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Systems Of Set-Valued Equations In Banach Spaces

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Introduction

In this paper we give sufficient conditions for the solvability of set-valued systems of the form

$$\begin{cases} 0 \in F(x,y) \\ 0 \in G(x,y) \end{cases} \tag{1}$$

where F and G are multivalued maps, defined in the following way

$$F(x,y) = y - \hat{F}(x,y)\ ,\ G(x,y) = x - \hat{G}(x,y)$$

where $x \in X$, $y \in Y$, with X, Y Banach spaces, and $\hat{F}$ and $\hat{G}$ are upper semicontinuous, compact multivalued maps, defined on the closure of a suitable open subset U of the $X \times Y$, taking values in X and Y respectively. For simplicity we will call F and G *multivalued compact vector fields*, even if in the literature this definition may be used also with different meaning in a different context. Roughly speaking, we solve the first equation in term of y as a function of "the parameter" x considering the application $x \multimap S(x) = \{y \in Y \ :\ 0 \in F(x,y)\}$ and hence we introduce the solution set $S(x)$ in the second one. The fixed points of the composite function $\hat{G}(x, S(x))$ are the solutions of system (1).

The paper is organized as follows. In Section 1 we give some definitions about set-valued maps and we recall some known results we will need in the sequel. In Section 2 we give a first result for solving system (1) by assuming F with convex, compact values and G single valued. Successively we solve system (1) under the hypotheses that G is a multivalued admissible map and the set of solutions $S(x)$ is acyclic. We want to note here that in general, in the convex case, the solution set $S(x)$ is not necessarily acyclic.

In Section 3 we give applications of our results. More precisely, we consider the problem of finding conditions for the solvability of two point boundary value problems for a multivalued differential system. We want to point out that while such problems can be formulated within the framework of the general theory of differential inclusions, they may also be viewed as models of problems in control theory.

1 Notations and definitions.

Definition 1.1. Let X and Y be topological spaces. A set-valued map M from X into Y is said to be *upper semicontinuous at* $x \in X$ if for every neighborhood V of $M(x)$ there exists a neighborhood U of x such that $M(x) \subset V$ for every $x \in U$. If, for every $x \in X$, M is upper semicontinuous at x and $M(x)$ is compact, then M is said to be upper semicontinuous on X. If M sends bounded sets into relatively compact sets, then it is said to be *compact*. M is said to be *proper* if, for each compact set K of Y, $M^{-1}(K)$ is compact. We will denote a multivalued map M from X to Y with the symbol $M : X \multimap Y$. By an r-neighborhood of a subset Ω of a metric space X we mean the set $B(\Omega, r) = \{y \in X : \exists\, x \in \Omega$ such that $d(x, y) < r\}$.

Definition 1.2. Let X and Y be Banach spaces and $M : X \multimap Y$ be a multivalued map. We say that a continuous map $\mu : X \to Y$ is a *ϵ-graph approximation of* M (shortly ϵ-approximation) if graph $\mu \subset B$ (graph M, ϵ). We will say that a map $\mu : X \to Y$ is a *ϵ-pointwise approximation for* $M : X \multimap Y$ if $\mu(x) \in B(M(x), \epsilon)$ for all $x \in X$.

The following result is due to Cellina [2].

Theorem 1.3. *Let X and Y be metric locally convex spaces and let $M : X \multimap Y$ be an upper semicontinuous map with compact and convex values. Then for any $\epsilon > 0$ there exists an ϵ-approximation of M.*

Using this result, in [3] Cellina and Lasota gave a definition of degree (we will denote it by Deg) for multivalued compact vector fields with convex values.

Definition 1.4. Let X and Y be topological Hausdorff spaces. An upper semicontinuous map, with a finite number of points as images, $M : X \multimap Y$ will be called a *weighted map* (shortly w-map) if to each x and $y \in M(x)$ a multiplicity or weight $m(y, M(x)) \in Z$ is assigned in such a way that the following property holds

a) if U is an open set in Y with $\partial U \cap M(x) = \emptyset$, then

$$\sum_{y \in M(x) \cap U} m(y, M(x)) = \sum_{y' \in M(x') \cap U} m(y', M(x'))$$

whenever x' is close enough to x, (see [5] and [11]).

Definition 1.5. The number

$$i(M(x), U) = \sum_{y \in M(x) \cap U} m(y, M(x))$$

will be called the *index or multiplicity* of $M(x)$ in U.

