A Study on Convergence of Competitive CNNs

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***Abstract*— In a series of papers published in the seventies, Grossberg has developed a geometric approach for analyzing the global dynamical behavior and convergence properties of a class of competitive dynamical systems. In this paper, Grossberg approach is extended to competitive standard Cellular Neural Networks (CNNs), and it is used to investigate convergence of classes of non-symmetric competitive CNNs under the hypothesis that they induce a globally consistent decision scheme.**

I. INTRODUCTION

In [1]–[3], Grossberg has developed a geometric approach for analyzing the global dynamical behavior and convergence properties of the class of nonlinear competitive systems

$$\dot{x}_i = \alpha(x_i)\mathcal{M}_i(x), \quad i = 1, 2, \ldots, n \tag{1}$$

where $x = (x_i)_{i=1,2,\ldots,n} \in \mathbb{R}^n$ is the vector of state variables. The continuously differentiable function $\mathcal{M}_i : \mathbb{R}^n \to \mathbb{R}$, which satisfies

$$\frac{\partial \mathcal{M}_i}{\partial x_j}(x) \le 0 \quad \forall j \ne i, \quad \forall x \in \mathbb{R}^n$$

represents the *competitive balance* at the state x_i. The continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ is such that $\alpha(x_i) > 0$ when $x_i > 0$, and $\alpha(0) = 0$, i.e., α is an *amplification function* that converts the competitive balance into the growth rate $\dot{x}_i$. Moreover, since $\alpha(0) = 0$, the hyperplanes $x_i = 0$ are invariant for the dynamics of (1). Hence there is a *subset* of $\mathbb{R}^n$, namely the positive orthant $O^+ = \{x \in \mathbb{R}^n : x_i > 0, i = 1, 2, \ldots, n\}$, where the state variables evolve: for initial conditions $x_i(0) > 0$, $i = 1, 2, \ldots, n$, the solution $x(t)$ of (1) is such that $x(t) \in O^+$ for all $t \ge 0$. An important special case is the classical Volterra-Lotka system for n competing species

$$\dot{x}_i = x_i\left(1 - \sum_{k=1}^{n} B_{ik}x_k\right), \quad i = 1, 2, \ldots, n \tag{2}$$

where $B_{ik} \ge 0$ for all $i \ne k$.

It has been proved in [1]–[3] that each competitive system (1) induces a *decision scheme*, and if the scheme is *globally consistent*, then each solution is forced through a series of local decisions (or jumps) which eventually lead to a final global decision (or global consensus). This corresponds to the fact that the solution has settled into an equilibrium point, i.e., (1) is *convergent*. In other circumstances, it may happen that the decision scheme is *globally inconsistent*, hence the series of local decisions never terminates and the system can sustain *non-vanishing oscillations*. This is true for example in the 'voting paradox' discussed by May and Leonard in 1975, i.e., a Volterra-Lotka model with three competing species x_1, x_2, and x_3, which has a contradictory decision scheme where x_1 beats x_2, x_2 beats x_3, and x_3 beats x_1.

The goal of this paper is to extend Grossberg approach in order to address convergence of standard competitive cellular neural networks (CNNs) [4], i.e., CNNs with inhibitory (non-positive) interconnections between distinct neurons. What makes really attractive Grossberg approach in the CNN framework, is that it does not require the existence of a Lyapunov function, hence it is applicable also to address convergence of *non-symmetric* CNNs. In this regard, we recall that the Lyapunov method developed by Chua and Yang [4] is applicable to symmetric (reciprocal) CNNs, only.

The paper shows that it is possible to associate a decision scheme with the competitive dynamical system satisfied by the CNN, and to globally analyze the CNN dynamics and convergence properties on the basis of the consistency or inconsistency of the scheme. In particular, the paper investigates the convergence properties implied by a globally consistent decision scheme, in the case where there are three competing neurons. The proofs of the main results in this paper are given in [5]. There, the convergence results are also extended to higher-order CNNs.

