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# A TRACKING PROBLEM FOR UNCERTAIN VECTOR SYSTEMS*

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## 1. PROBLEM STATEMENT

WE CONSIDER a nonlinear control system with deterministic uncertain dynamics, modelled by a differential inclusion in $\mathbb{R}^n$ of the form

$$\dot{x} \in F(t, x, u), \qquad t \geq 0 \qquad u \in U \subset \mathbb{R}^m, \qquad x \in \mathbb{R}^n, \tag{1}$$

where $U$ is a specified subset, $F$ is a $t$-measurable, $(x, u)$-continuous map and $F(t, x, u)$ is a closed, convex set for $(t, x, u) \in [0, \infty) \times \mathbb{R}^n \times U = \mathfrak{D}$. The initial state $x(0)$ is also uncertain, but bounded, we assume $|x(0)| \leq M$ for some constant $M \in \mathbb{R}^+$. Concerning the dynamics, for the moment we suppose the Caratheodory selection property, that is

($H1$) Every system dynamics is given by some Caratheodory function

$$f : \mathfrak{D} \to \mathbb{R}^n \text{ such that } f(t, x, u) \in F(t, x, u).$$

We now suppose we are given a fixed time-invariant linear control model

$$\begin{cases} \dot{y} = Ay + Bv \quad y(t) \in \mathbb{R}^n, \qquad v(t) \in W \subset \mathbb{R}^m \\ y(0) = y_0 \quad |y_0| \leq N \quad N \in \mathbb{R}^+ \end{cases}$$

where $W$ is a specified subset.

The problem that we want to study is that of determining a state feedback control under which the nonlinear system is asymptotically equivalent to a given state of the linear model.

More precisely, given an $\alpha > 0$ and any fixed control pair (y, $v$) for the linear model, we want to construct a feedback control law $\bar{u}$ such that every choice of dynamics $f(t, x, \bar{u}(t, x))$ from $F(t, x, \bar{u}(t, x))$ leads to a state $x(t)$ satisfying

$$|y(t) - x(t)| \leq a e^{-\alpha t} \quad (a \in \mathbb{R}^+), \qquad \text{for } t \geq T$$

for some $T$ independent from $x$ and estimated by known data.

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Since in general the control law $(t, x) \to \bar{u}(t, x)$ will not be continuous, the state $x$ corresponding to such a control will be a Filippov solution [5] on $[0, +\infty)$ of the equation

$$\begin{cases} \dot{x} = f(t, x, \bar{u}(t, x)), \\ x(0) = x_0, \qquad |x_0| \le M. \end{cases} \tag{2}$$

In Section 2 we solve this problem using an approach which relies on the theory of variable structure systems (V.S.S.) (see [2] and [11] for a complete introduction to the topic). Specifically, we define a sliding manifold $S$ in $\mathbb{R}^+ \times \mathbb{R}^n$, and we consider a nondifferentiable Liapunov function $V$ defined on $\mathbb{R}^+ \times \mathbb{R}^n$. By using the generalized gradient we can give a condition, $(H2)$, which will permit us to solve the problem (theorem 1). This result will generalize those given in [1, 6, 8].

In Section 3 we describe a class of dynamical systems satisfying $(H2)$. Finally, in Section 4 we give another condition, based on the dichotomy property of the dynamics for the error term $E(t) = y(t) - x(t)$.

## 2. RESULTS USING VARIABLE STRUCTURE CONTROL

In this section we will give conditions for the solution of the problem stated above based on a V.S.S. approach. To this end let us consider a $n \times n$ matrix $C$ such that its logarithmic norm $\mu(C) \le -\alpha < 0$ (see [4], for the definition of $\mu$ and its properties; of course $\mu(C)$ depends on our choice of norm in $\mathbb{R}^n$). However it will always be the case that under our assumption the spectrum of $C$ is contained in $\{z \in \mathbb{C} : \text{Re}\, z \le -\alpha < 0\}$. Let $\mathcal{A}$, $\mathcal{B}$ be open sets containing all the continuous states of the nonlinear system and of the model dynamics, respectively, and let $W$ be the set in which the control variable $v$ takes values.

