

An Optimal Control Problem in L^∞ ¹

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Abstract. We consider an optimal control problem in L^∞ , where the cost functional has a penalty term which involves the structure of the control law. This type of penalization allows us to restrict our attention only to piecewise constant controls, with assigned switching times, i.e., the controls belong to finite-dimensional control spaces. This fact and our assumptions on the dynamics give as a consequence the compactness of the minimizing sequences in $C([0, 1], \mathbb{R}^n) \times L^\infty([0, 1], \mathbb{R})$. The existence of a minimum of the cost functional is then obtained by a direct method. This result allows us to avoid the usual convexity assumption on the cost functional C and on the multivalued vector field associated to the dynamics when we have to consider the controls in all of L^∞ .

Key Words. Optimality, piecewise constant controls, controllability.

1. Introduction

In this paper, we consider the nonlinear control process described by the system of differential equations

$$\dot{x}(t) = f(t, x(t), u(t)), \quad \text{for almost all } t \in [0, 1]. \quad (1)$$

We assume that the function $f: [0, 1] \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ satisfies the following conditions:

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(F1) for any $(p, q) \in \mathbb{R}^n \times \mathbb{R}$, the map $f(\cdot, p, q)$ is Lebesgue measurable on $[0, 1]$; for a.a. $t \in [0, 1]$, the map $f(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^n \times \mathbb{R}$;

(F2) there exist nonnegative $\alpha, \beta, \gamma \in L^1([0, 1], \mathbb{R})$ such that

$$|f(t, p, q)| \leq \alpha(t) + \beta(t) \cdot |p| + \gamma(t) \cdot |q|,$$

for each $p \in \mathbb{R}^n$, for each $q \in \mathbb{R}$, and for a.a. $t \in [0, 1]$.

The function $t \rightarrow u(t)$, describing the control law, is assumed to belong to $L^\infty([0, 1], \mathbb{R})$, and the function $t \rightarrow x(t)$, describing the state of (1), is in $AC([0, 1], \mathbb{R}^n)$.

In order to formulate our optimization problem, we need some preliminaries. First, we recall that every essentially bounded function $u \in L^\infty([0, 1], \mathbb{R})$ can be regarded, except for a set of measure zero, as the uniform limit of a sequence of simple functions $\{\phi_n\}_{n \in \mathbb{N}}$, where, for any fixed $n \in \mathbb{N}$,

$$\phi_n = \sum_{j=1}^{n \cdot 2^n} a_j \cdot \chi_{E_j}, \quad a_j = \operatorname{ess\,inf}_{t \in E_j} u(t),$$

and the sets E_j , $1 \leq j \leq n \cdot 2^n$, are given by

$$E_j = \{t \in [0, 1] : j \cdot 2^{-n} \leq u(t) \leq (j+1) \cdot 2^{-n}\};$$

see Ref. 1. Obviously, if u is a simple function, $\phi_n = u$ for sufficiently large values of n . The convention $\chi_\emptyset = 0$ is assumed.

From the covering theorem of Vitali, we have that, for all $1 \leq j \leq n \cdot 2^n$, there exists a finite or countable family of nondegenerate closed intervals having no internal points in common $\{I_j^k\}_{k=1}^{K(j)}$ such that, except for a set of measure zero, we get

$$E_j = \bigcup_{k=1}^{K(j)} I_j^k.$$

Let $(\alpha_j^i, \beta_j^i)_{i=1}^{D(j)}$ be the connected components of the set $\bigcup_{k=1}^{K(j)} I_j^k$, where $D(j)$ is finite or infinite; and let D_n be the set

$$D_n = \{D(1), D(2), \dots, D(j), \dots, D(n \cdot 2^n)\}.$$

Let R be any positive constant. Let us denote by U_R the closed ball $\overline{B(0, R)} \subset L^\infty([0, 1], \mathbb{R})$. For all $u \in U_R$, let us consider now the sequence $\{\phi_n\}_{n \in \mathbb{N}}$ with

$$\phi_n = \sum_{j=1}^{n \cdot 2^n} a_j \cdot \chi_{E_j} = \sum_{j=1}^{n \cdot 2^n} \sum_{i \in D(j)} a_j^i \cdot \chi_{(\alpha_j^i, \beta_j^i)}, \quad a_j^i = a_j, \quad \forall i \in D(j).$$

The last equality holds except for a set of measure zero. Let

$$d_n = \sum_{j=1}^{n \cdot 2^n} D(j),$$

and define

$$d(u) = \limsup_{n \rightarrow +\infty} d_n.$$

We are now in the position to define the cost functional.

Definition 1.1. We define the cost functional $C: AC([0, 1], \mathbb{R}^n) \times U_R \rightarrow \mathbb{R} \cup \{\infty\}$ as follows:

$$C(x, u) = \int_0^1 f_0(t, x(t), u(t)) dt + g_0(x(1)) + h_0(x(0)) + \eta(d(u)); \quad (2)$$

here, $f_0: [0, 1] \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies hypotheses (F1)-(F2); $g_0, h_0: \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous functions; and $\eta: \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R} \cup \{\infty\}$ is a continuous, non-decreasing, nonnegative function with $\eta(\infty) = \infty$.

