Full-Range Cellular Neural Networks and Differential Variational Inequalities

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***Abstract*— We consider the Full-Range (FR) model of Cellular Neural Networks (CNNs) in the ideal case where the neuron nonlinearities are hard-comparator functions with two unbounded vertical segments. The dynamics of FR-CNNs is rigorously analyzed by using theoretical tools from set-valued analysis and differential inclusions. The fundamental property proved in the paper is that FR-CNNs are equivalent to a special class of differential inclusions named *differential variational inequalities.* On this basis, a sound foundation to the dynamics of FR-CNNs is given, by establishing results on the existence and uniqueness of the solution starting at a given point, and on the existence of equilibrium points. Moreover, some fundamental results on trajectory convergence towards equilibrium points (complete stability) for reciprocal standard CNNs are extended to reciprocal FR-CNNs by using a generalized Lyapunov approach.**

I. Introduction

The Standard (S) Cellular Neural Networks (S-CNNs) introduced by Chua and Yang [1], have been one of the most investigated neural paradigms in the last two decades. Significant advantages in the VLSI implementation of CNN chips with a large number of cells have been achieved by introducing the so called Full-Range (FR) model of CNNs [2]. These networks are obtained by modifying S-CNNs, namely in a FR-CNN each neuron is characterized by a self-loss hard-comparator nonlinearity with two unbounded *vertical* segments at $x_i = \pm 1$, which prevent the state trajectories from exceeding the (normalized) range $x_i \in [-1, 1]$. Almost all recently manufactured chips [3], [4], [5], [6] are based on the FR model. These include the popular ACE4k [4] and ACE16k [6] chips, implementing the CNN Universal Machine [7], i.e., a general purpose vision device that incorporates optical sensing and combines logic operations and spatial-temporal dynamics generated by a FR-CNN architecture.

Since the ideal neuron nonlinearities used in a FR-CNN possess two unbounded vertical segments, they correspond mathematically to set-valued maps. Therefore, differently from S-CNNs, where the dynamics obeys a (conventional) differential equation, FR-CNNs are described by a differential inclusion [8].

The goal of this paper is to rigorously analyze FR-CNNs by using appropriate tools from set-valued analysis and the theory of differential inclusions. The fundamental property here established is that the dynamics of FR-CNNs can be described by a specific class of differential inclusions named *differential variational inequalities*. Accordingly, the paper gives a sound theoretical foundation to the dynamics of FR-CNNs, by addressing the definition, existence, and uniqueness of the solution of FR-CNNs, and the existence of at least one stationary solution (an equilibrium point). The analysis is valid in the general case where the neuron interconnection matrix T is symmetric or non-symmetric. Moreover, under the hypothesis of symmetric T, it is shown that FR-CNNs obey a gradient differential inclusion, and a fundamental result on trajectory convergence towards equilibrium points (complete stability) is obtained via a generalized Lyapunov approach.

The results obtained in the paper can be considered as a generalization to the ideal case of nonlinearities with vertical segments, of previous results in the literature obtained by approximating the ideal nonlinearity of a FR-CNN by a less-hard single-valued comparator function possessing segments with finite slope, which leads to a neural network model described by a standard differential equation [3], [9].

II. Generalized Lyapunov Approach

Here, we develop appropriate mathematical tools which enable to apply a generalized Lyapunov approach for analyzing complete stability of FR-CNNs. Moreover, we recall some basic properties about differential variational inequalities.

1) Tangent and Normal Cones to Closed Convex Sets: Given a non-empty closed convex set $K \subset \mathbb{R}^n$, and a point $x \in K$, the tangent cone to K at x is defined as follows [10]

$$\mathcal{T}_K(x) = \{v \in \mathbb{R}^n : \liminf_{\rho \to 0^+} \frac{\text{dist}(x + \rho v, K)}{\rho} = 0\}$$

while the normal cone to K at x is

$$\mathcal{N}_K(x) = \{p \in \mathbb{R}^n : \langle p, v \rangle \le 0, \forall v \in \mathcal{T}_K(x)\}.$$

Both $\mathcal{T}_K(x)$ and $\mathcal{N}_K(x)$ are non-empty closed convex cones in $\mathbb{R}^n$. In particular, if x belongs to the interior of K, then $\mathcal{T}_K(x) = \mathbb{R}^n$ and so $\mathcal{N}_K(x) = \{0\}$.