We consider the error vector $E = y - x$, and the functions

$$s(t, y, x, C) = y - x - e^{Ct}c_0, \qquad p(t, y, v, C) = Ay + Bv - Ce^{Ct}c_0$$

$$V(s) = \sum_{i=1}^{n} |s_i|$$

where $t \in [0, \infty)$; $x$ and $x_0$ belong to $\mathcal{A}$; $y$ and $y_0$ belong to $\mathcal{B}$; $v \in W$, $c_0 = c_0(x_0, y_0) \in \mathbb{R}^n$, $|c_0| \le L$. We select an arbitrary control-state pair $y = y(t)$, $v = v(t)$ for the linear model and we fix this selection. In this context the sliding manifold is given by the set $S = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : s(t, y, x, C) = 0\}$. We posit the following condition:

$(H2)$ there exists a constant $k \ne 0$ such that for every $t \in [0, \infty)$, $x \in \mathcal{A}$, we can find $\bar{u} = \bar{u}(t, x) \in U$ such that

$$\inf_{w \in F(t, x, \bar{u}(t, x))} \zeta \cdot w \ge \zeta \cdot p(t, y, v, C) + k^2 \tag{3}$$

for any $\zeta \in \partial V(s)|_{s = s(t, y, x, C)}$ where $a \cdot b$, $a, b \in \mathbb{R}^n$ denotes the scalar product and $\partial V(s)$ is the generalized gradient of the map $V$ at the point $s$ (see [3]). We need some preliminary results concerning the map $V$.

LEMMA 1. Let $V : \mathbb{R}^n \to \mathbb{R}$ be defined as above. Then the following hold

(i) The function $V(s(t))$, where $s(t) = y(t) - x(t) - e^{Ct}c_0$ and $x(t)$ is any solution of (2), is absolutely continuous on $[0, +\infty)$;

(ii) When the derivative $(\mathrm{d}/\mathrm{d}t)V(s(t))$ exists, i.e. for a.a. $t \in [0, +\infty)$, it is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}V(s(t)) = \zeta \cdot \dot{s}(t) \qquad \forall \zeta \in \partial V(s(t)).$$

(iii) If

$$\frac{\mathrm{d}}{\mathrm{d}t}V(s(t)) \leq -\gamma^2 < 0$$

for a.a. $t \in [0, +\infty) \cap \{t : s(t) \neq 0\}$ then there exists $T > 0$ such that $s(t) = 0$ for any $t \geq T$.

*Proof.* (i) is obvious, while a proof of (ii)–(iii) can be found in [8]. ■

LEMMA 2. For any constant $k \neq 0$ the multivalued map

$$(t, x) \to \{u \in U : \inf_{w \in F(t,x,u)} \zeta \cdot w \geq \zeta \cdot p(t, y, v, C) + k^2\} = U_\zeta(t, x)$$

where $\zeta \in \partial V(s)|_{s=s(t,y,x,C)}$, has the following properties:
(a) $(t, x) \to U_\zeta(t, x)$ is measurable, where $\zeta = \zeta(t, x)$,
(b) $(t, x) \to \hat{U}(t, x) = \bigcap_\zeta U_\zeta(t, x)$ is measurable.
Therefore there exists a measurable selection $(t, x) \to \bar{u}(t, x)$ of $\hat{U}(t, x)$.

