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Deterministic LQ Regulator Design by Polynomial Equations

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Abstract - The standard LQ output regulation problem is directly approached and solved via spectral factorization and a pair of bilateral Diophantine equations. Stabilizability and/or detectability requirements in the Riccati context are expressed in terms of stability of greatest common left and right divisors of polynomial matrices.

Keywords - Optimal Control, Linear Systems, Multivariable systems, Polynomials.

1 Introduction

This paper deals with the classic Linear-Quadratic Output Regulation (LQOR) problem. A linear, continuous-time, state-space representation for the time-invariant plant to be output regulated is given

$$\begin{cases} \dot{x}(t) &= \Phi x(t) + Gu(t) \\ y(t) &= Hx(t) \end{cases} \qquad (1)$$

with $x(0) \in \Re^n$, $u(t) \in \Re^m$ and $y(t) \in \Re^m$. The problem is to find an input signal $u(t) \in l_2$, if it exists, minimizing the integral quadratic-cost

$$J = \int_0^\infty (\|y(t)\|^2_{\Psi_y} + \|u(t)\|^2_{\Psi_u})dt \qquad (2)$$

for any initial state $x(0)$ and stabilizing the plant (1). In (2) $\Psi_y = \Psi'_y > 0$, $\Psi_u = \Psi'_u > 0$. $\|v(t)\|^2_\Psi = v'(t)\Psi v(t)$ and the prime denotes transpose. The above index cost can be equivalently transformed, by means the Parseval's Lemma, in a form involving the Laplace transforms of $y(t)$ and $u(t)$

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} (\|y(s)\|^2_{\Psi_y} + \|u(s)\|^2_{\Psi_u})ds. \qquad (3)$$

As well known, under stabilizability and detectability assumptions on the triplet (Φ, G, H), problem (1)-(3) can be solved in state-feedback form by using the unique nonnegative definite solution of the relevant algebraic Riccati equation. Moreover, the resulting closed-loop system turns out to be asymptotically-stable.

The aim of this paper is to provide a direct matrix-fraction approach to the problem. In this way, the solution is obtained by first solving a spectral factorization problem and next finding the minimum-degree solution with respect to a "dummy" polynomial matrix of a pair of bilateral Diophantine equations.

This problem was previously addressed by Kucera (1983) for the special case of scalar input and a completely reachable pair (Φ, G). Thanks to these limitative assumptions, in (Kucera,1983) the solution is given in terms of a single Diophantine equation. A discrete-time version of the general case was approached by Mosca and Nistri (1989). They showed that two bilateral Diophantine equations must be simultaneously solved as referred above. In this paper the continuous-time version of the problem treated in (Mosca and Nistri, 1989) is presented.

One of the reasons for considering a polynomial solution for the standard deterministic LQOR problem is to show that Riccati-based and polynomial methods are fully conceptually equivalent, as far as steady-state (semi-infinite horizon) results are concerned. In particular, stabilizability and/or detectability requirements in the Riccati context are expressed in terms of stability of greatest common left and right divisors of polynomial matrices.

The reader is referred to (Kucera, 1979), whose notations are adopted as much as possible considering the difference between the discrete and continuous time domains. The following definitions are assumed hereafter: for any polynomial matrix $P(s)$ in the indeterminate s. $P(s)$ is said to be *row reduced* if the matrix of the coefficients of the highest power of s in each row of $P(s)$ has full row rank. Similarly, $P(s)$ is said to be *column reduced* if the matrix of the coefficients of the highest power of s in each column of $P(s)$ has full column rank. Further, for any real rational matrix $R(s)$ we define $R^*(s) := R'(-s)$.

2 Main Result

As well known, problem (1)-(3) only depends on a completely observable subsystem of (1) obtainable via Kalman's canonical decomposition. Thus, we assume from the outset that

(A.1) (Φ, H) is a completely observable pair.

Hereafter all quantities are assumed to be Laplace transforms. Let $y(s)$ be the output of (1) due to $x(0)$ and the input signal $u(s)$. It is given by

$$y(s) = HA^{-1}(s)[x(0) + Bu(s)] \qquad (4)$$

where $A(s)$ and B are the following polynomial matrices

$$\begin{aligned} A(s) &:= sI - \Phi & (5) \\ B &:= G. & (6) \end{aligned}$$

Hereafter, by the sake of simplicity, the argument s will be omitted unless required to avoid possible confusion. The quadratic cost (3) can be conveniently rewritten as

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} (y^* \Psi_y y + u^* \Psi_u u) ds. \qquad (7)$$