Observe that, in the definition of the cost functional, the function η appears with argument $d(u)$. The meaning of this function is to indicate that we pay a cost each time that we allow the control function to be nonconstant or jump.

In fact, it is clear that, from a practical point of view, the most convenient and physically realizable control law is that described by a piecewise constant control function with the least possible number of jumps (switching times). On the other hand, as will be proved in Section 2, it is possible to transfer a given state x_0 to another desired state x_1 along trajectories of the nonlinear system (1) corresponding to piecewise constant controls having a bound on the number of switching times. The proof of this fact is based on the results stated in Ref. 2.

For linear autonomous control processes, results in this direction are given in Refs. 3 and 4, while for linear nonautonomous systems, see Refs. 5 and 6. In Ref. 7 (see also the references therein), some properties of the reachable set at a fixed time, corresponding to the set of all piecewise constant controls, are stated for linear autonomous control processes. Lastly, for nonlinear autonomous control systems of the form

$$\dot{x} = f(x) + ug(x),$$

with f, g analytic vector fields defined on an analytic manifold and $u \in [-1, 1]$, we refer to Ref. 8. Actually, the results contained in the quoted Refs. 3, 4, 8, establishing bounds on the number of switching times, have been obtained in the bang-bang controllability context, and the bang-bang controls turn out to be also time-optimal controls. Moreover, we would like to point out that in Refs. 3, 4, 8, the switching times may depend on the control function; thus, the control set is not, in general, contained in a fixed

finite-dimensional linear space. In Refs. 2, 5, 6, the controls are contained in a linear space, since the switching times are the same for all the controls.

In conclusion, it appears reasonable to introduce the penalty η in the cost functional C in order to restrict ourselves to deal only with piecewise constant controls.

To the knowledge of the authors, there are no results in the literature concerning the existence of the minimum of a cost functional with a penalty term of the type considered here. In fact, the conditions on the controls are, in general, of a quantitative nature (e.g., conditions on the norm), rather than conditions involving the structure of the functions.

The paper is organized as follows. In Section 2, we give sufficient conditions (Theorem 2.1 and Corollary 2.1) for the system

$$\dot{x}(t) = f(t, x(t), u(t)), \quad \text{for a.a. } t \in [0, 1], \quad (3a)$$

$$x(0) \in A, \quad x(1) \in B, \quad (3b)$$

$$A, B \text{ given nonempty compact subsets of } \mathbb{R}^n, \quad (3c)$$

to be solvable using as control spaces N -dimensional subspaces PC_N of $L^\infty([0, 1], \mathbb{R})$ consisting of piecewise constant functions with at most N preassigned switching times. More precisely, we will give conditions which ensure that system (3) is solvable in a given control space PC_n and, for any positive integer $N > n$, we will study the structure of the solution set of (3) when the controls are in a (suitable) ball of any space $PC_N \supset PC_n$. We use techniques based on the Leray-Schauder continuation principle as well as on the properties of the topological degree of a map. Such techniques have been used by the first author to establish both local and global controllability properties for nonlinear control problems (Refs. 2 and 9) or optimization problems (Refs. 10 and 11).

In Section 3, we prove (Theorem 3.1) the existence of a minimum for the cost functional C . The proof of Theorem 3.1 is based on the fact that the solution set S is not empty (Section 2) and that the definition of C allows us to replace minimizing sequences $(x_n(\cdot), u_n(\cdot))$ with sequences in which the second component is contained in some finite-dimensional subspace of $L^\infty([0, 1], \mathbb{R})$, of piecewise constant functions.

In Section 4, we give some examples, illustrating our results.

In what follows, $(AC)^s$, $(C^r)^s$, $(L^r)^s$, $s \in \mathbb{N}$, $1 \leq r \leq +\infty$, will denote the Banach spaces $AC([0, 1], \mathbb{R}^s)$, $C^r([0, 1], \mathbb{R}^s)$, $L^r([0, 1], \mathbb{R}^s)$, equipped with the usual norms (we will omit the superscript if $s = 1$). Finally, $|\cdot|$ and $|\cdot|_X$ will denote the Euclidean norm in any finite-dimensional space and the norm in any infinite-dimensional Banach space X , respectively.