II. COMPETITIVE CNNS

The standard CNNs, introduced by Chua and Yang in 1988 [4], obey the system of nonlinear differential equations

$$\dot{x} = -Dx + TG(x) + I \tag{N}$$

where $x = (x_i)_{i=1,2,\ldots,n} \in \mathbb{R}^n$ is the vector of neuron state variables; $D = \mathrm{diag}(d_1, d_2, \ldots, d_n) \in \mathbb{R}^{n\times n}$ with $d_i > 0$, $i = 1, 2, \ldots, n$, is a diagonal matrix of neuron self-inhibitions; $T = (T_{ij})_{i,j=1,2,\ldots,n} \in \mathbb{R}^{n\times n}$ is the neuron interconnection matrix; $I = (I_i)_{i=1,2,\ldots,n} \in \mathbb{R}^n$ is the vector of biasing inputs, and $G(x) = (g(x_1), \ldots, g(x_n))' : \mathbb{R}^n \to \mathbb{R}^n$, where the prime means transpose, is a diagonal mapping where

$$g(\rho) = \frac{1}{2}(|\rho + 1| - |\rho - 1|) : \mathbb{R} \to \mathbb{R}$$

is the piecewise-linear neuron activation.

Definition 1: The CNN (N) is said to be convergent (or completely stable), if and only if for any solution $x(t)$ of (N) we have $\lim_{t\to+\infty} x(t) = x_e$, where x_e is an equilibrium point of (N).

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Assumption 1: The CNN (N) is competitive, i.e., we have $T_{ij} \leq 0$, for all $i \neq j$. ■

Competitive CNNs are of great importance for the applications, see [6]–[9] and references therein. Under the assumption that T is symmetric, it is well-known that a competitive CNN admits a global Lyapunov function and is convergent from the general Lyapunov theory developed in [4]. On the contrary non-symmetric competitive CNNs may exhibit non-convergent dynamics including non-vanishing oscillations and chaos. For example, in [10] a third-order non-symmetric competitive CNN has been studied, which displays Hopf bifurcations originating a large-amplitude and globally attracting stable limit cycle, in a wide range of parameters. In [11], a fourth-order non-symmetric competitive CNN is presented, which exhibits a cascade of period-doubling bifurcations leading to the birth of a complex attractor. Up to now, no general method is available for determining conditions under which a non-symmetric competitive CNN is convergent.

The goal of this paper is to extend Grossberg approach to competitive CNNs. This involves two main steps: (i) first of all we write the CNN equations with respect to the neuron outputs, and show that in this way we are brought back to a dynamical system that is structurally similar to the class of competitive systems (1) (Section III); (ii) we analyze the convergence properties of the dynamical system satisfied by the CNN outputs, by generalizing to this system Grossberg approach (Sections IV-V). We stress that the extended method does not require the existence of a Lyapunov function. As such it is applicable to address convergence in the general case where the interconnection matrix T of the competitive CNN is not necessarily symmetric, and the CNN possesses multiple equilibrium points.

III. DYNAMIC SYSTEM FOR CNN OUTPUTS

The CNN (N) is quite different in structure with respect to (1), and Grossberg approach is not directly applicable to analyze (N). In particular, a competitive balance and amplification function with properties analogous to those of model (1), are not identifiable for (N). Moreover, the state space of (N) is the whole $\mathbb{R}^n$ space. This notwithstanding, we prove in what follows that it is possible to put the CNN equations in a form structurally analogous to that of system (1), if we write the same equations with respect to the neuron outputs.

Let us consider the following system of differential inclusions

$$\dot{y} \in H(y)M(y), \quad y \in \mathbb{R}^n \tag{3}$$

where

$$M(y) = Ay + I = (-D + T)y + I, \quad y \in \mathbb{R}^n$$

is the affine vector field satisfied by the CNN (N) in the linear region $\{y \in \mathbb{R}^n : |y_i| < 1, i = 1, 2, \ldots, n\}$, extended to the whole $\mathbb{R}^n$ space. Furthermore, $H(y) = \operatorname{diag}(h(y_1), \ldots, h(y_n)) : \mathbb{R}^n \multimap \mathbb{R}^n$ is a diagonal set-valued map, such that $h(\rho) : \mathbb{R} \multimap \mathbb{R}$ is a non-negative set-valued map defined as

$$h(\rho) = \begin{cases} 1, & |\rho| < 1 \\ [0, 1], & |\rho| = 1 \\ 0, & |\rho| > 1. \end{cases} \tag{4}$$