If U is a connected set, the number $i(M(x),\ U)$ does not depend on $x \in X$. In this case the number $i(M) = i(M(x),\ U)$ will be called the *index of the weighted map* M, (see e.g. [11]).

Definition 1.6. ([13]) Let X be a metric space. An upper semicontinuous set valued map $M\ : X \multimap X$ is *admissible* if there are maps $G_i\ :\ Y_i \to Y_{i+1},\ i = 0,\ 1, ...,\ n$ (Y_i metric spaces, $Y_0 = Y_{n+1} =\ X$) satisfying

i) $F = G_n \circ ... \circ G_0$;

ii) G_i is upper semicontinuous with acyclic, compact values for each $i = 0,\ 1,\ ..., n$.

Each sequence $G_0, ...,\ G_n$ is called an admissible sequence for M.

We recall that the composition of upper semicontinuous maps is upper semicontinuous.

Definition 1.7. Let X be a Banach space and let $\overline{B}(0,r)$ be a closed ball in X of radius r. We will say that the upper semicontinuous map $M : \overline{B}(0,r) \multimap X$ verifies the *Borsuk-Ulam (B.U.) property on* ∂B if for all $x \in \partial B(0,r)$, $M(x)$ and $M(-x)$ are strictly separated by a hyperplane, i.e. for all $x\ \in \partial B(0,r)$ there exists a continuous functional $x^* \in X^*$, the dual space of X, such that $x^*(y) > 0$ for all $y \in M(x)$ and $x^*(y) < 0$ for all $y \in M(-x)$.

Definition 1.8. Let X, Y be metric spaces. Given $\mathcal{S} \subset X \times Y$ and $A\ \subset\ X$, denote by:

$$\mathcal{S}(x) = \{y \in\ Y\ : (x,y)\ \in \mathcal{S}\};$$
$$\mathcal{S}(A) = \{y\ \in\ Y\ : (x,y)\ \in\ \mathcal{S},\ x \in\ A\};$$
$$\mathcal{S}_x = \mathcal{S} \cap (\{x\} \times Y) \text{ and } \mathcal{S}_A = \mathcal{S} \cap (A \times Y) \text{ for } A \subset X.$$

Definition 1.9. Let X and Y be metric spaces. Let $U\ \subset\ X \times Y$ be open and locally bounded over X, i.e. for any $(x,y) \in\ U$ there exists a neighborhood $N \subset X$ of x such that U_N is a bounded set in $X \times Y$. We shall say that $F : \overline{U} \multimap Y$ is a *parametrized compact vector field* if $F(x,y) = y\ -\ \hat{F}(x,y)$ with $\hat{F}$ upper semicontinuous and $\hat{F}(D)$ relatively compact in Y for any bounded set D of $\overline{U}$. We shall denote by

$$S^F = \{(x,y) \in\ \overline{U}\ :\ y \in\ \hat{F}(x,y)\};$$
$$\mathcal{D}^F = \{x\ \in X\ :\ \mathcal{S}_x^F \cap \partial U = \emptyset\}.$$

Definition 1.10. Let $M\ :\ \overline{B}(0,r) \subset\ X \multimap X$ be an upper semicontinuous set valued map. The map M *satisfies the boundary condition "P"* if $x \in\ \ \partial B(0,r)$ and $\lambda x \in M(x)$ implies $\lambda \leq 1$.

Definition 1.11. A multivalued map $M : X \times Y \multimap Z$ is said to be *uniformly quasibounded with respect to x* if there exist $\alpha,\ \beta \in R^+$ such that:

$$\| M(x,y) \| = \sup_{z \in M(x,y)} \| z \| \leq \alpha \ \| y \| \ + \beta \text{ for any } x \in X.$$

In the sequel by $\epsilon M(\epsilon p)$ we will denote the set $B(M(B(p,\epsilon)),\ \epsilon)$.

2 Results

We want to investigate now the existence of solutions for the system

$$\begin{cases} y \in \hat{F}(x,y) \\ x = \hat{g}(x,y) \end{cases} \text{ or } \begin{cases} 0 \in y - \hat{F}(x,y) = F(x,y) \\ 0 = x - \hat{g}(x,y) = g(x,y) \end{cases} \tag{2}$$