2. Existence of Solutions

Let $N \geq n$ be an integer and let $\{v_i\}_{i=1}^N$ be a set of N linearly independent functions of L^∞ . Define

$$V_N = \text{span}\{v_i, i = 1, 2, \dots, N\}.$$

Let us consider the linear space $V_N = V_n \oplus V_\alpha$, where

$$V_n = \text{span}\{v_i, i = 1, 2, \dots, n\}, \quad V_\alpha = \text{span}\{v_i, i = n + 1, \dots, N\}.$$

Let us recall the standard isomorphism $J_N : \mathbb{R}^N \rightarrow V_N$ as follows:

$$J_N(b) = \sum_{i=1}^N b_i \cdot v_i, \quad \text{for any } b = (b_i)_{i=1}^N \in \mathbb{R}^N.$$

Finally let $\Pi_N : (\text{AC})^n \times V_N \rightarrow \{0\} \times V_\alpha$ be the projection defined, for any $(x, u) = (x, J_N(b)) \in (\text{AC})^n \times V_N$, by

$$\Pi_N(x, u) = \begin{cases} \left(0, \sum_{i=n+1}^N b_i \cdot v_i\right), & \text{if } N > n, \\ (0, 0), & \text{if } N = n. \end{cases}$$

Therefore, for each $(x, u) \in (\text{AC})^n \times V_N$, we have that

$$(x, u) = (I - \Pi_N)(x, u) + \Pi_N(x, u) = (x, u_n) + (0, u_\alpha),$$

where $(x, u_n) \in (\text{AC})^n \times V_n$ and $(0, u_\alpha) \in \{0\} \times V_\alpha$. In the sequel, we will denote $(0, u_\alpha)$ simply by u_α .

After these definitions and preliminaries, we can state the main result of this section.

Theorem 2.1. Let the control system (3) satisfy (F1)-(F2). Let $\mathcal{A}(\cdot)$, $\mathcal{B}(\cdot)$ be time-dependent $n \times n$, $n \times 1$ matrices with coefficients in L^1 ; and let V_n be an n -dimensional subspace of L^∞ . Assume that:

(i). For any pair $(x_0, x_1) \in A \times B$, the linear system

$$\dot{x}(t) = \mathcal{A}(t)x(t) + \mathcal{B}(t)u(t), \quad \text{for a.a. } t \in [0, 1], \tag{4a}$$

$$x(0) = x_0, \quad x(1) = x_1, \tag{4b}$$

has a solution $x \in (\text{AC})^n$ for some $u \in V_n$. In other words, there exists a pair $(x, u) \in (\text{AC})^n \times V_n$ which solves (4).

(ii) There exist positive constants Q, R such that

$$|x|_{(C)^n} < Q \quad \text{and} \quad |u|_{L^\infty} < R,$$

for all $(x, u) \in (AC)^n \times V_n$ which solve, for some $\lambda \in [0, 1]$, the system (homotopy)

$$\dot{x}(t) = \lambda f(t, x(t), u(t)) + (1 - \lambda)(\mathcal{A}(t)x(t) + \mathcal{B}(t)u(t)), \quad \text{for a.a. } t \in [0, 1], \quad (5a)$$

$$x(0) = x_0, \quad x(1) = x_1, \quad (x_0, x_1) \in A \times B. \quad (5b)$$

Then, for any N -dimensional subspace V_N of L^∞ with $V_N \supseteq V_n$ and any $u_\alpha \in \Pi_N((AC)^n \times V_N^R)$, where $V_N^R = U_R \cap V_N$, the system (3) has a solution $(x, u) \in (AC)^n \times V_N^R$, with $\Pi_N(x, u) = u_\alpha$.

Proof. The first part of the proof (i.e., the case where $N = n$) follows the lines of the proof of Theorem 2.1 in Ref. 2. For the sake of completeness, we will sketch it in the sequel.

First, let us define the bounded linear operator $L_n: (AC)^n \times V_n \rightarrow (L^1)^n \times \mathbb{R}^n \times \mathbb{R}^n$ and the Nemitskii operator $N_n: (AC)^n \times V_n \rightarrow (L^1)^n$ in the following way:

$$L_n(x, u)(t) = (\dot{x}(t) - \mathcal{A}(t)x(t) - \mathcal{B}(t)u(t), x(0), x(1)),$$

$$N_n(x, u)(t) = f(t, x(t), u(t)) - \mathcal{A}(t)x(t) - \mathcal{B}(t)u(t).$$

Hence, the system (5) can be rewritten in the abstract form

$$L_n(x, u) - y = \lambda H_n(x, u), \quad (6)$$

where $y = (0, x_0, x_1)$ and $H_n: (AC)^n \times V_n \rightarrow (L^1)^n \times \mathbb{R}^n \times \mathbb{R}^n$ is given by

$$H_n(x, u) = (N_n(x, u), 0, 0).$$

From assumptions (i) and (ii), it follows that the equation $L_n(x, u) = (0, x_0, x_1)$ has a unique solution in $(AC)^n \times V_n^R$. Consequently, it is not hard to show that L_n is an isomorphism, i.e., the linear system (4) is completely controllable by means of the control space V_n . Furthermore, using assumptions (F1)-(F2) and the Lebesgue's dominated convergence theorem, we can prove that N_n is a continuous and compact operator. Finally, by virtue of assumption (ii), we can apply, as in Theorem 2.1 of Ref. 2, the Leray-Schauder continuation principle (see Ref. 12) to Eq. (6), thus obtaining that

$$L_n(x, u) - y = H_n(x, u)$$

has a solution in Ω_n , where

$$\Omega_n = \{(x, u) \in (AC)^n \times V_n : |x|_{(C)}^n < Q \text{ and } |u|_{L^\infty} < R\}.$$

Thus, the system (3) is solvable.