*Proof.* (a) For any $(t, x) \in [0, +\infty) \times \mathcal{A}$ and any $\zeta \in \partial V(s)$ with $s = s(t, y, x, c)$, we have that

$$U_\zeta(t, x) = \{u \in U : \zeta \cdot w \in [\zeta \cdot (Ay(t) + Bv(t) - Ce^{Ct}c_0) + k^2, +\infty), \qquad \text{for any } w \in F(t, x, u)\}.$$

The multivalued map $(t, x, u) \to \{\zeta \cdot w, w \in F(t, x, u)\}$ is $(t, x)$-measurable and $u$-continuous, while $(t, x) \to [\zeta \cdot (Ay(t) + Bv(t) - Ce^{Ct}c_0) + k^2, +\infty)$ is measurable. Therefore, corollary 1Q in [9] applies to conclude that $(t, x) \to U_\zeta(t, x)$ is a measurable map with closed values.

(b) Since $\partial V(s)$ is a convex polyhedron with at most $2^n$ vertices $\zeta_i$, any vector $\zeta \in \partial V(s)$ can be written in the following way

$$\zeta = \sum_{i=1}^{2^n} \lambda_i \zeta_i, \qquad \lambda_i \geq 0, \qquad \sum_{i=1}^{2^n} \lambda_i = 1.$$

On the other hand, for any $\zeta \in \partial V(s)$ the following inclusion holds

$$U_\zeta(t, x) \supseteq \bigcap_{i=1}^{2^n} U_{\zeta_i}(y, x).$$

In fact let $u_0 \in \bigcap_{i=1}^{2^n} U_{\zeta_i}(t, x)$. Then

$$\zeta_i \cdot w \geq \zeta_i \cdot (Ay(t) + Bv(t) - Ce^{Ct}c_0) + k^2 i = 1, 2, \ldots, 2^n$$

for any $w \in F(t, x, u_0)$. For any scalar $\lambda_i > 0$ $i = 1, 2, \ldots, 2^n$ we have

$$\lambda_i \zeta_i \cdot w \geq \lambda_i \zeta_i \cdot (Ay(t) + Bv(t) - Ce^{Ct}c_0) + \lambda_i k^2.$$

Summing, we see that

$$\zeta \cdot w \geq \zeta \cdot (Ay(t) + Bv(t) - Ce^{Ct}c_0) + k^2$$

for any $w \in F(t, x, u_0)$, i.e. $u_0 \in U_\zeta(t, x)$, therefore for any $\zeta \in \partial V(s)$

$$\hat{U}(t, x) \supseteq \bigcap_{i=1}^{2^n} U_{\zeta_i}(t, x).$$

But

$$\hat{U}(t, x) \subseteq \bigcap_{i=1}^{2^n} U_{\zeta_i}(t, x).$$

This concludes the proof of (b). ■

*Remark* 1. The proof of lemma 2 shows, incidentally, that it is sufficient to check condition (3) only for the vertices $\zeta_i$ of $\partial V(s)$.

THEOREM 3. Let $(y, v)$ be a fixed response control pair for the linear model. Assume $(H_1)$ and $(H_2)$ and let $x$ be any Filippov solution of (2) on $[0, +\infty)$ for any system dynamics $f$. Then there exists $T \leq 2n\ (L + M + N)/k^2$ such that $s(t) = s(t, y(t), x(t), C) = y(t) - x(t) - e^{Ct}c_0 = \,= 0$ for any $t \geq T$.

*Proof*. Consider the function $V(s(t))$ for $t \in [0, +\infty) \cap \{t : s(t) \neq 0\}$, we know from lemma 1 that

$$\frac{d}{dt} V(s(t)) = \zeta \cdot (\dot{y}(t) - \dot{x}(t) - Ce^{Ct}c_0)\ \forall\ \zeta \in \partial V(s(t))$$

for every $x(t)$ which is a Filippov solution of the equation

$$\dot{x} = f(t, x, \bar{u}(t, x)) = g(t, x), \qquad \text{that is}$$

$$\dot{x}(t) \in G(t, x(t)) = \bigcap_{\delta > 0} \bigcap_{\mu(N) = 0} \overline{co}\{g(t, B(x(t), \delta)\backslash N)\}$$