Let us also rewrite (4)

$$y(s) = HA^{-1}(s)x(0) + HB_2(s)A_2^{-1}(s)u(s) \qquad (8)$$

where $B_2(s)A_2^{-1}(s)$ is a right coprime (rc) column reduced matrix-fraction description for $A^{-1}B$. Substituting (8) in (7), one finds

$$\begin{aligned} J = \tfrac{1}{2\pi j} \textstyle\int_{-j\infty}^{j\infty} \quad & u^* A_2^{-*}(A_2^* \Psi_u A_2 + B_2^* \Psi_x B_2) A_2^{-1} u + \\ & u^* A_2^{-*} B_2^* \Psi_x A^{-1} x(0) + x'(0) A^{-*} \Psi_x B_2 A_2^{-1} u + \\ & x'(0) A^{-*} \Psi_x A^{-1} x(0) ds \qquad (9) \end{aligned}$$

where $A_2^{-*} := (A_2^{-1})^*$ and $\Psi_x := H' \Psi_y H$.

Let $E(s)$ be the strictly Hurwitz column reduced (Kucera, 1980 - Lemma 4) polynomial matrix solving the following spectral factorization problem

$$E^* E = A_2^* \Psi_u A_2 + B_2^* \Psi_x B_2. \qquad (10)$$

Using (10) into (9), one finds

$$J = J_1 + J_2 \qquad (11)$$

with

$$\begin{aligned} J_1 &:= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} L^* L ds \\ J_2 &:= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} x'(0) A^{-*}(I - \Psi_x B_2 E^{-1} E^{-*} B_2^*) \Psi_x A^{-1} x(0) ds \\ L &:= E^{-*} B_2^* \Psi_x A^{-1} x(0) + E A_2^{-1} u. & (12) \end{aligned}$$

Note that J_2 does not depend on u.

In order to simplify the above expression, let us consider the following bilateral Diophantine equation

$$E^* Y + ZA = B_2 \Psi_x. \qquad (13)$$

It is temporarily assumed that a solution (Y, Z) of (12) exists with $\partial Z < \partial E^*$. Under this assumption, (12) becomes

$$L = Yx(s) + (E - YB_2)A_2^{-1} u + E^{-*} Z x(0). \qquad (14)$$

Next, to guarantee the closed loop stability the following equation has to be imposed

$$X := (E - YB_2) A_2^{-1} \qquad (15)$$

with X polynomial matrix satisfying

$$XA_2 + YB_2 = E. \qquad (16)$$

Substituting (15) in (14) we have

$$L = (Yx + Xu) + E^{-*} Z x(0).$$

Consequently the cost index J_1 can be split in the following three components

$$J = J_3 + J_4 + J_5 \qquad (17)$$

$$\begin{aligned} J_3 &:= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} x'(0) Z^* E^{-1} E^{-*} Z x(0) ds & (18) \\ J_4 &:= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \{(Yx + Xu)^* E^{-*} Z x(0) \\ & \quad + x'(0) Z^* E^{-1} (Yx + Xu)\} ds & (19) \\ J_5 &:= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} (Yx + Xu)^* (Yx + Xu) ds & (20) \end{aligned}$$

where J_3 does not depend on u. Further, if X and Y are constant matrices, i.e. $\partial X = \partial Y = 0$ (Lemma 1 below verifies that this properties holds true), it is easy to prove by using the Residue Theorem that the cost term J_4 is identically zero. This follows from the fact that $Z^* E^{-1}$ is a strictly proper stable transfer-matrix and the signal $u(t)$ and $x(t)$ are in L_2.

Last, premultiplying both sides of (16) by E^* and taking in to account

$$E^* X - ZB = A_2^* \Psi_u. \qquad (21)$$

The following lemma sums up the above results.

Lemma 1 - Provided that

i) Eq.(13) and (21) [or (13) and (16)] have a solution (X, Y, Z) with $\partial Z < \partial E^*$;

ii) $J_2 + J_3$ is bounded;

the LQOR problem is solved by

$$u(s) = -X^{-1} Y x(s) \qquad (22)$$

with X and Y, constant matrices, specified in i), and correspondingly,

$$J_{min} = J_2 + J_3. \; \square$$

Proof - It remains to be shown that $\partial X = \partial Y = 0$ and that X is nonsingular. From $A^{-1}B = B_2 A_2^{-1}$, or $GA_2 = (sI - \Phi)B_2$, it follows that $\partial A_2 > \partial B_2$ and consequently from (10)

$$\partial E = \partial A_2 = \partial B_2 + 1.$$

Thus, (13) and (21) imply respectively that $\partial X = 0$ and $\partial Y = 0$. Next, nonsingularity of X is implied by (21). □

A condition under which i) of Lemma 1 is fulfilled is given by the next lemma whose proof closely follows a similar proof in (Kucera, 1979).