2) Chain Rule for a Class of Extended-Valued Functions: Let $\Psi_K : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the indicator of the non-empty closed convex set $K \subset \mathbb{R}^n$, namely

$$\Psi_K(x) = \begin{cases} 0, & x \in K \\ +\infty, & x \notin K. \end{cases}$$

For any point $x \in K$, the generalized gradient of Ψ_K at x is given by [11, Proposition 2.4.12]

$$\partial \Psi_K(x) = \mathcal{N}_K(x).$$

In the paper, we will use extended-valued Lyapunov functions $W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of the form

$$W(x) = \phi(x) + \Psi_K(x) \tag{1}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is $C^1(\mathbb{R}^n)$ and $\Psi_K : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the indicator of a non-empty closed convex set $K \subset \mathbb{R}^n$. The calculus of the time derivative of W along solutions $x(t)$ of a given differential inclusion is of key importance in the Lyapunov approach used in this paper. To this end the next rule, generalizing to extended-valued functions an analogous rule given in [12, Property 1], will play a crucial role.[1]

Property 1 (Chain Rule): Consider function W defined in (1), and suppose that $x : [0, +\infty) \to \mathbb{R}^n$ is an absolutely continuous function such that $x(t) \in K$ for all $t \in [0, +\infty)$. Then, for a.a. $t \in [0, +\infty)$ we have

$$\frac{d}{dt} W(x(t)) = \langle \xi, \dot{x}(t) \rangle \quad \forall \xi \in \nabla \phi(x(t)) + \mathcal{N}_K(x(t)) \tag{2}$$

where $\nabla \phi(x) + \mathcal{N}_K(x) = \partial W(x)$, for any $x \in K$. ∎

Proof. Since $x(t) \in K$ for all $t \in [0, +\infty)$, we have $\Psi_K(x(t)) = 0$ and hence $W(x(t)) = \phi(x(t))$ for all $t \in [0, +\infty)$. Being $x(t)$ absolutely continuous, then it is differentiable for a.a. $t \in [0, +\infty)$. If $t \in [0, +\infty)$ is an instant at which $x(t)$ is differentiable, from the standard chain rule for the composition of differentiable functions we have that

$$\frac{d}{dt} W(x(t)) = \langle \nabla \phi(x(t)), \dot{x}(t) \rangle.$$

Moreover, it is possible to show that

$$\dot{x}(t) \in \mathcal{N}_K^{\perp}(x(t)) \tag{3}$$

where $\mathcal{N}_K^{\perp}(x(t))$ denotes the set of vectors of $\mathbb{R}^n$ which are orthogonal to the vectors of $\mathcal{N}_K(x(t))$. Hence, $\langle \xi, \dot{x}(t) \rangle = 0$ for any $\xi \in \mathcal{N}_K(x(t))$, and so (2) immediately follows.

Let us prove that (3) holds. Let $h > 0$, and note that since $x(t)$ and $x(t+h)$ belong to K, we have

$$\operatorname{dist}(x(t) + h\dot{x}(t), K) \le \|x(t) + h\dot{x}(t) - x(t+h)\|_2.$$

Dividing by h, and accounting for the differentiability of x at time t, we obtain

$$\lim_{h \to 0^+} \frac{\operatorname{dist}(x(t) + h\dot{x}(t), K)}{h} = 0.$$

Therefore, we have $\dot{x}(t) \in \mathcal{T}_K(x(t))$.

Now, suppose that $h < 0$. Since once more $x(t)$ and $x(t+h)$ belong to K, we have

$$\begin{aligned} 0 \;&\le\; \frac{\operatorname{dist}(x(t) + (-h)(-\dot{x}(t)), K)}{-h} \\ &\le\; \frac{\|x(t) + h\dot{x}(t) - x(t+h)\|_2}{-h}. \end{aligned}$$

[1] We will use a.a. as an abbreviation of the term 'almost all'.

