moreover

$$X(t) = X(0) + \int_0^t \mathcal{F}_0(s)\, ds ,$$

with $X(0) = (i^*, 0, 0)$ and $i^* = \lim_{q\to+\infty} i_q$.

The proof of the theorem will be completed once we prove the existence of a control $u^* \in U_M \subset (L^\infty)^m$ such that

$$\mathcal{F}_0(t) = F(t, x^*(t), u^*(t)) \qquad \text{a.e. in } [0,1],$$

since then

$$x^*(t) = i^* + \int_0^t f(s, x^*(s), u^*(s))\, ds,$$

$$y^*(t) = \int_0^t k(s, x^*(s), u^*(s))\, ds,$$

$$x_0^*(t) = \int_0^t f_0(s, x^*(s), u^*(s))\, ds$$

and $y^*(1) = -m(x^*(1)) - n(x^*(0))$. Thus we will have $(x^*, u^*) \in S$ and, by the continuity of C on $(C)^n \times (L^\infty)^m$, we will have $C(x^*, u^*) = I$ and the proof will be complete.

Therefore we turn to question of the existence of such a control u^*. To this end, we have that for any $q \in \boldsymbol{N}$

$$\mathcal{F}_q(t) \in F(t, x_q(t), B_M) \qquad \text{for a.a. } t \in [0,1].$$

By the weak convergence of $\{\mathcal{F}_q\}_{q \in \boldsymbol{N}}$ to $\mathcal{F}_0$ it follows that

$$\limsup_{q \to +\infty} \langle z, \mathcal{F}_q(t) \rangle \geq \langle z, \mathcal{F}_0(t) \rangle \geq \liminf_{q \to +\infty} \langle z, \mathcal{F}_q(t) \rangle,$$

for a.a. $t \in [0,1]$ and for any $z \in \boldsymbol{R}^{n+k+1}$. Thus, by the continuity of F with respect to x, passing to the limit as $q \to +\infty$ we get

$$\sup_{u \in B_M} \langle z, F(t, x^*(t), u) \rangle \geq \langle z, \mathcal{F}_0(t) \rangle \geq \inf_{u \in B_M} \langle z, F(t, x^*(t), u) \rangle.$$

Assumption (b) implies that

$$\mathcal{F}_0(t) \in F(t, x^*(t), B_M),$$

for a.a. $t \in [0,1]$. At this point we invoke a result of Roxin ([5], p. 111) concerning the existence of a control $u^* \in U_M$ such that

$$\mathcal{F}_0(t) = F(t, x^*(t), u^*(t)) \qquad \text{for a.a. } t \in [0,1].$$

This completes the proof. ■

REMARK 4. – Assumption (a) in Theorem 4 is satisfied, for example, if assumptions (i)-(ii) of Theorem 1 hold and the considerations, already made for $1 \leq r < +\infty$, on the continuous dependence of x on the parameters are adapted to this case.

We end this section with some other remarks.

REMARK 5. – Adding to the cost functional $C\colon (AC)^n \times (L^r)^m \to \boldsymbol{R}$ the functional $\gamma_0\colon (L^r)^m \to \boldsymbol{R} \cup \{\infty\}$, $1 \leq r < +\infty$, defined by

$$\gamma_0(u) = \sup_{h \in (-1,1)} \int_0^1 \left| \frac{u(t+h) - u(t)}{h} \right|^r dt\,,$$

where $u \in (L^r)^m$ is extended by 1-periodicity from $[0,1]$ to $\boldsymbol{R}$, we can show the following

(F1) Assume (i), (ii) and (iii) of Theorem 1. Then there exists a pair $(x,u) \in S$ such that $C_0(x^*, u^*) = \inf_{(x,u) \in S} C_0(x,u)$, where $C_0(x,u) = C(x,u) + \gamma_0(u)$.

The proof is based on the Riesz-Fréchet-Kolmogorov criterion of compactness in $(L^r)^m$ where $1 \leq r < +\infty$. ■

REMARK 6. – The following statement holds.

(F2) If $1 < r < +\infty$, the function f, l are linear in the control u and the function f_0 in the cost functional C is convex in u. Then there exists a pair $(x^*, u^*) \in S$ such that $C(x^*, u^*) = I$.