Theorem 2.1. *Let X,Y be Banach spaces, let $U \subset X \times Y$ be an open, locally bounded set over X. Let $\hat{F} : \overline{U} \multimap Y$ be an upper semicontinuous compact map with convex values and $\hat{g} : \overline{U} \to X$ be a continuous and compact map. Suppose that there exists $r > 0$ such that $\overline{B}(0,r) \subset \mathcal{D}^F$ and $Deg\,(F(0,\cdot),U(0),0) \neq 0$. Let $T : \overline{B}(0,r) \multimap X$ be defined by $T(x) = x - \hat{T}(x)$, where $\hat{T}(x) = \hat{g}(x,\ S(x))$ and $S(x) = \{y \in Y : y \in \hat{F}(x,y)\}$. Let us suppose that for all $x \in \partial B$, such that $0 \notin T(x)$, $T(x)$ and $T(-x)$ are strictly separated by an hyperplane. Then there exists $x \in \overline{B}(0,r)$ such that $0 \in T(x)$. Hence system (2) has a solution.*

Before proving theorem 2.1 we need some preliminary results. First observe that the map $S : \overline{B}(0,r) \subset X \multimap Y$ defined by $x \multimap S(x)$ is upper semicontinuous on $\overline{B}(0,r)$. In fact the local boundedness of U implies that $S(x_0)$, is compact for every $x_0 \in X$. Moreover, if V is an open neighborhood of $S(x_0)$, then there exists an open neighborhood N of x_0, $N \subset \mathcal{D}^F$, such that $S(x) \subset V, \forall x \in N$. To see that let $y \in S(x_0)$ and let us consider neighborhoods of the form $N_{x_0} \times V_y$, with N_{x_0} a neighborhood of x_0 in $\mathcal{D}^F$ and V_y a neighborhood of y in Y, such that

$$V_y \subset U(x_0) \cap V \text{ and } N_{x_0} \times V_y \subset U.$$

Let S be the set $\{(x,y) \in X \times Y \ : \ y \in \hat{F}(x,y)\}$. By the compactness of $S_{x_0} = S \cap (\{x_0\} \times Y)$ there exists a finite number, say s, of neighborhoods of the previous form covering S_{x_0}. Let

$$N_0 = \bigcap_{i=1}^{s} N_i \text{ and } V' = \bigcup_{i=1}^{s} V_i$$

Clearly for each neighborhood N of x_0, with $N \subset N_0$, we have that $N \times V' \subset U$. Let us prove that there exists a neighborhood N of x_0 such that $S(x) \subset V'$ for all $x \in N$. Suppose not, then there exists a bounded sequence $\{(x_n,\ y_n)\}$ with $x_n \to x_0$, $y_n \in \hat{F}(x_n,\ y_n)$ and $y_n \notin V'$. As $\hat{F}$ is compact and upper semicontinuous, we may assume (by passing to a subsequence, if necessary) that

$y_n \to y_0$. Then $y_0 \in \hat{F}(x_0, y_0)$, that is $y_0 \in S(x_0)$, contradicting $S(x_0) \subset V'$. □

Note that $S(x) \neq \emptyset$ because of the hypothesis on the degree of F.

Lemma 2.2. *Let $X = R^n$, $Y = R^m$ and let $\overline{B}(0,r) \subset \mathcal{D}^F$. Then for each neighborhood W of S_B^F there exists $\epsilon > 0$ such that if $\hat{f} : \overline{U}_B \to R^m$ is an ϵ-approximation of $\hat{F}_{|\overline{U}_B}$, then $S_B^f \subset W$.*

Proof. S_B^F is a closed and compact set. In fact, let $\{(x_n, y_n)\} \subset S_B^F$ and $(x_n, y_n) \to (x_0, y_0)$. As $\{(x_n, y_n)\} \subset S_B^F$, we have that $y_n \in \hat{F}(x_n, y_n)$ for any $n \in N$. As $\hat{F}$ is upper semicontinuous, then $y_0 \in \hat{F}(x_0, y_0)$, that is $(x_0, y_0) \in S_B^F$. Then S_B^F is closed and obviously compact.

Let W be a neighborhood of S_B^F, V a ϵ_1-neighborhood of S_B^F, $V \subset W$, with $\partial V \cap \partial W = \emptyset$. Let $\epsilon_2 = d(\partial V, \partial W)$ and $A = \overline{U}_B \backslash V$. Since A is a compact set,we have that

$$\inf_{(\tilde{x},\tilde{y})\in A} \{\| s - \tilde{y} \|,\ s \in \hat{F}(\tilde{x}, \tilde{y})\} = \epsilon_3 > 0.$$