Let $\rho = -h$. Then,

$$\lim_{\rho \to 0^+} \frac{\operatorname{dist}(x(t) + \rho(-\dot{x}(t)), K)}{\rho} = 0$$

and hence $-\dot{x}(t) \in \mathcal{T}_K(x(t))$.

It suffices now to observe that $\mathcal{T}_K(x) \cap -\mathcal{T}_K(x) = \mathcal{N}_K^{\perp}(x)$, for any $x \in K$. In fact, if $v \in \mathcal{T}_K(x) \cap -\mathcal{T}_K(x)$, then $\langle v, \gamma \rangle \le 0$ and $\langle -v, \gamma \rangle \le 0$ for any $\gamma \in \mathcal{N}_K(x)$. Then, $\langle v, \gamma \rangle = 0$ for any $\gamma \in N_K(x)$, i.e., $v \in \mathcal{N}_K^{\perp}(x(t))$. Conversely, if $\langle v, \gamma \rangle = 0$ for any $\gamma \in \mathcal{N}_K(x)$, then $\langle -v, \gamma \rangle = 0$ for any $\gamma \in \mathcal{N}_K(x)$ and so $v \in \mathcal{T}_K(x) \cap -\mathcal{T}_K(x)$. Finally, since ϕ and Ψ_K are regular functions at any $x \in K$, see [11, Corollary at p. 32 and Proposition 2.4.12], we have that $\partial W(x(t)) = \nabla \phi(x(t)) + \mathcal{N}_K(x(t))$. ∎

3) Differential Variational Inequalities: Let $K \subset \mathbb{R}^n$ be a non-empty closed convex set and $G : K \to \mathbb{R}^n$ be a given function. Following [8, Chapter 5], a differential variational inequality is a differential inclusion of the form

$$\dot{x} \in G(x) - \mathcal{N}_K(x). \tag{4}$$

A solution $x(t)$ to (4), $t \in [t_1, t_2]$, with $t_1 < t_2$, is an absolutely continuous function such that

$$\begin{cases} x(t) \in K, & \text{for all } t \in [t_1, t_2] \\ \dot{x}(t) \in G(x(t)) - \mathcal{N}_K(x(t)), & \text{for a.a. } t \in [t_1, t_2]. \end{cases} \tag{5}$$

If in addition

$$\dot{x}(t) = m\left(G(x(t)) - \mathcal{N}_K(x(t))\right) \tag{6}$$

for a.a. $t \in [t_1, t_2]$, then $x(t)$ is said to be a *slow* solution of (4). In (6), $m(Q)$ denotes the element of the non-empty closed convex set $Q \subset \mathbb{R}^n$ with the smallest norm.

III. FR Model of CNNs

We consider CNNs whose dynamics is described by the differential inclusion

$$\dot{x} \in F_{\text{fr}}(x) = Tx + I - S(x) \tag{fr}$$

where $x \in H \subset \mathbb{R}^n$ is the vector of neuron state variables, $F_{\text{fr}} : H \multimap \mathbb{R}^n$ is the set-valued vector field associating to the state x the set $F_{\text{fr}}(x)$ of feasible velocities at x, and $H = [-1, 1]^n = [-1, 1] \times [-1, 1] \times \cdots \times [-1, 1]$, is a hypercube in $\mathbb{R}^n$. Furthermore, $T \in \mathbb{R}^{n \times n}$ is the constant neuron interconnection matrix, $I \in \mathbb{R}^n$ is the constant input, and $S(x) = (s(x_1), \cdots, s(x_n))' : H \multimap \mathbb{R}^n$ is a set-valued map where $s(x_i) : [-1, 1] \multimap \mathbb{R}$ is defined as

$$s(x_i) = \begin{cases} (-\infty, 0], & x_i = -1 \\ 0, & x_i \in (-1, 1) \\ [0, +\infty), & x_i = 1. \end{cases} \tag{7}$$

Neural network (fr) corresponds to the so called FR model of CNNs. This model significantly differs from that of S-CNNs originally introduced by Chua and Yang [1]. The main difference is that in (fr) each neuron has a self-loss nonlinearity which is represented by a hard-comparator function s as in (7), while a standard piecewise-linear input-output neuron nonlinearity with unity gain in the linear region, and ± 1 saturation levels, is employed for S-CNNs. The physical implementation of the FR-CNN model necessarily shows a finite slope of the vertical segments of (7). It can be proven, however, that the solutions of the "physical" FR-CNN model tend to those of the theoretic FR-CNN model when the slope tends to infinity.