Lemma 2 - Let the greatest left divisors (GCLD) of A and B be strictly Hurwitz. Thus, there is a unique solution (X, Y, Z) of (13) and (21) [or (13) and (16)] such that $\partial Z < \partial E^*$. □

Proof - Let D be a GCLD of A and B. Then, there is a unimodular matrix U such that

$$[-B \;\; A]U = [D \;\; 0]$$

Next rearrange (13) and (21) as follows

$$E^{*}[X \; Y] + Z[-B \; A] = [A_2^{*}\Psi_u \;\; B_2^{*}\Psi_x]$$

Postmultiplying the latter equation by U and setting $\underline{X} := XU$ and $\underline{Y} := YU$, one gets

$$\begin{aligned} E^{*}\underline{X} + ZD &= A_2^{*}\Psi_u U_{11} + B_2^{*}\Psi_x U_{21} \qquad (23) \\ E^{*}\underline{Y} &= A_2^{*}\Psi_u U_{12} + B_2^{*}\Psi_x U_{22} \qquad (24) \end{aligned}$$

$$0 = -BU_{12} + AU_{22} = -BA_2 + AB_2.$$

Last equation shows that there is a D_2 such that $U_{12} = A_2 D_2$ and $U_{22} = B_2 D_2$. Thus (24) reduces to

$$\underline{Y} = ED_2. \qquad (25)$$

Further, (24) is solvable since E and D are strictly Hurwitz and hence E^{*} and D do not share any nonunit invariant polynomial matrix of compatible dimensions. Correspondingly,

$$[X \;\; Y]U = [TD \;\; 0] = [-TB \;\; TA]U.$$

Therefore all solutions of (13) and (21) are:

$$X = X_o - TB; \quad Y = Y_o + TA; \quad Z = Z_o - E^{*}T$$

Since E is column reduced, it follows that there is a unique solution (X, Y, Z) such that $\partial Z < \partial E^{*}$ (Kucera 1979,1980). In (Mosca, Giarre' and Casavola, 1989) it is shown that this solution coincides with the one of (13) and (21). □

Boundness of $J_2 + J_3$ is clearly guaranteed by the stability of the closed-loop system, that, in turn, if the plant has no unstable hidden modes, is fulfilled if and only if E is strictly Hurwitz. In fact, if the latter is true, the control law (18) can be rewritten as $E^{-1}Xu = -E^{-1}Yx$ with both $E^{-1}X$ and $E^{-1}Y$ stable left coprime transfer-matrices such that $E^{-1}XA_2 + E^{-1}YB_2 = I$.

Lemma 3 - If (A.1) holds then the spectral factor E is strictly Hurwitz. □

Proof - We first observe that if HB_2 and A_2 are rc then $\det E(j\omega) \neq 0, \forall\omega \in \Re$ and we conclude that E is strictly Hurwitz. It thus suffices to show that complete observability of (Φ, H) implies that HB_2 and A_2 are rc. In order to prove this, we begin by nothing that by PBH test (Kailath, 1980), complete observability of (Φ, H) is equivalent to H and A being rc. This, in turn, implies that H and A_1 are rc, if $A^{-1}B = A_1^{-1}B_1$ with A_1 and B_1 left coprime (lc). In fact, $A = \Delta A_1$ and $B = \Delta B_1$ with Δ a GCLD of A and B. Hence

$$I = XA + YH = (X\Delta)A_1 + YH.$$

We finally show that if H and A_1 are rc, then HB_2 and A_2 are rc. In fact consider the transfer-matrices

$$HA_1^{-1} = HB_2A_2^{-1}$$

for which $\partial detA_1 = \partial detA_2$. The expression on LHS can be minimally realized in state-space form with a state of dimension equal to $\partial detA_1$ since H and A_1 are rc. Thus $HB_2A_2^{-1}$ must be also minimally realized with the same state dimension. So we conclude that HB_1 and A_2 are rc. □

The previous three lemmas show that the LQ output regulation problem can be solved provided that (Φ, H) is a completely observable pair and the GCLD's of A and B are stable. This is the same as assuming that the given plant (Φ, G, H), which in general need not be completely observable, has all its observable-unreachable eigenvalues stable. All the results are summed up in the following

Theorem - Consider the LQ output regulation problem (1)-(3) for the arbitrary plant $\sum = (\Phi, G, H)$. Then,