We start now with the second part of the proof. For this, suppose that $N > n$, and consider the set

$$S_N = \{(x, u) \in (AC)^n \times V_N^R : (x, u) \text{ is a solution of (3)}\}.$$

We assert that S_N is compact in $(C)^n \times L^x$ for any $N > n$. Note that the x -components of elements of S_N are bounded in $(C)^n$ by the constant Q , so that, via assumption (F2), it is bounded in $(AC)^n$. This compactness property can be derived from the proof of Theorem 3 and Remark 1 of Ref. 9. In fact, these results apply under assumption (F2), the compactness of the set A , and the boundedness of the controls u (by the constant R , independent of N). Therefore, we have that, for any $u_\alpha \in \Pi_N((AC)^n \times V_N^R)$ and for any pair $(x_0, x_1) \in A \times B$, the following homotopy is admissible in $[0, 1] \times \Omega_N$:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), & \text{for a.a. } t \in [0, 1], \\ x(0) &= x_0, & x(1) = x_1, \\ \Pi_N(x, u) &= \mu \cdot u_\alpha, & \mu \in [0, 1]. \end{aligned}$$

Here,

$$\Omega_N = \{(x, u) \in (AC)^n \times V_N : |x|_{(C)^n} < Q \text{ and } |u|_{L^x} \leq R\}.$$

Therefore, since

$$\deg(L_n - y - H_n, \Omega_n, 0) = \deg(L_n - y, \Omega_n, 0) \neq 0,$$

we have

$$\begin{aligned} &\deg((L_N - y - H_N, \Pi_N), \Omega_N, (0, u_\alpha)) \\ &= \deg(L_n - y - H_n, \Omega_n, 0) \neq 0, & \text{for any } u_\alpha \in \Pi_N((AC)^n \times V_N^R). \end{aligned}$$

Here, $L_N, H_N : (AC)^n \times V_N \rightarrow (L^1)^n \times \mathbb{R}^n \times \mathbb{R}^n$ are defined in the same way as L_n, H_n . The solution property of the topological degree guarantees the existence of a solution $(x, u) \in \Omega_N$ of the problem:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), & \text{for a.a. } t \in [0, 1], \\ x(0) &= x_0, & x(1) = x_1, \\ \Pi_N(x, u) &= u_\alpha, \end{aligned}$$

for any $u_\alpha \in \Pi_N((AC)^n \times V_N^R)$. This completes the proof. □

Remark 2.1. We would like to point out that assumptions (i) and (ii) guarantee that the linear control process (4) is solvable in $(AC)^n \times V_n$ for any pair $(x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n$, i.e., the linear system is completely controllable by means of controls belonging to the n -dimensional space V_n . Therefore, assumptions (i), (ii) ensure that, if we can find a linear system which is homotopic to the nonlinear one in such a way that all the possible solutions of the homotopy (5) in $(AC)^n \times V_n$ are uniformly bounded, then we obtain the solvability (controllability) of both the systems (3) and (4) using the controls belonging to V_n .

Remark 2.2. Observe that, under our hypotheses on the system (3), in particular assumption (F2), Theorem 2.3 of Ref. 2 applies to the homotopy (5). Thus, we have that, in order to verify assumption (ii) of Theorem 2.1, it suffices to prove the boundedness in V_n of the controls of the solution pairs (x, u) of (5). Indeed, Theorem 2.3 of Ref. 2 ensures also the compactness in $(C)^n \times L^\infty$ of the set $S_n \subset (AC)^n \times L^\infty$, which is defined as the solution set S_N with $N > n$.

Let us consider now a set $\{t_1, t_2, \dots, t_N\}$ of N fixed switching times such that $0 < t_1 < \dots < t_N \leq 1$; and let us define the N -dimensional subspace PC_N of L^∞ , consisting of piecewise constant \mathbb{R} -valued functions with at most N discontinuities at the fixed switching times $\{t_1, t_2, \dots, t_N\}$, as follows:

$$PC_N = \text{span}\{\chi_i, i = 1, 2, \dots, N\},$$

where

$$\chi_i(t) = \chi_{(t_{i-1}, t_i)}(t) \quad \text{and} \quad t_0 = 0.$$

Observe that we could also define PC_N in the following way:

$$PC_N = \text{span}\{\chi_{[0, t_i]}, i = 1, 2, \dots, N\}.$$

Finally, for $N \geq n$, let $\{t_{i_1}, t_{i_2}, \dots, t_{i_n}\} \subset \{t_1, t_2, \dots, t_N\}$ be a fixed set of n switching times such that $t_{i_{k-1}} < t_{i_k}$, for $k = 1, 2, \dots, n$, with $t_{i_0} = 0$ and $t_{i_n} = t_N$. Define

$$PC_n = \text{span}\{\chi_{i_k}, k = 1, \dots, n\}.$$

Then,

$$V_\alpha = PC_\alpha = \text{span}\{\chi_{i_j}, j \in \{1, \dots, n\}, \text{ and } j \neq i_k\}.$$

We are now in a position to formulate the following corollary.