This fact can be proved by using weak topologies and the lower semicontinuity property of C. ■

As mentioned in the Introduction we shall describe in the last part of this note how all the previous results can be formulated in the case where D is not the singleton $\{0\}$.

Let $D \subset \boldsymbol{R}^k$ be a nonempty compact set and let $B_R \subset \boldsymbol{R}^k$ be a ball containing D. First we have to ensure that (2) is solvable for any $w \in D$. For this we have the following

PROPOSITION 1. – Let $1 \leq r \leq +\infty$ and $U_d \subset (L^r)^m$. Assume that assumptions (i)-(ii) of Theorem 1 are satisfied. Moreover, suppose that

(ii)′ the set $\{(x,u) \in (AC)^n \times U_d\colon (x,u)$ is a solution of (2) with $w \in B_R\}$ is bounded in $(C)^n \times (L^r)^m$.

Then (2) is solvable in $(AC)^n \times U_d$ for any $w \in D$.

PROOF. – For any $w \in D$, consider the homotopy

$$(\Lambda_d - \psi_d)(x,u) = (0, \mu w) \qquad \text{where } \mu \in [0,1]\,.$$

By (ii)′ this is an admissible homotopy in $(AC)^n \times (L^r)^m$, thus

$$\deg(\Lambda_d - \psi_d, \Omega_d^R, (0,w)) = \deg(\Lambda_d - \psi_d, \Omega_d, 0)\,,$$

where $\Omega_d^R \subset (AC)^n \times (L^r)^m$ is any open bounded set containing the set defined in (ii)$'$. But the topological degree of the right hand side of the previous equality is nonzero by the portion of the proof of Theorem 1 concerning the case $p = d$ (which, as already observed is valid for $r = +\infty$ too).

Therefore the assertion follows from the homotopy invariance and the solution properties of the topological degree. ■

Proposition 1 implies that S_d^D is a non empty (compact) subset of $(C)^n \times (L^r)^m$, whenever $1 \leqslant r \leqslant +\infty$, and $\deg(\Lambda_d - \psi_d, \Omega_d^R, (0, w)) \neq 0$ for any $w \in D$. Therefore it is immediate that, if $1 \leqslant r \leqslant +\infty$ and under the further assumption (iii) the conclusion of Theorem 1 holds for the nonlinear boundary value control problem (2) whenever $w \in D$.

Therefore, all the previous results can be easily reformulated in the case when the nonempty compact set D is not the singleton $\{0\}$.

5. – Examples.

The following examples illustrate how the results given above can be applied. We prefer to give simple examples in order that the meaning of our assumptions, in particular (i)-(iii) of Theorem 1, not be hidden by computations. For more general examples of nonlinear boundary value control problems we refer to [3] (see also [1]).

EXAMPLE 1. – Let $f\colon [0,1] \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ be a function satisfying the assumptions (f_1) and (f_2), with $\gamma \in L^\infty$, specified in the Introduction. Assume that the Cauchy problem

$$\text{(4)} \qquad \begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \text{for a.a. } t \in [0,1], \\ x(0) = a, \end{cases}$$

has a unique solution, defined on $[0,1]$, for every $u \in L^1$ and $a \in \boldsymbol{R}$.

Furthermore assume the following condition

(H$_1$) for any $\rho \in \boldsymbol{R}$, $f(t,p,q) \to +\infty$ as $q \to +\infty$ uniformly for $p \geqslant \rho$ and for a.a. $t \in [0,1]$; $f(t,p,q) \to -\infty$ as $q \to -\infty$ uniformly for $p \leqslant \rho$ and for a.a. $t \in [0,1]$.

Let I, J be given compact sets in $\boldsymbol{R}$ and consider the nonlinear boundary value control problem

$$\text{(5)} \qquad \begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \text{for a.a. } t \in [0,1], \\ x(0) \in I, \ x(1) \in J, \end{cases}$$

Thus $l(x,u) = \left(x(0), x(0) + \int\limits_0^1 f(t, x(t), u(t))\, dt \right) \in \boldsymbol{R} \times \boldsymbol{R}$ and $d = 1$.