According to the theory of differential inclusions [8], a solution of (fr) on $[0, \tilde{t}]$, $\tilde{t} > 0$, with initial condition $x(0) = x_0 \in H$, is an absolutely continuous function $x(t)$ on $[0, \tilde{t}]$ such that $x(t) \in H$ for $t \in [0, \tilde{t}]$ and for a.a. $t \in [0, \tilde{t}]$ we have $\dot{x}(t) \in Tx(t) + I - S(x(t))$. In particular, an equilibrium point $e \in H$ is a stationary solution of (fr), hence it satisfies the algebraic inclusion $0 \in F_{\rm fr}(e) = Te + I - S(e)$.

We begin by establishing a relationship between (fr) and differential variational inequalities. To this end, note that

$$S(x) = \mathcal{N}_H(x)$$

for any $x \in H$, i.e., $S(x)$ coincides with the normal cone to H at x. Hence, (fr) can be rewritten as

$$\dot{x} \in Tx + I - S(x) = Tx + I - \mathcal{N}_H(x)$$

for $x \in H$. This means that the stated Cauchy problem associated with (fr) is equivalent to the following differential variational inequality

$$\begin{cases} x(0) = x_0 \in H, \quad x(t) \in H, & \text{for all } t \in [0, \tilde{t}] \\ \dot{x}(t) \in Tx(t) + I - \mathcal{N}_H(x(t)), & \text{for a.a. } t \in [0, \tilde{t}]. \end{cases} \tag{8}$$

The next property holds.

Property 2: For any $x_0 \in H$, there is a unique solution $x(t)$, $t \in [0, +\infty)$, of (fr) with initial condition $x(0) = x_0$, which coincides with the slow solution

$$\dot{x}(t) = m\left(Tx(t) + I - \mathcal{N}_H(x(t))\right) = \mathcal{P}_{\mathcal{T}_H(x(t))}(Tx(t) + I) \tag{9}$$

for a.a. $t \in [0, +\infty)$, where $\mathcal{P}_{\mathcal{T}_H(x(t))}(Tx(t) + I)$ denotes the projection of $Tx(t) + I$ on $\mathcal{T}_H(x(t))$. Moreover, there is at least one equilibrium point $e \in H$ of (fr). ■

Proof. Since a Cauchy problem associated with (fr) is equivalent to the differential variational inequality (8), then from [8, Th. 1, p. 267], given any $x_0 \in H$, there is at least a solution $x(t) \in H$, $t \in [0, +\infty)$, of (fr) such that $x(0) = x_0$.

Let us prove that the solution is unique. Pick $x_0 \in H$. Let $x(t)$ and $y(t)$ be two solutions of (fr) which are defined on a common interval $[0, \tau]$, for some $\tau > 0$, and are such that $x(0) = y(0) = x_0$. Since $x(t)$ and $y(t)$ are absolutely continuous, then they are differentiable for a.a. $t \in [0, \tau]$ and

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\|x(t) - y(t)\|_2^2\right) &= \langle x(t) - y(t), \dot{x}(t) - \dot{y}(t)\rangle \\ &= \langle x(t) - y(t), Tx(t) - Ty(t)\rangle \\ &\quad -\langle x(t) - y(t), v_x(t) - v_y(t)\rangle \end{aligned}$$

for some $v_x(t) \in S(x(t))$ and $v_y(t) \in S(y(t))$.

Since $Tx + I$ is a Lipschitz vector field on $\mathbb{R}^n$, we obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\|x(t) - y(t)\|_2^2\right) &\leq \lambda\|x(t) - y(t)\|_2^2 \\ &\quad -\langle x(t) - y(t), v_x(t) - v_y(t)\rangle \end{aligned}$$

where $\lambda > 0$ is the Lipschitz constant.