where $B(x, \delta) = \{z \in \mathbb{R}^n : |z - x| < \delta\}$.
By lemma 1 of [5] we can conclude that there exists a set $N_0$, depending on $t$, such that

$$\dot{x} \in G(t, x(t)) = \bigcap_{\delta > 0} \overline{co}\{f(t, z, \bar{u}(t, z)), z \in B(x(t), \delta)\backslash N_0\}$$

$$\subseteq \bigcap_{\delta > 0} \overline{co}\, J(t, x(t), \delta)$$

where $J(t, x(t), \delta) = \{\cup F(t, z, \bar{u}(t, z)), z \in B(x(t), \delta)\backslash N_0\}$. For any vector $\chi \in \mathbb{R}^n$ and for any $\delta > 0$ we have that

$$\chi \cdot \dot{x}(t) \geq \inf_{r \in \overline{co} J(t, x(t), \delta)} \chi \cdot r = \inf_{r \in J(t, x(t), \delta)} \chi \cdot r. \tag{4}$$

For a given $\delta > 0$ and $\chi \in \mathbb{R}^n$ there exists $r_\delta \in J(t, x(t), \delta)$ and $z_\delta \in B(x(t), \delta)\backslash N_0$ such that

$$\inf_{r \in J(t, x(t), \delta)} \chi \cdot r \geq \chi \cdot r_\delta - \frac{k^2}{2} \text{ with } r_\delta \in F(t, z_\delta, \bar{u}(t, z_\delta)). \tag{5}$$

Since $F(t, z_\delta, \bar{u}(t, z_\delta))$ is a convex closed set we obtain

$$\chi \cdot r_\delta \geq \inf_{r \in F(t, z_\delta, \bar{u}(t, z_\delta))} \chi \cdot r. \tag{6}$$

Since $s(t) \neq 0$ and the function $s = s(t, y, x, C)$ is a continuous function of $x$, for $\delta > 0$ small enough we can find $z_\delta \notin N_0$ with $|z_\delta - x(t)| < \delta$ and $s = s(t, y, z_\delta, C) \neq 0$. Using (3) we get from (4)–(5) and (6) that

$$\zeta \cdot \dot{x}(t) \geq \inf_{r \in F(t, z_\delta, \bar{u}(t, z_\delta))} \zeta \cdot r - \frac{k^2}{2} \geq \zeta \cdot p(t, y, v, C) + \frac{k^2}{2} \tag{7}$$

for any $\zeta \in \partial V(s)|_{s=s(t,y,z_\delta,C)}$.

Let $\delta = 1/n$. For $n$ sufficiently large let us fix $\zeta_n \in \partial V(s)|_{s=s(t,y,z_\delta,C)}$.

Since the generalized gradient is an upper semicontinuous map, the sequenced $\{\zeta_n\}$ is bounded and so, by passing to a subsequence if necessary, we get

$$\zeta_n \to \zeta_0 \in \partial V(s(t)).$$

Therefore from (7) we have that

$$\zeta_0 \cdot x(t) \geq \zeta_0 \cdot p(t, y, v, C) + \frac{k^2}{2} \text{ with } \zeta_0 \in \partial V(s(t)).$$

Finally, by virtue of lemma 1

$$\frac{\mathrm{d}}{\mathrm{d}t} V(s(t)) = \zeta_0 \cdot p(t, y, v, C) - \zeta_0 \cdot \dot{x}(t) \leq -\frac{k^2}{2}$$

and so by (iii) of lemma 1, there exists $T > 0$ such that $s(t) = 0\ \forall t \geq T$. On the other hand we have

$$V(t) = \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t} |s_i(t)| \leq -\frac{k^2}{2}$$

that is

$$\frac{k^2}{2} \leq \sum_{i=1}^{n} |s_i(0)| = \sum_{i=1}^{n} |y_i(0) - x_i(0) + c_{0_i}| \leq n(L + M + N) \qquad \text{and } T \leq 2n\frac{L + M + N}{k^2}. \quad \blacksquare$$

We are now in a position to prove the following.