Let $\epsilon = \min \{\epsilon_1, \epsilon_2, \epsilon_3\}$ and let $\hat{f} : \overline{U} \to Y$ be an ϵ-approximation of $\hat{F}_{|\overline{U}_B}$. Let $(x, y) \in \overline{U}_B \backslash W$ and let $y = \hat{f}(x, y)$, that is (x, y) in S_B^f. Then there exists $(\overline{x}, \overline{y}) \in \overline{U}_B$ such that:
$\| (x, y) - (\overline{x}, \overline{y}) \| + \| \hat{f}(x, y) - z \| < \epsilon$ for some $z \in \hat{F}(\overline{x}, \overline{y})$.
As $\| (x, y) - (\overline{x}, \overline{y}) \| < \epsilon$ it follows that $(\overline{x}, \overline{y}) \notin S_B^F$. Since $y = \hat{f}(x, y)$ we get that $\| \overline{y} - z \| < \epsilon$. Hence $(\overline{x}, \overline{y}) \notin A$. Thus $(\overline{x}, \overline{y}) \in V \backslash S_B^F$.This is absurd, since $(x, y) \in \overline{U} \backslash W$ and $\| (x, y) - (\overline{x}, \overline{y}) \| < \epsilon$ and so $(\overline{x}, \overline{y}) \notin V$. □

Lemma 2.3. *Let $X = R^n$, $Y = R^m$ and let $\overline{B}(0,r) \subset \mathcal{D}^F$. There exists an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ there exists $\hat{f} : \overline{U} \to R^m$, such that $\hat{f}_{|\overline{U}_B}$ is an ϵ-approximation of $\hat{F}_{|\overline{U}_B}$ and*
a) S^f is a finite subset of $U(x), \forall x \in \overline{B}(0, r)$;
b) $Deg(F(0,\cdot),\ U(0),\ 0) = Deg\ (\ f(0,\cdot),\ U(0),\ 0)$.

Proof. The definition of degree given by Cellina and Lasota, [3], ensures the existence of a positive number, say ϵ_0, with the property that for any $\epsilon < \epsilon_0$ every ϵ-approximation of $F_{|\overline{U}_B}$ has the same degree as $F_{|\overline{U}_B}$. Fix $\epsilon < \epsilon_0$ and let $\tilde{f} : \overline{U}_B \to R^m$ be an $\epsilon/2$ approximation of $\hat{F}_{|\overline{U}_B}$. Using the same arguments as in Lemma 2.4 of [11] we can prove that there exists $\tilde{f}_1 : \overline{U}_B \to R^m$, $\epsilon/2$-pointwise approximation of $\tilde{f}$, that satisfies property a). Then $\tilde{f}_1$ is an ϵ-approximation of $\hat{F}_{|\overline{U}_B}$ that satisfies property a). The map $\hat{f}$ is then any continuous extension to $\overline{U}$ of $\tilde{f}_1$, therefore b) follows immediately from the choice of ϵ. □

Lemma 2.4. *Let X and Y be Banach spaces. Let U be a locally bounded, open subset in $X \times Y$. Let us suppose that for all $x \in \mathcal{D}^F$ the equation $y \in \hat{F}(x, y)$*

has only isolated solutions. Then the application $x \multimap S(x)$ *is a w-map and* $i(S) = Deg(F(x,\cdot),\ U(x),\ 0)$.

Proof. We have already seen that the map $x \multimap S(x)$ is an upper semicontinuous map. We want to prove that to any $y \in S(x)$ it is possible to associate an integer $m(y,\ S(x))$ with the property of Definition 1.4. If y is an isolated solution for $\hat{F}$ then there exists a neighborhood Ω of y such that $\Omega \cap S(x) = \{y\}$.

Let us define $m(y, S(x)) = Deg\ (\ F(x,\cdot),\ \Omega,\ 0)$. Using the excision property, $m(y,\ S(x))$ does not depend on the choice of Ω provided that $\Omega \cap S(x) = \{y\}$. Let W be an open set in Y such that $S(x) \cap \partial W = \emptyset$. As S is upper semicontinuous there exists a ball $\overline{B}(x,r)$ such that $\forall x' \in \overline{B}(x,r)$ we get $S(x') \cap \partial W = \emptyset$.

Let us consider now the following homotopy, $H : [0,1] \times W \multimap Y$, defined by

$$H(t,y) = F(tx + (1-t)x', y).$$

As $tx + (1-t)x' \in \overline{B}(x,r)$ for all $t \in [0,1]$, H is an admissible homotopy between $F(x,\cdot)_{|W}$ and $F(x',\cdot)_{|W}$. From this fact, using the additivity property of the degree, we get:

$$\sum_{y \in S(x) \cap W} m(y, S(x)) = Deg(\ F(x,\cdot),\ W, 0) =$$

$$Deg(\ F(x',\cdot),\ W,\ 0) = \sum_{y \in S(x') \cap W} m(y, S(x')).$$

□

The following lemmata, whose proofs can be found in [11], hold.