1. The problem is solvable if and only if the GCLD's of A_o and B_o are stable, where $A_o := sI - \Phi_o$ and $B_o := G_o$ and $\sum_o = (\Phi_o, G_o, H_o)$ is completely observable subsystem of $\sum_o$ obtained via Kalman's canonical observability decomposition of $\sum$.
2. Provided that the solvability condition is fulfilled, the optimal input sequence is given as in (22) where X and Y are obtained by first solving the spectral factorization problem (10) and next finding the minimum degree solution w.r.t. Z of the pair of bilateral Diophantine equations (13) and (21) [or (13) and (16)].
3. The LQ optimal feedback-gain matrix
$$F := -XY^{-1} \qquad (26)$$
is a root of the bilinear matrix polynomial equation
$$Z(A - BF) - A_2^{*}\Psi_u F = B_2^{*}\Psi_x \qquad (27)$$
with $\partial Z < \partial E^{*}$.
4. The overall closed-loop system is asymptotically stable if and only if the plant $\sum$ has no unstable hidden modes. □

Proof - Eq. (27) is readily obtained by rewriting (21) as $E^{*} = (ZB + A_2^{*}\Psi_u)X^{-1}$ and using it into (13). □

3 Conclusion and Further Remarks

Remark 1 - There are in general, two bilateral Diophantine equations, viz. (13) and (21) [or, as shown in (Mosca, Giarre' and Casavola 1989), equivalently (13) and (21)], to must be solved with $\partial Z < \partial E^{*}$ in order to finding the LQ optimal feedback-gain matrix $F = X^{-1}Y$. In particular, the reader is referred to (Mosca, Giarre' and Casavola, 1989) and (Hunt, Sebek and Grimble, 1987) where the above issue is thoroughly studied. Nevertheless, for the sake of better understanding, an example is discussed hereafter. □

Example 1 - Let $\Phi = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$; $G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $H = \begin{bmatrix} 1 & 1 \end{bmatrix}$; $\Psi_u = 1$; $\Psi_y = 1$. Notice that (Φ, G) is not completely reachable, whereas (Φ, H) is completely observable. We find $A = \begin{bmatrix} s-1 & 0 \\ 0 & s+\frac{1}{2} \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. A GCLD of A and B is $\begin{bmatrix} 1 & 0 \\ 0 & s+\frac{1}{2} \end{bmatrix}$ which is stable. Thus, according to Theorem - part 1) the problem is solvable, and, according to part 4), the resulting LQ optimal feedback stabilizes the plant, being the only plant hidden eigenvalue $\lambda = -\frac{1}{2}$.

We also find: $A_2 = [s-1]$; $B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; and, via spectral factorization, $E = [s+\sqrt{2}]$. Further: $A_2^* = [-s-1]$; $B_2^* = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $E^* = [-s+\sqrt{2}]$, which implies $\partial Z = 0$.

Eq. (13) and (21) [or (13) and (16)], give $X = 1$; $Y = [y_1 = \sqrt{(2)}+1 \quad y_2 = \frac{2(2\sqrt{2}-1)}{7}]$; $Z = [z_1 = \sqrt{2}+1 \quad z_2 = \frac{2(2\sqrt{2}-1)}{7}]$. Hence, $F = -[\sqrt{2}+1 \quad \frac{2(2\sqrt{2}-1)}{7}]$ and $\Phi + GF = \begin{bmatrix} -\sqrt{2} & -\frac{2(2\sqrt{2}-1)}{7} \\ 0 & -\frac{1}{2} \end{bmatrix}$. Eq. (16) alone yields $X = 1$, $y_1 = \sqrt{2}+1$, $z_1 = \sqrt{2}+1$. However, it does not yield any information on y_2 and z_2. □

Remark 2 - Eq. (27) is an interesting and somewhat unexpected equation in that it could be directly used to compute the LQ optimal feedback-gain matrix without solving the spectral factorization problem (10) except for the easy task of determining ∂E^*, provided that together with the condition $\partial Z < \partial E^*$ it yields only a single stabilizing F among all its solutions whenever the GCLD's of A and B are stable. This can be admittedly advanced at this time only as conjecture. □

Example 2 - Consider again the LQ output regulation problem of Example 1. For this case, (27) yields for $\partial Z = 0$ two solutions (Z, F). The first coincides with the one found in Ex. 1. The second one is given by $Z = [\sqrt{2}-1 \quad \frac{2\sqrt{2}-5}{2}]$ and $F = [\sqrt{2}-1 \quad \frac{2\sqrt{2}-5}{2}]$. Note that the latter yields $\Phi + GF = \begin{bmatrix} 2-\sqrt{2} & -\frac{2\sqrt{2}-5}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}$ and, hence, an unstable closed-loop system. □

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