Corollary 4. Assume $(H_1)$ and $(H_2)$. Let $(y, v)$ be a fixed state-control pair for the linear model. If $x$ is any system state for $\bar{u}$, that is any Filippov solution of (2) on $[0, +\infty)$ for any system dynamics corresponding to the feedback control $\bar{u}$, we have

$$|y(t) - x(t)| \leq \mathrm{e}^{\mu(C)t} c_0 \leq \mathrm{e}^{-\alpha t} c_0$$

for all $t \geq T$.

*Proof*. By theorem 3 for any $t \geq T$ we have

$$s(t) = 0 \qquad \text{that is } y(t) - x(t) = \mathrm{e}^{Ct} c_0.$$

Now the fact that $|\mathrm{e}^{Ct}| \leq \mathrm{e}^{\mu(C)t}$ for $t \geq 0$ implies the assertion. $\blacksquare$

*Remark* 2. Theorem 3 generalizes the results obtained in [1] for scalar differential equations of order $n$ and those of [8] for vector systems that are linear in the control. In [8] a reference trajectory was given, rather than a reference linear system. The changes to make in the definitions of $E$, $s$ and $p$ in this case, are obvious. In [6] a robust controller is obtained to keep the tracking error $E = y - x$ within a desired tolerance. Here $x$ is the output of a nonlinear control system in which the control $u$ appears affinely and $y$ is the desired reference trajectory. The controller is robust in the sense that the tracking error is ultimately bounded in the presence of modelling errors. We refer to the papers quoted in [1, 6, 8] for a wide bibliography on tracking.

## 3. A CLASS OF SYSTEMS SATISFYING CONDITION $(H_1)$

We give now conditions on the multivalued map $F$ which ensure that condition $(H_1)$ is satisfied.

Let $\mathcal{O}$ be the collection of all nonempty, compact, convex subsets of $\mathbb{R}^n$

$$\mathcal{O} = \{\theta \subset \mathbb{R}^n \mid \theta \text{ nonempty, compact, convex}\}.$$

Let $\Omega$ be the collection of all ordered orthonormal bases in $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ let us consider the following set

$$V_1(\theta, x) = \{a \in \theta : a \cdot x = q(\theta, x)\}$$

where $q(\theta, \cdot\,)$ denotes the support function of the convex set $\theta$. We need the following.

*Definition* 1. For any $\omega = (v_1, v_2, \ldots, v_n) \in \Omega$, let $\omega_i = (v_1, v_2, \ldots, v_i)$, $i = 1, 2, \ldots, n$. (Hence $\omega_n = \omega$.) (Note that $v_i$ is a vector not a component of vector.) Let $V_i$ be the set defined as follows

$$V_i(\theta, \omega_i) = V_1(V_{i-1}(\theta, \omega_{i-1}), v_i)$$

for any $i = 1, 2, \ldots, n$.

Finally, let $\eta : \mathcal{O} \times \Omega \to \mathbb{R}^n$ be the map given by

$$\eta(\theta, \omega) = V_n(\theta, \omega_n).$$

We have the following lemmata.

LEMMA 5. ([10]). Im $\eta(\theta, \cdot\,) = \text{Ext}\,\theta$, where Ext $\theta$ denotes the set of the extreme points of the set $\theta$.

LEMMA 6. ([10]). The map $h_n : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ defined by

$$h_n(\theta_1, \theta_2) = \sup_{\omega \in \Omega} |\eta(\theta_1, \omega) - \eta(\theta_2, \omega)|$$

is a translation invariant distance in $\mathcal{O}$ and the space $(\mathcal{O}, h_n)$ is complete.

Observe that if $h_0$ denotes the Hausdorff distance then $h_0(\theta_1, \theta_2) \le h_n(\theta_1, \theta_2)$.