Note that under Assumption 1 we have

$$\frac{\partial M_i}{\partial y_j}(y) = A_{ij} = T_{ij} \leq 0 \;\; \forall i \neq j, \;\; \forall y = (y_i)_{i=1,2,\ldots,n} \in \mathbb{R}^n.$$

Property 1: Let $x(t)$, $t \geq 0$, be a solution of the CNN (N), and $y(t) = G(x(t))$, $t \geq 0$, the corresponding output solution. Then, $y(t)$, $t \geq 0$, is also a solution of (3). ■

By writing (3) in components we obtain

$$\dot{y}_i \in h(y_i)M_i(y) = h(y_i)\left(\sum_{k=1}^{n} A_{ik}y_k + I_i\right), \;\; i = 1, 2, \ldots, n \tag{5}$$

for any $y = (y_i)_{i=1,2,\ldots,n} \in \mathbb{R}^n$, whose form is analogous to that of the class of competitive systems (1), and in particular to the Volterra-Lotka system (2). Indeed, (5) is characterized by: (a) functions M_i, $i = 1, 2, \ldots, n$, which are the competitive balance at each CNN neuron output y_i; (b) a non-negative function h, which plays the role of an amplification function converting the competitive balance into the growth rate $\dot{y}_i$, and (c) a subset of the space $\mathbb{R}^n$, namely the hypercube $K_n = [-1, 1]^n$, where the solutions of (5) starting in K_n are constrained to evolve.

Although (5) is structurally analogous to model (1), there is however a basic difference. In fact, the amplification h in (4) is a *set-valued* map assuming multiple values at the saturation levels $\rho = \pm 1$,[1] while the amplification function α in model (1) is a conventional single-valued function. Consequently, we need ad hoc techniques to analyze the dynamics of the differential inclusion (5), which differ substantially from those used to analyze the ordinary differential equation in model (1).

IV. DECISION SCHEME FOR COMPETITIVE CNNS

By means of the competitive balance M, we define for the CNN the following relevant functions and subsets of K_n. Let

$$M^+(y) = \max_{i=1,2,\ldots,n} \left\{ M_i(y) = \sum_{j=1}^{n} A_{ij}y_j + I_i \right\} : K_n \to \mathbb{R}$$

be the maximal balance function, and

$$M^-(y) = \min_{i=1,2,\ldots,n} \left\{ M_i(y) = \sum_{j=1}^{n} A_{ij}y_j + I_i \right\} : K_n \to \mathbb{R}$$

the minimal balance function. Furthermore, consider the subsets of K_n

$$\begin{aligned} R^+ &= \{y \in K_n : M^+(y) \geq 0\} \\ R^- &= \{y \in K_n : M^-(y) \leq 0\} \\ R^\star &= R^+ \cap R^-. \end{aligned}$$

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[1] This agrees with the fact that the velocity $\dot{y}_i(t)$ of the CNN output solution $y(t)$ is not uniquely defined at the saturation levels $y_i(t) = \pm 1$, but rather there are multiple feasible velocities when $y_i(t) = \pm 1$.

A. Ignition property

Property 2: If the CNN (N) satisfies Assumption 1, then R^+ is positively invariant for the output solutions of (N). This means that, given any output solution $y(t)$ of (N) such that $y(t_+) \in R^+$ for some $t_+ \geq 0$, then we have $y(t) \in R^+$ for all $t \geq t_+$. The sets R^- and $R^\star$ are positively invariant for the output solutions of (N) as well. ■

Property 2 represents an ignition property for the competitive CNN. Suppose that at some instant t_+ the output of neuron i is enhanced, i.e., $dy_i(t)/dt|_{t=t^+} \geq 0$. Then, there is at least a neuron $j = j(t) \in \{1, 2, \ldots, n\}$, the index j depending on t, which is enhanced for any time $t \geq t_+$. Said another way, when the competition between neurons starts, it thereafter never turns off.