Lemma 2.5. *Let* $B = B(0,r)$ *be an open ball in* R^n *and let* $M : \overline{B} \multimap R^n$ *be a w-map with* $i(M) \neq 0$. *If* M *verifies the B.U. condition on* ∂B, *then there exists* $x \in \overline{B}(0,r)$ *such that* $0 \in M(x)$.

Lemma 2.6. *Let* $M\ :\ \overline{B} \multimap X$ *be a compact vector field satisfying the B.U. property. Then there exists* $\epsilon > 0$ *such that every* ϵ*-approximation of* M *satisfies the B.U. property.*

We can now give the proof of Theorem 2.1

Proof. The result is obviously true if there exists $x \in \partial B$, $B = B(0,r)$, such that $0 \in T(x)$. In any other case, assume that system (2) does not have solutions (x,y) with $y \in S(\overline{B})$, i.e. $0 \notin T(x)$ for all $x \in \overline{B}$. Since T is an upper semicontinuous and "compact" vector field, its image is closed. The assumption $0 \notin T(x)$ implies the existence of $\epsilon_1 > 0$ such that $B(0,\epsilon_1) \cap T(\overline{B}(0,r)) = \emptyset$. On the other hand from Lemma 2.6 there exists ϵ_2 such that every ϵ_2-approximation T^ϵ of T, $T^\epsilon : \overline{B} \multimap X$ verifies the B.U. property.

Let $\delta = \min\ \{\epsilon_1,\epsilon_2\}$ and let $V \subset U$ be defined by: $V = \{(x,y) \in U\ :\ (x, g(x,y)) \in \delta\ \ \mathrm{Gr}\ T\}$, where Gr T stands for the graph of T and δA is the

δ-neighborhood of the set A. Clearly V is an open set being the inverse image of the open set δ Gr T, through the continuous map g.

We divide the proof into three parts.

First part. $X = R^n$, $Y = R^m$.

Let ϵ^* be that one given in Lemma 2.2, i.e. every ϵ^* approximation $\hat{f}$ of $\hat{F}$ has the property that $S_B^f \subset V$. Let ϵ_0 given by Lemma 2.3 and let $\epsilon' = \min\{\epsilon^*, \epsilon_0, \delta\}$. By Lemmata 2.3 and 2.4 there exists a continuous map $\hat{f} : \overline{V} \to R^n$ which is an ϵ'-approximation for $\hat{F}$ on $\overline{V}_B$ and such that the set valued map $S' : \overline{B} \multimap R^n$ defined by $x \multimap S'(x) = S^f(x)$, $f = I - \hat{f}$, is a w-map such that $S'_B \subset V$. The index of the map is given by

$$i(S') = Deg(F(0,\cdot), V(0), 0) = Deg(F(0,\cdot), U(0), 0) \neq 0.$$

The set valued map $T'(x) = g(x, S'(x))$ is a w-map with index $i(T') = i(S') \neq 0$, (see [5]). As $S'_B \subset V$ we have that Gr $T' \subset \epsilon'$ Gr T. Since T' verifies the B.U. property then, by Lemma 2.5, there exists $x \in \overline{B}$ such that $0 \in T'(x)$. Then $0 \in \epsilon' T(\epsilon' x)$. This contradiction establishes the result.

Second part. $X = R^n$, *Y Banach space.*

Let $\tilde{f}$ be a $\frac{\epsilon'}{2}$-approximation of $\hat{F}$ on $\overline{U}_B$, and let $\hat{f}$ be a $\frac{\epsilon'}{2}$-pointwise approximation of $\tilde{f}$ whose range is contained in a finite dimensional subset K_1 of Y. By Lemma 2.2 and the properties of the degree, we get

$$\begin{aligned} 0 \neq Deg(F(0,\cdot), V(0), 0) &= deg(f(0,\cdot), V(0), 0) \\ &= deg(f(0,\cdot)_{|V_1}, V(0) \cap K_1, 0) \end{aligned}$$

with $V_1 = V \cap (X \times K_1)$ and $S_B^f \subset V_1$. Then $S_B^f \neq \emptyset$. Let $g_1 = g_{|V_1}$ and $\hat{f}_1 = \hat{f}_{|V_1}$. These two maps satisfy the hypotheses of Theorem 2.1. Then, for the first part of the proof, the set valued map $T''(x) = g_1(x, S^{f_1}(x))$ has a zero in $\overline{B}$. As $S_B^{f_1}(x) \subset V$ we have that Gr $T'' \subset \epsilon'$ Gr T, contradicting the fact that $0 \notin \epsilon_1 T(\epsilon_1 x)$.