THEOREM 7. Let $F : \mathfrak{D} \to \mathcal{O}$ be a Caratheodory multivalued function with respect to the $h_n$ metric (i.e. $F$ is a $t$-measurable, $(x, u)$-continuous multivalued map, with respect to the $h_n$

metric). Then the multivalued function

$$\eta_\omega \circ F : \mathfrak{D} \to \mathbb{R}^n \text{ defined by}$$

$$(t, x, u) \to \eta(F(t, x, u), \omega)$$

is Caratheodory.

*Proof.* The proof is trivial, once one observes that the map $\beta_\omega : \mathcal{O} \times \mathcal{O} \to \mathrm{R}^n$ given by $\beta_\omega(A, B) = \eta(A, \omega) - \eta(B, \omega)$ is linear and continuous. ■

We can prove the following selection theorem.

THEOREM 8. If $F$: $\mathfrak{D} \to \mathcal{O}$ is a Caratheodory multivalued function with respect to the $h_n$-metric then $F$ has the Caratheodory selection property.

*Proof.* $F(t_0, x_0, u_0) = \text{coExt } F(t_0, x_0, u_0) = \text{coIm } \eta(F(t_0, x_0, u_0), \cdot)$, thus for any $y_0 \in F(t_0, x_0, u_0)$, there exists $m \in \mathbb{N}$, $\lambda_j \in \mathbb{R}^+$, $\omega_j \in \Omega$, $j = 1, \ldots, m$ such that $\sum_{j=1}^m \lambda_j = 1$, $y_0 = \sum_{j=1}^m \lambda_j \eta(F(t_0, x_0, u_0), \omega_j)$. Then the map $f : \mathfrak{D} \to \mathbb{R}^n$ defined by $f(t, x, u) = \sum_{j=1}^m \lambda_j \eta(F(t, x, u), \omega_j)$, is a Caratheodory selection of $F$ such that $y_0 = f(t_0, x_0, u_0)$. ■

As we are looking for approximation results, having in mind the results on the density of the attainable set of the continuous selections of a continuous multivalued map, with respect to the attainable set for all possible measurable selections (see [7, 10]), we replace $(H_1)$ by the following condition:

$(H_1')$ $F$ is Caratheodory with respect to the $h_n$ metric.

Therefore as a consequence of $(H_1')$ and the above noted density results we can confine ourselves to consider only the dynamics given by the Caratheodory selections $f : \mathfrak{D} \to \mathbb{R}^n$ of $F$ of the form

$$f(t, x, u) = \sum_{j=1}^{m} \lambda_j \eta(F(t, x, u), v_j) \qquad \text{for some } \lambda_j \in \mathbb{R}^+, \sum_{j=1}^{m} \lambda_j = 1,$$

$$\omega_j \in \Omega, j = 1, \ldots, m \in N.$$

In this case hypothesis $(H_2)$ can be replaced by the following:

$(H_2')$ There exists a constant $k \neq 0$ such that for every $t \geq 0$ and $x \in \mathcal{C}$, we can find $\bar{u} = \bar{u}(t, x) \in U$ such that

$$\inf_{\omega \in \Omega} \zeta \cdot \eta(F(t, x, \bar{u}), \omega) \geq \zeta \cdot p(t, y, v, C) + k^2 \qquad \forall_\zeta \in \partial V(s)|_{s=s(t,y,x,C)}.$$

*Remark* 3. Condition $(H_2')$ is particularly simple to verify when $\forall (t, x, u)$, $F(t, x, u)$ is a convex polyhedron, since in this case the extreme points of $F(t, x, u)$ are the vertices of the polyhedron.

