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A fixed point theorem for non compact acyclic-valued maps *

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RIASSUNTO. — Si dimostra un teorema di punto fisso per applicazioni multivoche non compatte a valori aciclici, e si dà un lemma che caratterizza le trasformazioni compatte di uno spazio di Banach riflessivo in sé.

SUMMARY. — We prove a fixed point theorem for multivalued non compact acyclic-valued maps. Moreover we give a characterization of the compact maps defined on a reflexive Banach space.

INTRODUCTION.

In a recent paper (see [8]) S. P. SINGH proved the following theorem:

Let E *be a reflexive Banach space and* C *a non empty, bounded, closed and convex subset of* E.

Let

$$f : C \to C$$
$$g : C \to E$$
$$h : C \to E$$

be such that

a) $f = g + h$;

b) g *is* α*-nonexpansive and* $(I - g)$ *is demiclosed;*

c) h *is strongly continuous.*

Then f *has at least one fixed point.*

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In this paper we extend the above theorem to the class of upper-semicontinuous multi-valued maps. Our results extend the one of S. P. SINGH also in the single-valued case.

Moreover we give a characterization of compact maps in reflexive Banach spaces. This characterization improves the result of D. G. DE FIGUEIREDO (see [2]) since he gave only a sufficient condition for f to be compact, while our condition is also necessary.

NOTATIONS AND DEFINITIONS.

We recall that a *multivalued map* f from a set X into a set Y is a triple (G, X, Y) where G, the *graph* of f, is a subset of $X \times Y$ such that $f(x) = \{y \in Y : (x, y) \in G$ is non empty for each $x \in X\}$.

$f(X) = \cup \{f(x) : x \in X\}$ is the *range* of f while X is its *domain*.

We will use the symbol $f : X \multimap Y$ to indicate a multi-valued map and $f : X \to Y$ for the single-valued ones.

Let $A \subset X$ and $B \subset Y$ be subsets; we shall use the following notations: $f(A) = \cup \{f(x) : x \in A\}$; $f^-(B) = \{x \in X : f(x) \cap B \neq \emptyset\}$; $f^+(B) = \{x \in X : f(x) \subset B\}$. $f^-(B)$ and $f^+(B)$ are called the *lower inverse image* and the *upper inverse image* of B respectively.

For $f : X \to Y$ single-valued we have $f^-(B) = f^+(B) = f^{-1}(B)$.

Let $f : X \multimap Y$ be a multivalued map from a topological space X into a topological space Y.

We say that f is *upper semicontinuous (u.s.)* on X if for any open set $0 \subset Y$ the set $f^+(0)$ is open or equivalently if for any closed set $C \subset Y$, $f^-(C)$ is closed.

We say that $f : X \multimap Y$ has *closed values* if $f(x)$ is closed for every $x \in X$, and f has *closed graph* if G is a closed subset of $X \times Y$.

If X and Y are metric spaces and $f : X \multimap Y$ has closed graph then $x_n \to x$, $y_n \to y$ and $y_n \in f(x_n)$ implies that $y \in f(x)$.

If Y is compact then $f : X \multimap Y$ is u.s. with closed values if and only if f has closed graph.

We say that $f : X \multimap Y$ is *compact* if for any bounded set $A \subset X$, $f(A)$ is relatively compact.

We say that $f : X \multimap Y$ is *completely continuous* if f is u.s. and $f(X) \subset K$, K being a compact subset of Y.

If Y is a locally convex topological vector space and $f(x)$ is convex for every $x \in X$, then f is called *convex-valued*.

If for every $x \in X$, $f(x)$ is acyclic in the Vietoris homology theory with coefficients in Q, then f is *acyclic-valued*.

A *fixed point* of a multi-valued map $f : X \multimap X$ is a point $x \in X$ such that $x \in f(x)$.

Let X be a normed space and $f : X \multimap X$ be a multi-valued map. We recall that

$$d(x, f(x)) = \inf \{ \| x\text{-}y \|, \ y \in f(x) \} .$$

Let A be a bounded subset of a metric space X.

Following KURATOWSKI [4] we define $\alpha(A)$ as the infimum of all $\varepsilon > 0$ such that A can be covered by a finite family of subsets with diameter less than ε. Since $\alpha(A) = 0$ if and only if $\bar{A}$ is compact, the number $\alpha(A)$ is frequently called a measure of non compactness of A.

Let X be a linear normed space, and $f : X \multimap X$ be an upper semicontinuous multi-valued map; f is said to be *α-contractive* (*α-nonexpansive*) [1]) if for any bounded set $A \subset X$ we have

$$\alpha(f(A)) \leq k\, \alpha(A) \qquad \text{with} \qquad 0 \leq k < 1 , \quad (k = 1)$$

If for any bounded set $A \subset X$, with $\alpha(A) > 0$, we have:

$$\alpha(f(A)) < \alpha(A)$$

then f is said to be *condensing* (or *densifying*).

RESULTS.