Property 3: Suppose that the CNN (N) satisfies Assumption 1, and let $y(t)$ be an output solution of (N) such that $y(t) \notin R^\star$ for all $t \geq 0$. Then, all outputs $y_i(t)$, $i = 1, 2, \ldots, n$, are either monotonically increasing or monotonically decreasing for $t \geq 0$, hence there exists the $\lim_{t\to+\infty} y(t) = y(\infty)$. ■

Property 3 implies that only within $R^\star$ there is an interesting dynamics for (N) where the neuron outputs are not necessarily monotone increasing or monotone decreasing functions. In what follows we thus restrict our analysis to any output solution $y(t)$ to (N) that hits $R^\star$ at some finite instant $t_\star$, and is such that $y(t) \in R^\star$ for all $t \geq t_\star$ (see Property 2).

B. Jumps and jump sets

Let $y(t)$ be an output solution of the competitive CNN (N), such that $y(t) \in R^\star$ for $t \geq t_\star$. In analogy to Grossberg approach, we want to analyze the dynamical behavior of $y(t)$ in $R^\star$ by keeping track of which neuron is winning the competition, i.e., by tracking the index $w = w(t) \in \{1, 2, \ldots, n\}$, in general depending on t, such that we have $M_{w(t)}(y(t)) = M^+(y(t))$. To this end, we will follow the jumps of $y(t)$ between the regions

$$R_i^+ = \{y \in R^\star : M_i(y) = M^+(y)\}, \quad i = 1, 2, \ldots, n.$$

Definition 2: We say that an output solution $y(t)$ of (N) makes no jump (between regions R_i^+) in the time interval $(t_a, t_b) \subset (t_\star, +\infty)$, if and only if there exists an index $w \in \{1, 2, \ldots, n\}$ such that $M_w(y(t)) = M^+(y(t))$ for all $t \in (t_a, t_b)$. ■

Let

$$t_1 = \sup\{\tau > t_\star : y(t) \text{ makes no jump in } (t_\star, \tau)\}$$

where $t_\star < t_1 \leq +\infty$. Moreover, let $w(1) \in \{1, 2, \ldots, n\}$ be such that $M_{w(1)}(y(t)) = M^+(y(t))$ for all $t \in (t_\star, t_1)$.

If $t_1 = +\infty$, then $w(1)$ is the winning neuron for all $t \geq t_\star$. Otherwise, if $t_1 < +\infty$ we let

$$t_2 = \sup\{\tau > t_1 : y(t) \text{ makes no jump in } (t_1, \tau)\}$$

where $t_1 < t_2 \leq +\infty$. Also, let $w(2) \in \{1, 2, \ldots, n\}$ be such that $M_{w(2)}(y(t)) = M^+(y(t))$ for all $t \in (t_1, t_2)$. Of course, we have $w(2) \neq w(1)$. If the case $t_1 < +\infty$ occurs, we will say that $y(t)$ *jumps from the winning region* $R_{w(1)}^+$ *to the winning region* $R_{w(2)}^+$ *at the instant* t_1.

If $t_2 = +\infty$, then $w(2)$ is the winning neuron for all $t \geq t_1$. Otherwise, if $t_2 < +\infty$ we let

$$t_3 = \sup\{\tau > t_2 : y(t) \text{ makes no jump in } (t_2, \tau)\}$$

where $t_2 < t_3 \leq +\infty$. Moreover, let $w(3) \in \{1, 2, \ldots, n\}$ be such that $M_{w(3)}(y(t)) = M^+(y(t))$ for all $t \in (t_2, t_3)$. We have $w(3) \neq w(2)$. If $t_2 < +\infty$, we will say that $y(t)$ jumps from region $R_{w(2)}^+$ to region $R_{w(3)}^+$ at the instant t_2.