Corollary 2.1. Assume that all the hypotheses of Theorem 2.1 are satisfied, with the spaces V_N, V_n replaced by PC_N and PC_n , respectively. Let us assume, furthermore, that $\tilde{x}(1) \neq 0$, where \tilde{x} is the solution of the Cauchy problem

$$\dot{x}(t) = \mathcal{A}(t)x(t) + \mathcal{B}(t)u(t),$$

$$x(0) = 0,$$

which corresponds to the constant control $u(t) = 1$, for every $t \in [0, 1]$. Then, the conclusion of Theorem 2.1 remains valid if we put $t_N = 1$; i.e., the control space PC_N consists of piecewise constant functions having at most $N - 1$ fixed switching times in the open interval $(0, 1)$.

Proof. Taking into account that, under the assumptions of Theorem 2.1, the linear control system (4) is completely controllable by means of controls in PC_n , the proof can be easily derived from Theorems 2.1 and 2.2 of Ref. 5 and the above Theorem 2.1. \square

Remark 2.3. The fact that a linear control system may or may not be controlled by means of

$$PC_n = \text{span}\{\chi_k, k = 1, 2, \dots, n\}$$

can be checked numerically. In fact, in order to obtain complete controllability, it suffices to verify that

$$\det(X^{-1}(t_1)\tilde{x}(t_1), \dots, X^{-1}(t_n)\tilde{x}(t_n)) \neq 0,$$

where \tilde{x} is the function defined in Corollary 2.1 and $X(t)$ is the fundamental matrix of the linear system (see Ref. 5).

3. Optimization

In this section we will solve the following optimization problem: Prove the existence of a pair $(x^*, u^*) \in (AC)^n \times U_R$ such that

$$C(x^*, u^*) = \inf_{(x,u) \in S} C(x, u),$$

where $C(x, u)$ is defined by (2), $(x^*, u^*) \in S$, and

$$S = \{(x, u) \in (AC)^n \times U_R \text{ such that } \dot{x}(t) = f(t, x(t), u(t)), \\ \text{for a.a. } t \in [0, 1], x(0) \in A, x(1) \in B\}.$$

Theorem 3.1. Assume that all the hypotheses of Theorem 2.1 are satisfied. Then, the cost functional C assumes the minimum on the solution set S of (3).

Proof. Theorem 2.1 implies that the solution set S is nonempty. Moreover, from assumption (F2), the boundedness in L^x of the controls by the constant R , and the compactness of the sets $A, B \in \mathbb{R}^n$, it follows that S is a bounded set in $(AC)^n \times L^x$. Thus, by the definition of C , we have that

$$-\infty < c = \inf_{(x,u) \in S} C(x, u) < +\infty;$$

the right-hand inequality follows from the fact that, for any $N \geq n$, we have $S_N \subset S$ and $S_N \neq \emptyset$, where the solution set $S_N, N \geq n$, is that of Theorem 2.1.

Consider now a minimizing sequence $\{(x_m, u_m)\}_{m \in \mathbb{N}} \subseteq S$, $m \in \mathbb{N}$, that is,

$$\lim_{m \rightarrow +\infty} C(x_m, u_m) = c.$$

We assert that the function η in the definition of C guarantees that there exists a positive integer M_1 , such that

$$d(u_m) \leq M_1, \quad \text{for every } m \in \mathbb{N}.$$

In fact, let us assume the contrary. Then, there exists a subsequence $\{u_{m_k}\}_{k \in \mathbb{N}}$ of $\{u_m\}_{m \in \mathbb{N}}$ such that

$$d(u_{m_k}) \rightarrow +\infty, \quad \text{as } k \rightarrow +\infty.$$

This implies that

$$C(x_m, u_m) \rightarrow +\infty, \quad \text{as } m \rightarrow +\infty,$$

contradicting the fact that c is finite.

Therefore, for sufficiently large m , we get $d(u_m) \leq M_1$, and this implies that each function u_m assumes only a finite number of values on $[0, 1]$, except for a set of measure zero. In fact, the sequence $\{d_n^m\}_{n \in \mathbb{N}}$, such that

$$\limsup_{n \rightarrow +\infty} d_n^m = d(u_m),$$

defined for each $m \in \mathbb{N}$ as in the introduction, is bounded by the constant M_1 , which is independent from $m \in \mathbb{N}$. Thus, for every $m \in \mathbb{N}$, u_m (except for a set of measure zero) is a simple function with a finite number of jumps (bounded by a constant, say $M \in \mathbb{N}$, independent from $m \in \mathbb{N}$); i.e., u_m is a piecewise constant function which can be represented (not necessarily in the standard form, i.e., with values b_s distinct) as follows:

$$u_m = \sum_{s=1}^M b_s \cdot \chi_s,$$

where

$$\chi_s(t) = \chi_{(t_{s-1}, t_s)}(t) \quad \text{and} \quad t_0 = 0, \quad t_M \leq 1.$$

Note that, in general, the sequence $\{u_m\}_{m \in \mathbb{N}}$ is not necessarily contained in some finite-dimensional space PC_N , $N \geq 1$. Thus, for clarity in the sequel, we will denote b_s , (t_{s-1}, t_s) by b_s^m and (t_{s-1}^m, t_s^m) , respectively.