*Remark* 4. The form of $(H_2)$ depended on our choice of Liapunov function, $V(s) = \sum_{i=1}^n |s_i|$. It is clear that we could choose another Liapunov map, for instance the map $\tilde{V}(s) = \frac{1}{2}\sum_{i=1}^n s_i^2$. In this case we can avoid the complication of dealing with the generalized gradient of $\tilde{V}$ but the condition (3) in $(H_2)$ must be replaced with the following:

$$\begin{cases} \sup\limits_{z \in F(t,x,\bar{u})} x \cdot z \le x \cdot p(t, y, v, C) - k^2 \quad \text{and} \\ \inf\limits_{z \in F(t,x,\bar{u})} (y - e^{Ct}c_0) \cdot z \ge (y - e^{Ct}c_0) \cdot p(t, y, v, C) + k^2. \end{cases} \tag{8}$$

We observe that the two multivalued maps

$$(t, x) \to \{u \in U : \sup_{z \in F(t,x,u)} x \cdot z \le x \cdot p(t, y, v, C) - k^2\}$$

and

$$(t, x) \to \{u \in U : \inf_{z \in F(t,x,u)} (y - e^{Ct}c_0) \cdot z \ge (y - e^{Ct}c_0) \cdot p(t, y, v, C) + k^2\}$$

are both measurable and so is the intersection (we would need to suppose that this intersection is not empty) and thus admits a measurable selection $\bar{u}(t, x)$.

*Remark* 5. If we suppose that, for any dynamics $f(t, x, \bar{u}(t, x))$ with $\bar{u}(t, x) \in U$, the set $f(t, x, U) \subset \mathbb{R}^n$ is convex for a.a. $t \in [0, +\infty)$, and any $x \in \mathbb{R}^n$, then the theory of the equivalent control for nonlinear systems developed in [2] is available. We recall, in this context, the definition of the equivalent control. Let $S = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : s(t, y, x, C) = 0\}$. Assume the following condition.

$(H_3)$ There exists a neighbourhood $I$ of $S$ such that for every $(t, x) \in [T, +\infty) \times I$ the map $Ay(t) + Bv(t) - Ce^{Ct}c_0 + \partial s/\partial x \cdot f(t, x, \cdot)$ is on to one on $U$ and its range contains 0, where $\partial s/\partial x = -I_d (I_d$ stands for the identity $n \times n$ matrix). Then by $(H_3)$ the algebraic equation

$$Ay(t) + Bv(t) - Ce^{Ct}c_0 + \frac{\partial s}{\partial x} f(t, x, u) = w$$

has a unique solution (if any) and it will be denoted by $u^*(t, x, w)$.

*Definition* 2. The equivalent control for (2) is the mapping

$$(t, x) \to u^*(t, x, 0),$$

$t \in [T, +\infty)$, $x \in I$.

Theorems 1 and 2 of [2] guarantee that a state trajectory on the sliding manifold $S$ corresponding to a discontinuous $\bar{u} = \bar{u}(t, x)$ may be obtained as an absolutely continuous solution of (2) corresponding to the equivalent control $u^*$, which is continuous in $x$, and conversely.

*Remark* 6. Conditions on the dynamics $f(t, x, \bar{u}(t, x))$ which imply that the corresponding Filippov solutions are defined on all of $[0, +\infty)$ can be found in [5].

## 4. FURTHER RESULTS

In this section we always deal with the tracking problem formulated in Section 1, but here we look at it in a slightly different way. Specifically, we will consider directly the dynamics for the error $E(t) = y(t) - x(t)$ and using some well known facts about solutions of differential equations on $(0, +\infty)$ we will search for conditions ensuring that $E(t)$ tends to zero as $t \to +\infty$. Such conditions will be expressed as before in terms of the existence of a feedback control law fulfilling given inequalities which will also determine the rate at which the error tends to zero.