Proceeding in this way, we can construct a sequence of instants $t_\star < t_1 < t_2 < t_3 < \ldots$, and a corresponding sequence of indexes $w(1), w(2), w(3), \ldots$, in the set $\{1, 2, \ldots, n\}$, such that $y(t)$ does not jump in the intervals (t_{k-1}, t_k), $k = 1, 2, \ldots$ (we have let $t_0 = t_\star$), whereas $y(t)$ jumps from region $R_{w(k)}^+$ to region $R_{w(k+1)}^+$ at the instants t_k, $k = 1, 2, \ldots$ Such a sequence of jumps may be *finite* or *infinite*. If it is finite, then there exist an index $w \in \{1, 2, \ldots, n\}$ and an instant $t_w < +\infty$, such that $M_w(y(t)) = M^+(y(t))$ for all $t \geq t_w$, namely y_w is the eventual winning neuron.

C. Decision scheme

Suppose that $y(t)$ jumps from region R_i^+ to region R_j^+, $j \neq i$, at the instant $t = t_{\mathrm{j}}$. We equivalently say that the CNN takes a *local decision* where the CNN decides to maximally enhance neuron j instead of neuron i, at $t = t_{\mathrm{j}}$. Of course, the jump can only occur on the positive *jump set* between the winning regions R_i^+ and R_j^+, which is given by

$$J_{ij}^+ = R_i^+ \cap R_j^+ = \{y \in R^\star : M_i(y) = M_j(y) = M^+(y)\}.$$

Definition 3: We say that the positive jump set J_{ij}^+, $j \neq i$, is crossable from i to j, if and only if there exists at least an output solution $y(t)$ of the CNN (N), such that $y(t)$ jumps from R_i^+ to R_j^+ on J_{ij}^+ at some instant $t = t_{\mathrm{j}}$. Otherwise, we say that the jump set J_{ij}^+ is non-crossable from i to j. ■

It is now possible to associate with the competitive CNN (N) a positive *directed decision graph* $\mathcal{G}^+$. First, consider a directed graph specified by the set of nodes $\{1, 2, \ldots, n\}$, each node corresponding to a neuron, which is fully connected. Then, we construct a (reduced) positive directed decision graph $\mathcal{G}^+$ as follows: for each $i, j \in \{1, 2, \ldots, n\}$, with $j \neq i$, we remove the branch oriented from node i to node j if and only if J_{ij}^+ is non-crossable from i to j. The graph $\mathcal{G}^+$ is identified with the positive decision scheme induced by (N).

Definition 4: Suppose that the CNN (N) satisfies Assumption 1. The positive directed decision graph $\mathcal{G}^+$ associated with (N) is said to be globally consistent, if and only if $\mathcal{G}^+$ is acyclic, i.e., $\mathcal{G}^+$ has no directed jump cycle. ■

V. CONVERGENCE OF COMPETITIVE CNNS

A. A general result

The next general theorem on the asymptotic behavior of the solutions of a competitive CNN holds.

Theorem 1: Suppose that the CNN (N) satisfies Assumption 1, and that the positive directed decision graph $\mathcal{G}^+$ associated with the CNN is globally consistent. Let $y(t)$, $t \geq 0$, be an output solution of (N) such that $y(t) \in R^{\star}$ for all $t \geq t_{\star}$. Then, the following hold.

a) The solution $y(t)$ undergoes at most $n-1$ jumps between regions R_i^+ for $t \geq t_{\star}$, hence there exist an index $w \in \{1, 2, \ldots, n\}$ and $t_w > t_{\star}$, such that we have $y(t) \in R_w^+$, $t \geq t_w$ (neuron w is the eventual winning neuron). Furthermore, there exists the $\lim_{t \to +\infty} y_w(t) = y_w(\infty) \in [-1, 1]$.

b) If we have $-1 < y_w(t) < 1$, $t \geq t_w$, then there exists the $\lim_{t \to +\infty} y(t) = y(\infty) \in K_n$. Otherwise, we have $y_w(t) = 1$ or $y_w(t) = -1$, $t \geq t_{w1}$ for some $t_{w1} \geq t_w$. ■

B. Second-Order and Third-order Competitive CNNs

For competitive CNNs with two neurons the next basic result can be proved.