By passing to a subsequence if necessary, we can assume that, for any $s = 1, 2, \dots, M$, the sequence $\{t_s^m\}_{m \in \mathbb{N}}$ is convergent to some $\tau_s \in [0, 1]$. Obviously,

$$\tau_s \leq \tau_{s+1}, \quad \text{for any } s.$$

Let $\{s_1, s_2, \dots, s_{M_0}\}$ be the increasing subset of $\{1, 2, \dots, M\}$ such that

$$\tau_{s_{k-1}} < \tau_{s_k}, \quad k = 1, 2, \dots, M_0, \quad \text{and} \quad \tau_{s_0} = 0.$$

In order to simplify the notation, we will relabel $\{s_1, s_2, \dots, s_{M_0}\}$ as $\{1, 2, \dots, M_0\}$. For sufficiently large m , we obtain that

$$\alpha_l^m = \max\{\tau_{l-1}, t_{l-1}^m\} < \min\{\tau_l, t_l^m\} = \beta_l^m,$$

for any $l = 1, 2, \dots, M_0$, with $\tau_0 = 0$.

We define now a sequence $\{v_m\}_{m \in \mathbb{N}}$ as follows:

$$v_m(t) = b_l^m, \quad t \in (\tau_{l-1}, \tau_l), \quad l = 1, 2, \dots, M_0,$$

and m sufficiently large,

where

$$b_l^m = u_m(t), \quad \text{for any } t \in (\alpha_l^m, \beta_l^m).$$

On the other hand,

$$\{v_m\}_{m \in \mathbb{N}} \subset \text{PC}_{M_0} = \text{span}\{\chi_l, l = 1, 2, \dots, M_0\},$$

with

$$\chi_l = \chi_{(\tau_{l-1}, \tau_l)} \quad \text{and} \quad \|v_m\|_{L^\infty} \leq R.$$

Observe that the space PC_{M_0} not necessarily contains an n -dimensional space PC_n satisfying the conditions of Theorem 2.1.

Thus, passing to a subsequence if necessary, we have

$$v_m(t) \rightarrow u^*(t), \quad \text{for a.a. } t \in [0, 1],$$

where

$$u^* = \sum_{l=1}^{M_0} b_l \cdot \chi_l \quad \text{and} \quad \|u^*\|_{L^\infty} \leq R.$$

Let us now prove that

$$u_m(t) \rightarrow u^*(t), \quad \text{for a.a. } t \in [0, 1].$$

In fact,

$$|u_m(t) - u^*(t)| \leq |u_m(t) - v_m(t)| + |v_m(t) - u^*(t)| \leq 2 \cdot R \cdot \delta_m + \epsilon_m,$$

where

$$\delta_m = \max_l \{|t_{l-1}^m - \tau_{l-1}|, |t_l^m - \tau_l|\} \quad \text{and} \quad \epsilon_m = \max_l |b_l^m - b_l|.$$

Taking the limit as $m \rightarrow +\infty$, we obtain that

$$u_m(t) \rightarrow u^*(t), \quad \text{for a.a. } t \in [0, 1].$$

Since the sequence $\{(x_m, u_m)\}_{m \in \mathbb{N}} \subset S$ is bounded in $(AC)^n \times L^\infty$, passing to a subsequence if necessary, we get $x_m \rightarrow x^*$ in $(C)^n$. Hence,

$$(x_m, u_m) \rightarrow (x^*, u^*), \quad \text{in the } (C)^n \times L^\infty \text{ norm.}$$

Lebesgue's dominated convergence theorem together with our assumptions on C yield

$$c = \lim_{m \rightarrow +\infty} C(x_m, u_m) \geq C(x^*, u^*).$$

It remains only to prove that $(x^*, u^*) \in S$ in order to conclude that (x^*, u^*) is the minimum of C on S . For this, let $\{x_m\}_{m \in \mathbb{N}} \subset A$ and $\{y_m\}_{m \in \mathbb{N}} \subset B$ be sequences such that

$$\dot{x}_m(t) = f(t, x_m(t), u_m(t)), \quad x_m(0) = x_m, \quad x_m(1) = y_m.$$

The sequence $\{u_m\}_{m \in \mathbb{N}}$ converges to u^* in L^∞ . Define, for any $m \in \mathbb{N}$, the function $g_m: [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ as follows:

$$g_m(t, p) = f(t, p, u_m(t)), \quad \text{for a.a. } t \in [0, 1].$$

By virtue of assumptions (F1)-(F2) on the nonlinearity f , it is easy to see that the sequence $\{g_m\}_{m \in \mathbb{N}}$ satisfies all the hypotheses of Theorem 2.4 of Ref. 2. This theorem states the continuous dependence of the solutions of (1) on the parameters and on the initial conditions, when the initial-value problem does not have the uniqueness property and f is a Caratheodory function (and is just the main tool used to prove the results of Refs. 2 and 9 quoted throughout this paper). Therefore, there exists a subsequence $\{x_{m_k}\}_{k \in \mathbb{N}}$ of $\{x_m\}_{m \in \mathbb{N}}$ converging uniformly in $[0, 1]$ to an absolutely continuous function \hat{x} satisfying