Third part. X,Y *Banach spaces.*

Let $\hat{g}_2 : \overline{U}_B \to X$ be an ϵ-pointwise approximation of $\hat{g}$ on $\overline{U}_B$ with finite dimensional range. Let $X_1 \subset X$ be the subspace containing the range of $\hat{g}_2$ and let $g_2 = 1 - \hat{g}_{2|(X_1 \times Y) \cap \overline{U}_B}$ and $\hat{F}_2 = \hat{F}_{|(X_1 \times Y) \cap \overline{U}_B}$. Let $T' : \overline{B}' = \overline{B} \cap X_1 \multimap X_1$ defined by $T'(x) = g_2(x, S(x))$. T' is an ϵ-approximation of $T_{|\overline{B}}$ and by Lemma 2.6, T' satisfies the B.U. property on $\partial B'$. By the second part of the proof T' has a zero on $\overline{B}' \subset \overline{B}$, contradicting the fact that $0 \notin \epsilon\, T(\epsilon x)$. □

With a similar proof to the one of Theorem 1.4 of [11] we can prove the following

Theorem 2.7 *Let X and Y be as in Theorem 2.1. Let U be a locally bounded open set in $X \times Y$ and let $\hat{F} : \overline{U} \multimap Y$ be an upper semicontinuous convex valued mapping. Consider a compact, convex set $Q \subset X$ such that for every $x \in Q$ we* *have $y \notin \hat{F}(x,y)$ on $\partial U(x)$. Assume that for some (and hence for all) $x \in Q$ we have Deg $(F(x,\cdot),\ U(x),0) \neq 0$. If $\hat{g}\ :\ S_Q \to X$ is any continuous map such that $\hat{g}(x,S(x)) \subset Q$ for any $x \in Q$, then there exists a solution $(x,y) \in U$ of (2) with $x \in Q$ and $y \in S(x)$.*

In the next theorem we give an existence result for system (2) under the assumption that F and G are both multivalued maps and G is admissible. However we have to assume that the set $S(x)$ is acyclic for every $x \in \mathcal{D}^F$. Notice that this assumption holds in many cases (see e.g. [7] and [8]).

Theorem 2.8 *Let X, Y be Banach spaces, let $U \subset X \times Y$ be an open, locally bounded set. Let $\hat{F}\ :\ \overline{U} \multimap Y$ be an upper semicontinuous, uniformly quasibounded with respect to x, compact map. Let us suppose that there exists $r > 0$ such that $\overline{B}(0,r) \subset \mathcal{D}^F$ and for any $x \in \mathcal{D}^F$ the set $S(x) = \{y \in Y : y \in \hat{F}(x,y)\}$ is non empty and acyclic. Let $\hat{G}\ :\ \overline{U} \multimap X$ be a compact, admissible map, and $\hat{T}\ :\ \overline{B}(0,r) \multimap X$ be the map defined by:*

$$x \multimap \hat{T}(x) = \hat{G}(x, S(x)).$$

Suppose that the map $\hat{T}(x)$ satisfies property "P" of Definition 1.10. Then the system

$$\begin{cases} y \in \hat{F}(x,y) \\ x \in \hat{G}(x,y) \end{cases} \tag{1}$$

has a solution.

Proof. We have already proved that the map S is upper semicontinuous. Therefore, $\hat{T}$ is an admissible map with compact values. Furthermore, being $\hat{G}$ compact and $\hat{F}$ uniformly quasibounded, we get that $\hat{T}$ is a compact map. We want to show that $\hat{T}$ has fixed point in $\overline{B}(0,r)$.