Theorem 2: Consider a second-order CNN (N) satisfying Assumption 1. Then, the positive directed decision graph $\mathcal{G}^+$ is globally consistent, moreover (N) is convergent. ■

A third-order CNN in general does not have a globally consistent decision scheme ($\mathcal{G}^+$ is in general not acyclic), and it may be non-convergent, as it is shown in the next example.

Example. In [10], the following third-order competitive CNN has been considered

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = -\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & -\alpha & -\beta \\ -\beta & 0 & -\alpha \\ -\alpha & -\beta & 0 \end{pmatrix} \begin{pmatrix} g(x_1) \\ g(x_2) \\ g(x_3) \end{pmatrix} \tag{6}$$

where $\alpha, \beta > 0$. It can be verified that when parameters α, β belong to the open region

$$R_{\text{osc}} = \left\{ (\alpha, \beta) : \alpha + \beta > 2; \alpha < \frac{\beta^2 + 1}{\beta + 1}; \beta < \frac{\alpha^2 + 1}{\alpha + 1} \right\} \tag{7}$$

then (6) induces the positive directed decision scheme $\mathcal{G}^+ : 1 \to 2$, $2 \to 3$, and $3 \to 1$, which has a directed jump cycle. It is shown in [10] that for these parameters the solutions of (6) display large-size non-vanishing oscillations. ■

In the next basic result it is shown that, when the third-order CNN has a globally consistent decision scheme, then it also enjoys the property of convergence.

Theorem 3: Consider a third-order CNN (N) satisfying Assumption 1, and assume that the positive directed decision graph $\mathcal{G}^+$ is globally consistent. Then, (N) is convergent. ■

It is possible to analytically characterize some classes of third-order competitive CNNs satisfying the hypotheses of Theorem 3. Consider the third-order CNN (N) defined by $A = (A_{ij})_{i,j=1,2,3} = -D + T$, and suppose that $d_i = 1$, $i = 1, 2, 3$, and that the input $I = 0$. The following holds.

Property 4: Let $\mathcal{A} \subset \mathbb{R}^9$ be the set of interconnection parameters A_{ij}, $i, j \in \{1, 2, 3\}$, satisfying

$$A_{13} < A_{23} \leq 0; A_{12} < A_{32} \leq 0; A_{21} < A_{31} \leq 0; A_{11} \in \mathbb{R}$$

$$A_{22} < A_{12} - |A_{11} - A_{21}| - |A_{13} - A_{23}| - 2I_{\text{M}}$$

$$A_{33} < \min\{a_1, a_2\}$$

where

$$a_1 = A_{13} - |A_{11} - A_{31}| - |A_{12} - A_{32}| - 2I_{\text{M}}$$

$$a_2 = A_{23} - |A_{22} - A_{32}| - |A_{21} - A_{31}| - 2I_{\text{M}}$$

and $I_{\text{M}} = \max_{i=1,2,3} |I_i|$. Then, within $\mathcal{A}$ the third-order CNN has the globally consistent decision scheme $\mathcal{G}_1^+ : 2 \to 1, 3 \to 1, 3 \to 2$. Moreover, $\mathcal{A}$ is the union of a finite number of convex polyhedra with non-empty interior in $\mathbb{R}^9$. ■

Property 4 has the following consequence. Given a nominal third-order CNN whose interconnection parameters belong to the interior of region $\mathcal{A}$ defined in Property 4, then: (a) the nominal CNN has a globally consistent decision scheme and is convergent; (b) global consistency of decisions and convergence are properties that are *robust* with respect to (small) perturbations of the nominal CNN interconnections. We stress that within region $\mathcal{A}$ there are CNNs with non-symmetric interconnection matrices T, for which convergence cannot be proved to the authors' knowledge by means of other existing methods.

VI. CONCLUSION

The paper has extended to standard competitive CNNs a geometric approach previously developed by Grossberg for analyzing convergence of a different class of competitive dynamical systems. The approach permits to associate with a competitive CNN a decision scheme, and to globally analyze its dynamical behavior and convergence properties under the hypothesis that the decision scheme is globally consistent. An application has been considered to convergence of a class of non-symmetric third-order competitive CNNs.

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