$$\dot{\hat{x}}(t) = f(t, \hat{x}(t), u^*(t)), \quad \text{for a.a. } t \in [0, 1],$$

and such that $\hat{x}(0) \in A$ and $\hat{x}(1) \in B$, since the sets $A, B \subset \mathbb{R}^n$ are compact. Thus, we have that $\hat{x} = x^*$. This completes the proof. \square

Note that, in the above proof, the fact that $(x^*, u^*) \in S$ can be also proved by using the Lebesgue's dominated convergence theorem. We preferred to base our proof on Theorem 2.4 of Ref. 2, since it is in the spirit of the techniques that we have used throughout this paper.

Remark 3.1. In this paper, we have considered only scalar controls. To consider the case of vector-valued control functions $u \in (L^\infty)^m = L^\infty([0, 1], \mathbb{R}^m)$, $m > 1$, we can define

$$d(u) = \max\{d(u_r): r = 1, 2, \dots, m\}, \quad u = (u_r)_{r=1}^m,$$

where $d(u_r)$, $1 \leq r \leq m$, is defined as in the introduction. Furthermore, it is easy to see that we can restate all the previous results, except Corollary 2.1. Obviously, in this case, the N -dimensional \mathbb{R}^m -valued space PC_N is given by

$$(PC)_N = \text{span}\{\chi_i^r, i = 1, 2, \dots, N\},$$

where

$$\chi_i^r(t) = \chi_{(t_{i-1}, t_i)}(t) \cdot e_r, \quad r_i \in \{1, 2, \dots, m\},$$

and $\{e_1, e_2, \dots, e_m\}$ is the standard base of \mathbb{R}^m .

Remark 3.2. Finally, we would like to point out that assumption (F2) can be weakened as follows:

(F2') For each $\rho > 0$, there exists $\gamma_\rho \in L^1$ such that, for a.a. $t \in [0, 1]$ and every $(p, q) \in \mathbb{R}^n \times \mathbb{R}$, with $|p| + |q| \leq \rho$, we have that

$$|f(t, p, q)| \leq \gamma_\rho(t).$$

In this case, the observations in Remark 2.2 remain valid if it is possible to verify directly from the dynamical system (i.e., from the differential equations) the conditions of both Theorem 2.3 of Ref. 2 and Theorem 3 of Ref. 9. Otherwise, as Example 4.2 in the next section proves, we have to check both the boundedness of the states and of the controls.

4. Examples

Example 4.1. Consider the first-order differential equation

$$\dot{x}(t) = f(t, x(t), u(t)), \quad \text{for any } t \in [0, 1], \tag{7}$$

with $f: [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ continuous, satisfying (F2) and the following condition:

(H1) for any $\rho \in \mathbb{R}$, $f(t, p, q) \rightarrow +\infty$, as $q \rightarrow +\infty$, uniformly for $p \geq \rho$ and $t \in [0, 1]$; $f(t, p, q) \rightarrow -\infty$, as $q \rightarrow -\infty$, uniformly for $p \leq \rho$ and $t \in [0, 1]$.

We prove in the sequel that Eq. (7) is globally controllable by using only constant controls, i.e.,

$$u \in PC_1 = \text{span}\{\chi_{(0,1)}\}.$$

That is, for any given $x_0, x_1 \in \mathbb{R}$, there exists $u \in PC_1$ such that (7), together with $x(0) = x_0, x(1) = x_1$, has a solution in C^1 .

In order to do this, we will prove that, for any $\Gamma \geq 0$, all the possible solutions (x, u) of the following homotopy:

$$\begin{aligned}\dot{x} &= \lambda f(t, x(t), u) + (1 - \lambda)u, & \lambda \in [0, 1], \\ x(0) &= x_0, & x(1) = x_1,\end{aligned}$$

with $|x_0| + |x_1| \leq \Gamma$, are bounded in $C \times L^x$.

By Remark 2.2 it suffices to prove that there exists $R > 0$ such that $|u| < R$ for any pair (x, u) which solves the homotopy for some $\lambda \in [0, 1]$. Assume the contrary. Thus, there exist sequences $\{\lambda_n\}_{n \in \mathbb{N}} \subset [0, 1]$, $\{u_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$, $\{x_n^0\}_{n \in \mathbb{N}} \subset \mathbb{R}$, $\{x_n^1\}_{n \in \mathbb{N}} \subset \mathbb{R}$, such that

$$\begin{aligned}\dot{x}_n(t) &= \lambda_n f(t, x_n(t), u_n) + (1 - \lambda_n)u_n, & t \in [0, 1], \\ x_n(0) &= x_n^0, & x_n(1) = x_n^1,\end{aligned}$$