Using Lemma 2 in [10] we know that there exists a compact convex set $K \subset \overline{B}(0,r)$ such that $\overline{co}(\pi \circ \hat{T}(K)) = K$, where π is the radial projection of X on $\overline{B}(0,r)$. Since $\pi \circ \hat{T}$ is admissible , there exists $x \in K$ such that $x \in \pi \circ \hat{T}(x)$. Condition "P" implies that $x \in \hat{T}(x)$. □

3 Applications

In this section we present two applications of our results to problems involving convex-valued differential inclusions. While such problems can be formulated within the framework of the general theory of differential inclusions, a topic of independent interest, they may also be viewed as models of problems of a very different nature, for example: control theory. To be definite, consider a nonlinear control process described by a system of ordinary differential equations of the form:

$$\dot{x} = f(t,x,u) \tag{C}$$

where $f : [0,1] \times R^n \times R^m \to R^n$ satisfies Caratheodory's conditions and the control u is in Ω, with Ω a non-empty compact subset of R^m. If for $(t,x) \in [0,1] \times R^n$ we set

$$f(t,x,\Omega) = \Phi(t,x)$$

and assume that the multivalued map Φ is convex, then the trajectories of system (C) corresponding to controls from the set of functions $\mathcal{U} = \{u \in L^\infty((0,1),\ R^m) : u(t) \in \Omega \text{ for a.a. } t \in [0,1]\}$ are precisely those corresponding to the multivalued differential equation

$$\dot{x} \in \Phi(t,x).$$

One of the advantages of dealing with this equation rather than the original equation (C) is in the fact that in various situations it is easier to use differential inclusions in order to determine conditions sufficient to guarantee specified behaviour of the trajectories. On differential inclusions and their relationships with other fields see, for example, [1]. As mentionated above, the following two examples may also be viewed as control problems. Specifically, example 1 may be viewed as a problem of periodic controllability. Such problems have been treated in [9] and in [12] using degree theory (see also references therein). Example 2 may be considered as a reachability problem between two given sets. On this subject, see for example [4].

Example 1. Consider the initial value problem

$$\begin{cases} \dot{y} \in \Phi(t,y) \\ y(0) = y_0 \end{cases} \tag{E1}$$

where $\Phi : [0,1] \times R^n \to R^n$ is a Caratheodory function with compact, convex values, i.e. Φ satisfies the following conditions

(f1) Φ is a t-measurable, y-upper semicontinuous function;

(f2) for each $\rho > 0$ there exists $\alpha_\rho,\ \beta_\rho \in L^1((0,1),R)$ such that $|\ \Phi(t,p)\ | \leq \alpha_\rho(t) + \beta_\rho(t)\ |\ p\ |$, for a.a. $t \in [0,1]$ and every $p \in R^n$ with $|p| \leq \rho$, where

$$|\ \Phi(t,p)\ | = \sup_{z \in \Phi(t,p)} |\ z\ |\ .$$

Under our assumptions (E1) is equivalent to the integral form $y \in \hat{F}(y_0, y)$, where $\hat{F} : R^n \times (C)^n \to (C)^n$ is defined by:

$$(y_0, y) \multimap y_0 + \int_0^t \Phi(s, y(s))\ ds$$

where the integral is intended in the Aumann sense and $(C)^n$ stands for the Banach space $C([0,1],\ R^n)$.

Formulation of the problem: We want to give conditions on the map $\Phi(t,y)$ in order to prove, using Theorem 2.8, the existence of an initial state y_o corresponding to a 1-periodic solution of (E1).

First of all observe that, since Φ satisfies (f1) - (f2), for any $y_0 \in R^n$ the solution set $S(y_0)$ is a non empty, compact, acyclic subset of $(C)^n$. Moreover the solution map $S : R^n \to (C)^n$ is upper semicontinuous and sends bounded sets of R^n into bounded sets of $(C)^n$, which in turn, by (f2) are bounded in $AC([0,1], R^n) = (AC)^n$, and so they are compact in $(C)^n$.
Assume the following condition
(C1)-there exists $r > 0$ such that

$$\int_o^1 s(t)dt \leq 0,$$

where $s(t) = \sup_{|y| \geq r} \sigma(t,y)$ for almost all $t \in [0,1]$ and $\sigma(t,y) = \sup_{z \in \Phi(t,y)} < y, z >$.
Here $< \cdot, \cdot >$ denotes the standard Euclidean inner product in R^n.

In what follows we will rewrite our problem in a suitable form to apply Theorem 2.8. Since the set

$$\bigcup_{y_0 \in \overline{B}(0,r)} S(y_0)$$

is bounded in $(C)^n$, say by $\rho > 0$, we define a bounded, open set $U \subset R^n \times (C)^n$ by $U = B(0,r) \times B(0,\rho)$.
Let $\hat{G} : \overline{U} \multimap R^n$ be the map defined by $\hat{G}(y_0, y) = y(1)$. It is easy to see that $\hat{G}$ is a compact, continuous map.