Now, from (7) it can be easily verified that $\langle x(t) - y(t), w_x(t) - w_y(t)\rangle = \sum_{i=1}^n (x_i(t) - y_i(t))(w_{x,i}(t) - w_{y,i}(t)) \geq 0$, for any $w_x(t) \in S(x(t))$ and $w_y(t) \in S(y(t))$. Therefore

$$\frac{d}{dt}\left(\frac{1}{2}\|x(t) - y(t)\|_2^2\right) \leq \lambda\|x(t) - y(t)\|_2^2. \tag{10}$$

Let $d(t) = \|x(t) - y(t)\|_2^2$. By using Gronwall's inequality, we obtain from (10) that $d(t) \leq d(0)e^{2\lambda t}$, $t \in [0, \tau]$. Hence, being $d(0) = \|x(0) - y(0)\|_2 = 0$, we have $d(t) = \|x(t) - y(t)\|_2 = 0$, for any $t \in [0, \tau]$. Since this holds for any $\tau > 0$, the uniqueness of the solution on $[0, +\infty)$ is proved.

The fact that $x(t)$ is the slow solution to (fr), and a solution to the projected differential equation $\dot{x}(t) = \mathcal{P}_{\mathcal{T}_H(x(t))}(Tx(t) + I)$, follows from [8, Ch. 5, Sect. 6], while the existence of at least an equilibrium point $e \in H$ of (fr) is a consequence of [8, Th. 1, p. 267]. ■

IV. Complete Stability of FR-CNNs

Suppose that matrix T is symmetric, i.e., $T' = T$. To address trajectory convergence of (fr), we introduce the (candidate) extended-valued Lyapunov function $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ given by

$$V(x) = -\frac{1}{2}x'Tx - x'I + \Psi_H(x) \tag{11}$$

where $\Psi_H(x)$ is the indicator of H.

The next properties hold.

Property 3: Suppose that the neuron interconnection matrix T is symmetric. Then, (fr) is described by the gradient differential inclusion

$$\dot{x} \in F_{\rm fr}(x) = Tx + I - S(x) = Tx + I - \mathcal{N}_H(x) = -\partial V(x)$$

for any $x \in H$. ■

Proof. Since H is a non-empty compact convex set, we have $\partial\Psi_H(x) = \mathcal{N}_H(x)$, for any $x \in H$. Therefore, the result in the property follows by taking into account that for a symmetric T we have $\partial[(1/2)(x'Tx)] = Tx$, and that $\partial(x'I) = I$ and the functions $\Psi_H(x)$ and $\phi(x) = -(1/2)x'Tx - x'I$ are regular at any $x \in H$. ■

Property 4: Suppose that the neuron interconnection matrix T is symmetric. Let $x(t)$, $t \in [0, +\infty)$, be the solution of (fr) starting at $x(0) = x_0 \in H$. Then, $V(x(t))$ is differentiable for a.a. $t \in [0, +\infty)$, and

$$\frac{d}{dt}V(x(t)) = -\|\dot{x}(t)\|_2^2 = -\|m(Tx(t) + I - \mathcal{N}_H(x(t)))\|_2^2. \tag{12}$$

■

Proof. Since $x(t) \in H$ and it is absolutely continuous for $t \in [0, +\infty)$, then Property 1 ensures that for a.a. $t \in [0, +\infty)$ we have

$$\frac{d}{dt}V(x(t)) = \langle \xi, \dot{x}(t)\rangle \qquad \forall\, \xi \in \partial V(x(t)). \tag{13}$$

Now, from Property 3 we obtain $\dot{x}(t) \in -\partial V(x(t))$ for a.a. $t \in [0, +\infty)$. Since the scalar product $\langle \xi, \dot{x}(t)\rangle$ is independent of the choice of ξ, we can take $\xi = -\dot{x}(t)$, which yields $dV(x(t))/dt = -\|\dot{x}(t)\|_2^2$. Finally, by recalling that $x(t)$ is the slow solution of (fr), and using (9), we have $dV(x(t))/dt = -\|m(Tx(t) + I - \mathcal{N}_H(x(t)))\|_2^2$. ■

Let

$$E = \{x \in H : 0 \in Tx + I - S(x)\} \neq \emptyset$$

be the set of equilibrium points of (fr). The main result on trajectory convergence of (fr) is as follows.