with $|x_n^0| + |x_n^1| \leq \Gamma$ and $|u_n| \rightarrow +\infty$ as $n \rightarrow +\infty$. Suppose that $u_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Taking $\rho = -\Gamma$, by (H1) there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, $p \geq \rho$, $t \in [0, 1]$, we get

$$\lambda_n f(t, p, u_n) + (1 - \lambda_n)u_n > 0.$$

This implies that $x_n(t) \geq \rho$, for any $t \in [0, 1]$. Thus,

$$x_n(1) = x_n^0 + \int_0^1 [\lambda_n f(t, x_n(t), u_n) + (1 - \lambda_n)u_n] dt \rightarrow +\infty, \quad \text{as } n \rightarrow +\infty,$$

contradicting the fact that

$$|x_n(1)| = |x_n^1| \leq \Gamma.$$

Moreover, it is immediate to see that

$$\dot{x}(t) = u, \quad u \in PC_1,$$

is completely controllable in \mathbb{R} on the interval $[0, 1]$.

In conclusion, Theorem 2.1 applies with any $PC_{\mathbb{N}} \supseteq PC_1$, for all compact sets $A, B \subset \mathbb{R}$. Furthermore, Theorem 3.1 ensures the existence of a minimum $(x^*, u^*) \in C^1 \times L^x$ for any cost functional defined as in (2).

Example 4.2. Consider the second-order differential equation

$$\ddot{x}(t) = f(t, x(t), u(t)), \quad \text{for any } t \in [0, 1], \quad (8)$$

with $f: [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ continuous, satisfying the following conditions:

- (H1) for any $\rho \in \mathbb{R}$, $f(t, p, q) \rightarrow +\infty$, as $q \rightarrow +\infty$, uniformly for $p \geq \rho$ and $t \in [0, 1]$; $f(t, p, q) \rightarrow -\infty$, as $q \rightarrow -\infty$, uniformly for $p \leq \rho$ and $t \in [0, 1]$;

(H2) for any $M_0 > 0$ and $\rho \in \mathbb{R}$, there exist M_1, M_2 such that

$$f(t, p, q) \geq M_1, \quad p \geq \rho, |q| \leq M_0, t \in [0, 1],$$

$$f(t, p, q) \leq M_2, \quad p \leq \rho, |q| \leq M_0, t \in [0, 1].$$

It is possible to show that Eq. (8) is globally controllable on the interval $[0, 1]$ by means of piecewise constant controls of the form

$$u(t) = a \cdot \chi_{(0,1/2)}(t) + b \cdot \chi_{(1/2,1)}(t), \quad a, b \in \mathbb{R}. \tag{9}$$

Here, the set of switching times is represented by the singleton $\{1/2\}$ and

$$u \in PC_2 = \text{span}\{\chi_{(0,1/2)}, \chi_{(1/2,1)}\}.$$

Actually, in Ref. 2 it is proved that, for any $\Gamma \geq 0$, all the possible solutions (x, u) of the homotopy

$$\ddot{x}(t) = \lambda f(t, x(t), u(t)) + (1 - \lambda)(x(t) + u(t)),$$

$$\lambda \in [0, 1], t \in [0, 1],$$

$$x(0) = c_0, \quad \dot{x}(0) = d_0,$$

$$x(1) = c_1, \quad \dot{x}(1) = d_1,$$

with

$$(|c_0| + |c_1| + |d_0| + |d_1|) \leq \Gamma \quad \text{and} \quad u \in PC_2,$$

are bounded in $C^1 \times L^\infty$ independently from λ .

Roughly speaking, condition (H1) ensures the boundedness of controls, while (H2) ensures the boundedness of the state x when the controls are bounded. Therefore, given any two compact nonempty sets $A, B \subset \mathbb{R}^2$, taking Γ sufficiently large in such a way that

$$(|c_0| + |c_1| + |d_0| + |d_1|) \leq \Gamma, \quad \text{for any } (c_0, d_0) \in A \text{ and any } (c_1, d_1) \in B,$$

we obtain the existence of the positive constants Q, R of assumption (ii) of Theorem 2.1. On the other hand, with

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

by Remark 2.3 it is easy to see that

$$\ddot{x} = x(t) + u(t), \quad t \in [0, 1],$$

is completely controllable in \mathbb{R}^2 by means of controls in PC_2 . Thus, the conclusion of Theorem 2.1 applies with $V_N = PC_N$, so that PC_N^R is the space of bounded (by the constant R) piecewise constant functions obtained from PC_2^R by adding any $N - 2$ distinct switching times to $t_1 = 1/2$.

Finally, by Theorem 3.1, it follows that any cost functional C as defined in (2) assumes the minimum on the solution set S of (3). Indeed, it assumes

the minimum in a point $(x^*, u^*) \in (AC)^n \times PC_{M_0}^R$, where PC_{M_0} is the M_0 -dimensional space determined in the proof of Theorem 3.1. Observe that, if we assume (F2), then by the Remark 2.2 it follows that assumption (H2) can be dropped.

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