Recall that we have chosen as Schauder basis of L^1 the following one:

$$v_1(t) \equiv 1, \quad v_2(t) = t, \quad v_i(t) = \begin{cases} 0 & \text{for } t \notin \left(\dfrac{2n-2}{2^{m+1}}, \dfrac{2n}{2^{m+1}}\right), \\ 1 & \text{for } t = \dfrac{2n-1}{2^{m+1}}, \\ \text{linear in} & \left[\dfrac{2n-2}{2^{m+1}}, \dfrac{2n-1}{2^{m+1}}\right] \text{ and } \left[\dfrac{2n-1}{2^{m+1}}, \dfrac{2n}{2^{m+1}}\right], \end{cases}$$

where $i = 2^m + n + 1$, $n = 1, 2, \ldots, 2^m$ and $m = 0, 1, 2, \ldots$. Observe that, in particular, $v_i(t) \geq 0$ for any $t \in [0, 1]$ and for any $i \in \boldsymbol{N}$.

Assume for the moment that $I = J = \{0\}$.

By the uniqueness property of the initial value problem (4), in order to verify assumption (i) of Theorem 1 it is sufficient to prove the boundedness of the controls in $U_1 = \operatorname{span}\{v_1\}$. In fact, in this case, the continuous dependence property of the solutions of (5) on the controls implies that the $x = x(t)$ are bounded.

To prove the boundedness of the controls by contradiction, assume that there exists a sequence $\{(x_n, u_n)\}_{n \in \boldsymbol{N}} \subset R_1$ with corresponding $\{\lambda_n\}_{n \in \boldsymbol{N}} \subset [0, 1]$ such that $|u_n| \to +\infty$ as $n \to +\infty$. Thus

$$\begin{cases} \dot{x}_n(t) = \lambda_n f(t, x_n(t), u_n) & \text{for a.a. } t \in [0, 1], \\ x_n(0) = 0, \\ \displaystyle\int_0^1 f(t, x_n(t), u_n)\, dt = 0, \end{cases}$$

where $\{\lambda_n\}_{n \in \boldsymbol{N}} \subset [0, 1]$ and $|u_n| \to +\infty$ as $n \to +\infty$.

It is easy to see that, for sufficiently large n, the condition (H_1) implies that $x_n(t) \geq 0$ for any $t \in [0, 1]$ when $\lim\limits_{n \to +\infty} u_n = +\infty$, and that $x_n(t) \leq 0$ for any $t \in [0, 1]$ if $\lim\limits_{n \to +\infty} u_n = -\infty$. Thus, by (H_1) for n sufficiently large, the condition

$$\int_0^1 f(t, x_n(t), u_n)\, dt = 0 \qquad \text{for any } n \in \boldsymbol{N},$$

cannot be satisfied, indeed $\lim\limits_{n \to +\infty} \left| \displaystyle\int_0^1 f(t, x_n(t), u_n)\, dt \right| = +\infty$.

Therefore assumption (i) of Theorem 1 is verified.

Moreover, assumption (iii) of Theorem 1 can be verified by the same argument as before taking into account the boundedness in C of the controls u_α, $\alpha \geq 1$, by the same constant and the properties of the considered Schauder basis.

In order to verify assumption (ii), observe that the map $l_1^0 \colon \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R} \times \boldsymbol{R}$

given by

$$l_1^0(a,b) \equiv \left(a, \int_0^1 f(t,a,b)\,dt\right),$$

is such that the homotopy G: $[0,1]\times \boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}\times\boldsymbol{R}$ defined as follows

$$G(\lambda,a,b) = \left(a, \int_0^1 f(t,\lambda a,b)\,dt\right),$$

is admissible, i.e. the set

$$T = \{(a,b)\in \boldsymbol{R}\times\boldsymbol{R}\colon G(\lambda,a,b)=0 \text{ for some } \lambda\in[0,1]\},$$

is bounded in $\boldsymbol{R}^2$. Therefore

$$\deg(G(0,a,b),V,0) = \deg(G(1,a,b),V,0),$$

where $V\subset \boldsymbol{R}^2$ is a bounded set containing T.

On the other hand

$$\deg(G(0,a,b),V,0)\neq 0,$$

since

$$\left\langle (a,b), \left(a, \int_0^1 f(t,0,b)\,dt\right)\right\rangle = a^2 + \int_0^1 bf(t,0,b)\,dt > 0,$$

when $a^2+b^2>R$ for some positive constant R (see [3], Proposition 1).