For this, let $y = y(t)$ and $v = v(t)$ be the assigned state-control model for $t \in [0, +\infty)$ satisfying the linear differential system

$$\dot{y}(t) = Ay(t) + Bv(t)$$

with $y(0) = y_0$ for some $|y_0| \leq N$. Let $x = x(t)$ be a Filippov solution on $[0, +\infty)$ of the nonlinear problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t, x(t)) \\ x(0) = x_0 \qquad |x_0| \leq M \end{cases}$$

corresponding to some feedback control law $u = u(t, x)$. Consider the following dynamnics for the error term $E = y - x$:

$$\begin{cases} \dot{E} - AE = Ax(t) + Bv(t) - f(t, x(t), u(t, x(t)) \qquad t \in [0, +\infty) \\ E(0) = c_0 \end{cases} \tag{9}$$

where the right hand side is a measurable function of $t$. Consider an $n \times n$ real matrix $C$ such that $\mu(C) \leq -\alpha < 0$ and $A - C = Q$ is a matrix for which all the eigenvalues have nonpositive real part and those with zero real part are semisimple. Consider the differential system

$$\dot{E} - QE = Qx(t) + Bv(t) + Cy(t) - f(t, x(t), u(t, x(t))) = r(t) \tag{10}$$

with $E(0) = c_0$ and $t \in [0, +\infty)$.

LEMMA 9. Let the matrix $Q$ be as above. If $\int_0^{+\infty} |r(t)| \mathrm{d}t < +\infty$, then there is a one to one affine mapping between the bounded solutions on $[0, +\infty)$ of the homogeneous system $E - QE = 0$ and the bounded solutions on $[0, +\infty)$ of (10) such that the difference between corresponding solutions tends to zero as $t \to +\infty$.

*Proof.* Under our assumptions on $Q$ the homogeneous system $\dot{E} - QE = 0$ is uniformly stable. Thus it has an ordinary dichotomy with $P = I$ on $[0, +\infty)$, so that the result immediately follows from proposition 3 in [4]. ■

As a consequence of the previous result we have the following.

LEMMA 10. If $\mu(Q) \leq -\beta < 0$ and $r(t)$ is measurable with

$$\int_0^{+\infty} |r(t)| \mathrm{d}t < +\infty,$$

then any solution of (9) tends to zero as $t \to +\infty$.

*Proof.* By virtue of lemma 9, for any solution $E_r$ of (9) there is a solution $E$ of $\dot{E} - AE = 0$ such that

$$|E(t) - E_r(t)| \to 0 \qquad \text{as } t \to +\infty.$$

On the other hand we know that $E(t) \to 0$ for any solution of the homogeneous system since $\mu(Q) \leq -\beta < 0$. Thus $E_r(t) \to 0$ as $t \to +\infty$. ■

If $F$ is a multivalued map with values in a finite dimensional space by $|F(a)|$ we mean the $\sup_{b \in F(a)} |b|$. We are now in a position to prove the following.

THEOREM 11. Let $\varepsilon \in L^1([0, +\infty), \mathbb{R}^+)$. Assume that for any $(t, x) \in [0, +\infty) \times \mathcal{C}$ there exists $\bar{u} = \bar{u}(t, x)$ such that

$$|Qx + Bv(t) + Cy(t) - F(t, x, \bar{u}(t, x))| \leq \varepsilon(t).$$

Assume that $\mu(Q) \leq -\beta < 0$. Then for any system dynamics $f(t, x, \bar{u}) \in F(t, x, \bar{u})$ and for any $c_0 \in \mathbb{R}$ the corresponding solution $E_r = E_r(t)$ of (9) tends to zero as $t \to +\infty$.

The proof is immediate.

Note that the multivalued map defined for any $(t, x) \in [0, +\infty) \times \mathcal{C}$ as follows

$$(t, x) \to \{u \in U : |Qx + Bv(t) + Cy(t) - F(t, x, u)| \leq \varepsilon(t)\}$$

is measurable and so if it is has non empty values then there exists a measurable selection $\bar{u} = \bar{u}(t, x)$.

Finally, it is easy to see that if $\varepsilon \in L_1([0, +\infty), \mathbb{R}^+)$ can be chosen in such a way that

$$\varepsilon(t) \leq e^{-\delta t}, \qquad t \geq 0, \qquad \delta > 0$$

then $E_r(t) \to 0$ exponentially as $t \to +\infty$.

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