We give the following lemma that characterizes the compact mappings in a reflexive Banach space.

LEMMA 1. *Let* E *be a reflexive Banach space. Then* $h : E \to E$ *is compact if and only if the following property holds*:

i) *if* $x_n \rightharpoonup x$ (*converges weakly to* x) *then the sequence* $\{h(x_n)\}$ *is compact* (i.e. *any subsequence has cluster points*).

PROOF. *Only if*. Assume that h is compact and let $x_n \rightharpoonup x$. Clearly $\{x_n\}$ is bounded, therefore $\{h(x_n)\}$ is compact.

If. Let A be bounded. To prove that $h(A)$ is relatively compact it is enough to show that any sequence in $h(A)$ has cluster points.

Let $\{y_n\}$ be a sequence in $h(A)$. There exists $x_n \in A$ such that $h(x_n) = y_n$. Since $\{x_n\}$ is bounded and X is reflexive, we can find $x_{n_k} \rightharpoonup x$. Therefore $\{h(x_{n_k})\}$ is compact.

This implies that $\{y_n\}$ has cluster points.

Q.E.D.

This lemma generalizes a result due to D. G. DE FIGUEIREDO [2] which states that a strongly continuous mapping (i.e. $x_n \rightharpoonup x$ implies $f(x_n) \to f(x)$) in a reflexive Banach space is compact.

Let C be a bounded, closed and convex subset of a Banach space E. Assume that 0 is an interior point of C.

It is then possible to define the continuous radial retraction $\pi : E \to C$ in the following way.

First consider

$$p(x) = \begin{cases} \sup\{\lambda, \lambda \geq 0 : \lambda x \in C\} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Set $q(x) = \min\{p(x), 1\}$. Finally, define $\pi(x) = x \cdot q(x)$. The map π is obviously continuous.

Moreover π is α-nonexpansive (see R. D. NUSSBAUM [7]).

We give here a proof of this fact for completeness sake.

LEMMA 2. *The radial retraction* $\pi : E \to C$ *is* α*-nonexpansive.*

PROOF. Let $A \subset E$ be a bounded set. Since $\pi(A) \subset \overline{co}(A \cup \{0\})$ it follows that

$$\alpha(\pi(A)) \leq \alpha(\overline{co}(A \cup \{0\})) = \alpha(A \cup \{0\}) = \alpha(A)\,.$$

Q.E.D.

Let C be a bounded, closed and convex subset of a Banach space E and $f : C \multimap E$.

Assume that $0 \in C$. We recall that f satisfies the Leray-Schauder condition if $\lambda x \in f(x)$ for some $x \in \partial C$ implies $\lambda \leq 1$.

LEMMA 3. *Let* C *be a bounded, closed and convex subset of a Banach space* E, *with* $0 \in \text{int}\, C$.

Let $f : C \multimap E$ *be an* α*-nonexpansive acyclic-valued mapping.*

If f *satisfies the Leray-Schauder condition then*

$$\inf\{d(x, f(x))\,;\ x \in C\} = 0\,.$$

PROOF. (compare [3] and [6]).

Let $\{k_n\}$ be a sequence of real numbers such that $0 \leq k_n < 1$ and $k_n \to 1$.

The map $k_n f : C \multimap E$ is clearly α-contractive with constant $k_n < 1$.

Lemma 2 and a result proved in [5] imply that $k_n f$ has at least one fixed point $x_n \in C$.

Let $y_n \in f(x_n)$ be such that $x_n = k_n y_n$.

We have

$$d(x_n, f(x_n)) \leq \| y_n - x_n \| = \| y_n - k_n y_n \| = \| y_n \| (1 - k_n) .$$

Since f is α-nonexpansive and C is bounded there exists $R > 0$ such that $f(C)$ is contained in the ball of radius R centered at the origin.

Thus

$$d(x_n, f(x_n)) \leq R(1 - k_n) .$$

Q.E.D.

THEOREM 1. *Let* C *be a bounded, closed and convex subset of a reflexive Banach space* E *such that* 0 *belongs to the interior of* C.

Let

$$f : C \multimap E$$
$$g : C \multimap E$$
$$h : C \to E$$

be such that

a) $f = g + h$

b) g *is* α*-nonexpansive acyclic-valued mapping and such that:* $x_n \rightharpoonup x$, $z_n \in x_n - g(x_n)$ *and* $z_n \to z$ *implies that* $z \in x - g(x)$ (*i.e.* $(I\text{-}g)$ *is demiclosed*);

c) h *is continuous in the weak topology and satisfies the property* i) *of Lemma 1;*

d) f *satisfies the Leray-Schauder condition.*

Then f *has at least one fixed point.*

PROOF. First let us prove that f is u.s.; to this end it sufficies to show that the map h is continuous in the strong topology.

Let $x_n \to x$; from the weak continuity of h we have $h(x_n) \rightharpoonup h(x)$.