Consider the map $\hat{T} : \overline{B}(0,r) \multimap R^n$, defined by

$$y_0 \multimap \hat{T}(y_0) = \hat{G}(y_0, S(y_0)).$$

Clearly $\hat{T}$ is an admissible map and a fixed point of $\hat{T}$ is the initial condition of a 1-periodic solution to (E1). We will prove the existence of a fixed point of $\hat{T}$ by showing that, via the condition (C1), $\hat{T}$ satisfies property "P". For this, let us prove the following

Proposition 3.1. *Assume condition (C1). For any $y \in S(y_0)$ with $|y_0| = r$ we have that $| y(1) | \leq r$.*

Proof :. Let $\tau_o = sup\{t \in [0,1] \ : | y(t) | \leq r\}$. If $\tau_0 = 1$, we are done.
On the other hand, if $\tau_0 < 1$, then $| y(t) | > r$ for any $t \in (\tau_0, 1)$. By integrating on $(\tau_0, 1)$ the inequality

$$\frac{d}{dt} \frac{| y(t) |^2}{2} = < y(t), \ \dot{y}(t) > \leq s(t) \ \text{ for a.a. } t \in [0,1],$$

and using (C1), we obtain that

$$\frac{1}{2} [| y(1) |^2 - | y(\tau_0) |^2] \leq 0 \text{ and so } | y(1) | \leq r.$$

□

Now it is immediate to see that Proposition 3.1 implies that the map $\hat{T}$ satisfies property "P", in fact from $\lambda y_0 \in \hat{T}(y_0)$ it follows $\lambda \leq 1$.

Notice that the problem treated in Example 1, under different assumptions, can be solved by using Theorem 2.1.

Example 2. We consider the following system:

$$\begin{cases} \dot{y} \in \Phi(t, y) \\ y(0) = y_0 \\ y(1) \in K \subset R^n \end{cases} \tag{E2}$$

where Φ is the map defined as in Example 1.

Formulation of the problem: Given $K \subset R^n$, K compact and acyclic, we want to give conditions ensuring the existence of y_0 in a suitable ball $B(0,r) \subset R^n$, such that (E2) is solvable.

Assume the following condition.

(C2)- there exists $r > 0$ such that

$$r^2 = \int_0^1 i(t, y_0)\, dt \geq \sup_{y_1 \in K} < y_0, y_1 > \quad \text{for any } | y_0 |= r,$$

$$\text{where } i(t, y_0) = \inf_{|y| \leq \rho} \inf_{z \in \Phi(t,y)} < y_0, z >,$$

for a.a.$t \in [0,1]$, $|y_0| = r$, and ρ is determined as in Example 1.

Let $U = B(0,r) \times B(0,\rho) \subset R^n \times (C)^n$ and let $\hat{G} : \overline{U} \multimap R^n$ be the map defined by

$$\hat{G}(y_0, y) = \{y_0 + y_1 - y(1),\ y_1 \in K\}.$$

$\hat{G}$ is a compact, continuous map with compact, acyclic values.

Consider the map $\hat{T} : \overline{B}(0,r) \multimap R^n$ defined by $y_0 \multimap \hat{T}(y_0) = \hat{G}(y_0, S(y_0))$. Clearly $\hat{T}$ is an admissible map and a fixed point of $\hat{T}$ will be a solution to problem (E2).

We prove now the following proposition.

Proposition 3.2. *Assume condition (C2). Then* $< y_0,\ x > \geq 0$ for any $x \in T(y_0) = (I - \hat{T})(y_0)$.

Proof. Let $| y_0 |= r$. For a given $x \in T(y_0)$ we have,

$$< y_0,\ x >= | y_0 |^2 - < y_0,\ y_1 > +$$
$$< y_0, \int_0^1 \dot{y}(t)dt > \text{ for some } y \in S(y_0) \text{ and } y_1 \in K.$$

By using (C2) we obtain the assertion. □

This result ensures that the property "P" is satisfied by the map $\hat{T}$. In fact, if for some $| y_0 |= r$ we have that

$$\lambda y_0 = -\int_0^1 \dot{y}(t)\, dt + y_1, \quad \text{for some } y \in S(y_0) \text{ and } y_1 \in K,$$

or equivalently $(1-\lambda)y_0 \in (I - \hat{T})(y_0)$, then using Proposition 3.2 we obtain $(1-\lambda)|y_0|^2 \geq 0$ and so $\lambda \leq 1$.

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