Theorem 1: Suppose that the neuron interconnection matrix T is symmetric. Then, (fr) is quasi-convergent, i.e., for any $x_0 \in H$, the solution $x(t)$ of (fr) with initial condition $x(0) = x_0$ satisfies $\lim_{t\to+\infty} \mathrm{dist}(x(t), E) = 0$. If, in addition, the equilibrium points of (fr) are isolated, then (fr) is convergent, i.e., for any $x_0 \in H$, the solution $x(t)$ of (fr) with initial condition $x(0) = x_0$ satisfies $\lim_{t\to+\infty} x(t) = \tilde{e}$, for some equilibrium point $\tilde{e} \in E$. ■

Proof. Pick $x_0 \in H$, and let $x(t) \in H$, $t \in [0, +\infty)$, be the solution of (fr) with $x(0) = x_0 \in H$. Since $\Psi_H(x) = 0$ for $x \in H$, it follows that V is bounded from below on the compact set H, hence $V(x(t))$ is bounded from below for $t \in [0, +\infty)$.

Assume, for purpose of contradiction, that $x(t)$ does not converge to E as $t \to +\infty$, i.e., $\limsup_{t\to+\infty} \mathrm{dist}(x(t), E) = 2\alpha > 0$. Therefore, there exists a sequence $\{t_n\}$ such that $t_n \to +\infty$ as $n \to +\infty$ and $\lim_{n\to+\infty} \mathrm{dist}(x(t_n), E) = 2\alpha$.

Let us consider the compact set

$$K_\alpha = \mathrm{cl}\left(H \backslash \left(E + \frac{\alpha}{p} B(0,1)\right)\right)$$

where the symbol $\mathrm{cl}(\cdot)$ denotes the closure, $B(0,1) = \{x \in \mathbb{R}^n : \|x\| < 1\}$, and $p > 1$ can be chosen such that $K_\alpha \cap H \neq \emptyset$. We want to evaluate $dV(x(t))/dt$ when $x(t) \in K_\alpha$. To this end, consider the map $\nu(x) : H \to \mathbb{R}$ defined by

$$\nu(x) = \|m\left(Tx + I - \mathcal{N}_H(x)\right)\|_2.$$

Since the set-valued map $x \multimap Tx + I - \mathcal{N}_H(x)$ has nonempty closed convex values for $x \in H$, and it is upper semicontinuous on H, then $\nu(x)$ is lower semicontinuous on H, hence the minimum exists in any compact set contained in H [10, Lemma 9.3.1, p. 361]. In particular, being $K_\alpha \cap E = \emptyset$, we have $\xi \neq 0$ for any $\xi \in Tx + I - \mathcal{N}_H(x)$ and any $x \in K_\alpha$, hence we obtain $\min_{x\in K_\alpha} \nu(x) = \nu_\alpha > 0$. Therefore, from (12) we have

$$\frac{d}{dt} V(x(t)) \leq -\nu_\alpha^2 < 0 \tag{14}$$

for a.a. $t \geq 0$ such that $x(t) \in K_\alpha$.

On the basis of (14), it is easy to reach a contradiction to the fact that $V(x(t))$ is bounded from below for $t \in [0, +\infty)$. Indeed, since for sufficiently large n, say $n > \overline{n}$, we have $\mathrm{dist}(x(t_n), E) \geq \alpha(2p - 1)/p$, it follows that $x(t) \in K_\alpha$ for any $t \in (t_n, t_n + \delta)$, and for all $n > \overline{n}$, provided $0 < \delta \leq 2\alpha(p-1)/p\mu_\alpha$, where

$$\begin{aligned}\sup_{x\in K_\alpha} \|m(Tx + I - \mathcal{N}_H(x))\|_2 &= \sup_{x\in K_\alpha} \nu(x) \leq \mu_\alpha \\ &= \max_{y\in K_\alpha} \|Ty + I\|_2.\end{aligned}$$

Hence, on any of the infinitely many time intervals $(t_n, t_n + \delta)$, where $n > \overline{n}$, function V undergoes a negative jump which is lower than $-\delta\,\nu_\alpha^2$. By noting that, from Property 4, V is nonincreasing along $x(t)$ for all $t \geq 0$, we reach a contradiction to the fact that $V(x(t))$ is bounded from below for all $t \geq 0$. This contradiction implies that $\lim_{t\to+\infty} \mathrm{dist}(x(t), E) = 0$, hence (fr) is quasi-convergent.