Now by Lemma 1 we know that $\{\overline{h(x_n)}\}$ is compact and we are done, since weak and strong topologies coincide on compact sets. To prove that f is α-nonexpansive just recall that $f = g + h$, where g is α-nonexpansive and h is compact.

Lemma 3 implies that $\inf \{d(x, f(x)),\ x \in C\} = 0$.
Thus it sufficies to prove that $0 \in Im(I-f)$.
Let $y_n \to 0$ be a sequence in $Im(I-f)$.
We have

$$y_n \in x_n - f(x_n) = x_n - h(x_n) - g(x_n), \quad \text{i.e.}$$

$$y_n + h(x_n) \in x_n - g(x_n) \quad \text{with} \quad x_n \in C, \quad n = 1, 2 \ldots$$

Since C is weakly compact we may assume, without loss of generality, that $x_n \rightharpoonup x \in C$.

If follows that $\{h(x_n)\}$ is compact.

Let $\{h(x_{n_k})\}$ be a convergent subsequence, say to z.

Clearly $x_{n_k} \rightharpoonup x$ and so, by the continuity of h in the weak topology, $h(x_{n_k}) \rightharpoonup h(x)$.

It follows that $h(x) = z$. Thus for a subsequence $\{\bar{x}_n\}$ of $\{x_n\}$ we have

$$\bar{x}_n \rightharpoonup x \quad \text{and} \quad h(\bar{x}_n) \to h(x).$$

Since $\bar{y}_n + h(\bar{x}_n) \to h(x)$ and $\bar{y}_n + h(\bar{x}_n) \in \bar{x}_n - g(\bar{x}_n)$ for any integer n, it follows that

$$0 \in x - h(x) - g(x) = x - f(x).$$

Q.E.D.

Let E be a reflexive Banach space and $f : E \multimap E$ a multi-valued mapping.

Then, as in lemma 1, we can prove that f is compact if and only if the property i) of Lemma 1 holds, i.e. $\overline{\bigcup_n f(x_n)}$ is compact for every $x_n \rightharpoonup x$.

THEOREM 2. *Let* C *be a bounded, closed and convex subset of a reflexive Banach space* E, *with* $0 \in \operatorname{int} C$.

Let

$$f : C \multimap E$$

$$g : C \multimap E$$

$$h : C \multimap E$$

be such that

a) $f = g + h$;

b) *g is an α-nonexpansive, convex-valued mapping and* $(I-g)$ *is demiclosed;*

c) *h is a convex-valued mapping satisfying the property* i) *of Lemma 1 and with weakly closed graph;*

d) *f satisfies the Leray-Schauder condition.*
Then f has at least one fixed point.

PROOF. As in theorem 1 we first prove that f is u.s. By Lemma 1 h is compact, by hypotheses it has weakly closed graph. Therefore h has closed graph (in the strong topology). Since $\overline{h(C)}$ is compact then h has closed graph if and only if h is u.s. with closed values. Thus f is u.s. being a sum of two u.s. maps.

Moreover f is α-nonexpansive since g is α-nonexpansive and h is compact.

Lemma 3 insures that $\inf\{d(x, f(x)),\ x \in C\} = 0$.

Now we must prove that $0 \in Im(I-f)$.

Let $\{y_n\}$ be a sequence in $Im(I-f)$ which converges to 0.

Clearly $y_n = x_n - z_n - w_n$, with $z_n \in h(x_n)$ and $w_n \in g(x_n)$.

As in theorem 1 we can assume $x_n \rightharpoonup x$, thus $M = \overline{\bigcup_n h(x_n)}$ is compact.

The sequence $\{z_n\}$ is contained in M and so it has a convergent subsequence $z_{n_k} \to z$.

Therefore $z \in h(x)$. Hence for a subsequence $\{\bar{x}_n\}$ of $\{x_n\}$ we have

$$\bar{x}_n \rightharpoonup x\,,\ \bar{z}_n \to z\,,\ \bar{z}_n \in h(\bar{x}_n)$$

and $z \in h(x)$.

On the other hand $\bar{y}_n + \bar{z}_n = \bar{x}_n - \bar{w}_n \in \bar{x}_n - g(\bar{x}_n)$ and $\bar{y}_n + \bar{z}_n \to z$.

It follows that $z \in x - g(x)$, i.e. $0 \in x - z - g(x)$.

Since $z \in h(x)$ we finally have

$$0 \in x - h(x) - g(x)\,.$$

Q.E.D.

Remark 1. Clearly f satisfies Leray-Schauder condition if $f(C) \subset C$ or, more generally, $f(\partial C) \subset C$, where ∂C is the boundary of C.

Remark 2. Clearly the result we mentioned in the Introduction (see S. P. SINGH [8]) is a consequence of our theorems.

Remark 3. It can be easily verified that f satisfies Leray-Schauder condition if the following property holds.

$$g(x) \cap \{(1+\lambda)x - h(x)\} = \emptyset, \qquad \forall \lambda > 0 \quad \text{and} \quad x \in \partial C.$$

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