Finally, by using the same argument employed above to verify assumption (i) and the compactness of the sets I, $J\subset \boldsymbol{R}$, it is not hard to show that the homotopy

$$\begin{cases} \dot{x}(t) = f(t,x(t),u) & \text{for a.a. } t\in[0,1],\\ x(0)=\mu x_0, \quad x(1)=\mu x_1 & \mu\in[0,1], \end{cases}$$

where $(x,u)\in(AC)^n\times U_1$, is an admissible homotopy for any $x_0\in I$ and any $x_1\in J$.

Furthermore assumption (iii) when $(x_0,x_1)\in I\times J$ is also verified.

In conclusion all the results of Section 3 apply. ■

EXAMPLE 2. – Let f: $[0,1]\times\boldsymbol{R}^n\times\boldsymbol{R}^m\to\boldsymbol{R}^n$ be a function satisfying (f_1) and (f_2) with $\gamma\in L^1$. Let us consider the space $(L^\infty)^m$ as the control space.

Given $M>0$, define

$$I_M(t,p) = \inf_{|q|\leq M} \frac{\langle p, f(t,p,q)\rangle}{|p|^2}.$$

Assume the following condition

$$(\mathrm{H}_2) \qquad \int_0^1 \liminf_{|p| \to +\infty} I_M(t, p)\, dt > 0, \quad \text{whenever } M > 0\,.$$

Consider the nonlinear boundary value control problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \text{for a.a. } t \in [0, 1]\,, \\ \displaystyle\int_0^1 f(t, x(t), u(t))\, dt = 0\,, \\ \displaystyle\int_0^1 u(t)\, dt \in K\,, \end{cases}$$

where K is a compact set in $\boldsymbol{R}^m$. Thus

$$l(x, u) = (l_1(x, u), l_2(x, u)) = \left(\int_0^1 f(t, x(t), u(t))\, dt, \int_0^1 u(t)\, dt \right) \in \boldsymbol{R}^n \times \boldsymbol{R}^m \quad \text{and } d = m\,.$$

Let $U_m = \operatorname{span}\{e_1, te_2, \ldots, t^{m-1} e_m\}$, where $e_1, e_2, \ldots, e_m$ is the standard basis of $\boldsymbol{R}^m$. It is immediate that the set of controls belonging to U_m such that $\int_0^1 u(t)\, dt \in K$ is bounded in $(L^\infty)^m$, say, by the constant M. Therefore, by using arguments already employed in [3] and condition (H_2) we can prove that the solutions of the homotopy

$$\dot{x}(t) = \lambda f(t, x(t), u(t)) \qquad \text{for a.a. } t \in [0, 1]\,, \quad \lambda \in [0, 1]\,,$$

$$\int_0^1 f(t, x(t), u(t))\, dt = 0\,,$$

are bounded in $(AC)^n$, where $u \in U_m^M = \{u \in U_m : u(t) \in B_M \text{ for a.a. } t \in [0, 1]\}$. Thus $R_m \subset (AC)^n \times U_m$ is bounded. Furthermore, the homotopy $G: [0, 1] \times \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}^{n+m}$ defined as follows

$$G(\lambda, a, b) = \left(\int_0^1 f(t, a, J(\lambda b)(t))\, dt, \int_0^1 J(b)(t)\, dt \right),$$

is admissible. In fact, by virtue of (H_2), it can be easily proved that the set

$$T = \{(a, b) \in \boldsymbol{R}^n \times \boldsymbol{R}^m : G(\lambda, a, b) = 0 \text{ for some } \lambda \in [0, 1]\}\,,$$

is bounded in $\boldsymbol{R}^n \times \boldsymbol{R}^m$. Thus

$$\deg(G(1,a,b),V,0) = \deg(G(0,a,b),V,0),$$

where V is an open bounded set containing T.

But

$$\deg(G(0,a,b),V,0) \neq 0,$$

since the scalar product

$$\left\langle (a,b), \left(\int_0^1 f((t,a,0)\,dt, \int_0^1 J(b)(t)\,dt \right) \right\rangle,$$

is positive for $|a|^2 + |b|^2 > R$ for some positive constant R.

Thus assumptions (i)-(ii) of Theorem 1 are satisfied, so that taking the controls u in all of U_M, assumption (a) of Theorem 4 is satisfied (see Remark 4).

In conclusion, adding to (a) the convexity condition (b), Theorem 4 of Section 4 applies. ■

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