Then, suppose that the equilibrium points of (fr) are isolated. The ω-limit set of each trajectory $x(t)$, $t \geq 0$, of (fr) is a nonempty closed connected set. Hence, the fact that $x(t) \to E$ as $t \to +\infty$, immediately implies that $x(t) \to \tilde{e}$ as $t \to +\infty$, for some $\tilde{e} \in E$. ■

V. Conclusion

By using theoretical tools from set-valued analysis and differential variational inequalities, the paper has given a rigorous analytic foundation to the FR-CNN model where the neuron self-loss nonlinearities are ideal hard-comparator functions with two unbounded vertical segments. In particular, some fundamental results on complete stability for reciprocal S-CNNs have been extended to reciprocal FR-CNNs by using a Lyapunov approach generalized to a class of set-valued maps.

References

[1] L. O. Chua and L. Yang, "Cellular neural networks: Theory," *IEEE Trans. Circuits Syst.*, vol. 35, no. 10, pp. 1257–1272, October 1988.

[2] A. Rodríguez-Vázquez, S. Espejo, R. Domínguez-Castro, J. L. Huertas, and E. Sánchez-Sinencio, "Current-mode techniques for the implementation of continuous- and discrete-time cellular neural networks," *IEEE Trans. Circuits Syst. II*, vol. 40, pp. 132–146, March 1993.

[3] S. Espejo, R. Carmona, R. Domínguez-Castro, and A. Rodríguez-Vázquez, "A VLSI oriented continuous time CNN model," *Int. J. Circuit Theory Applicat.*, vol. 24, pp. 341–356, 1996.

[4] G. Linán, S. Espejo, R. Domínguez-Castro, and A. Rodríguez-Vázquez, "ACE4k: An analog I/O 64 × 64 visual microprocessor chip with 7-bit analog accuracy," *Int. J. Circuit Theory Applicat.*, vol. 30, pp. 89–116, 2002.

[5] G. L. Cembrano, A. Rodríguez-Vázquez, S. E. Meana, and R. Domínguez-Castro, "ACE16k: a 128X128 focal plane analog processor with digital I/O," *Int. J. Neural Systems*, vol. 13, no. 6, pp. 427–434, December 2003.

[6] A. Rodríguez-Vázquez, G. Liñán-Cembrano, L. Carranza, E. Roca-Moreno, R. Carmona-Galán, F. Jiménez-Garrido, R. Domínguez-Castro, and S. E. Meana, "ACE16k: the third generation of mixed-signal SIMD-CNN ACE chips towards VSoCs," *IEEE Trans. Circuits Syst. I*, vol. 51, no. 5, pp. 851–863, May 2004.

[7] T. Roska and L. O. Chua, "The CNN universal machine: An analogic array computer," *IEEE Trans. Circuits Syst. II*, vol. 40, pp. 163–173, March 1993.

[8] J. P. Aubin and A. Cellina, *Differential Inclusions*. Berlin: Springer-Verlag, 1984.

[9] F. Corinto and M. Gilli, "Comparison between the dynamic behaviour of Chua-Yang and full-range cellular neural networks," *Int. J. Circuit Theory Applicat.*, vol. 31, pp. 423–441, September/October 2003.

[10] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*. Boston: Birkauser, 1990.

[11] F. H. Clarke, *Optimization and Non-Smooth Analysis*. New York: John Wiley & Sons, 1983.

[12] M. Forti, P. Nistri, and M. Quincampoix, "Generalized neural network for nonsmooth nonlinear programming problems," *IEEE Trans. Circuits Syst. I*, vol. 51, no. 9, pp. 1741–1754, September 2004.