Proceedings of the Fourth Conference on Differential Equations and Applications, Rousse, 1989

THE APPLICATION OF TOPOLOGICAL METHODS TO THE PREDICTION OF PERIODIC OSCILLATIONS IN SYSTEMS WITH DISCONTINUITIES AND HYSTERESIS

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Abstract: We present a rigorous justification of the method of harmonic balance for systems with discontinuities and hysteresis. We use the Schauder fixed point theorem for multifunctions, followed by a degree-preserving homotopy on a multivalued vector field. The hypotheses needed for this abstract approach are very similar to the assumptions made in engineering practice.

1. INTRODUCTION

The method of harmonic balance is a useful method for predicting the existence of periodic oscillations in nonlinear systems which are low-pass filters. The determination of precise conditions under which the method is valid has been an area of considerable successful research in recent years ([14],[16],[17],[18]).

For systems with discontinuities and systems with hysteresis, there are special problems in justifying the method, however there has been some success ([9],[10], [15],[18],[19],[20],[21],[22]).

We present a technique which reduces the justification of the method to the question of justifying a homotopy between a finite-dimensional approximation (the harmonic balance equation) and the original problem. By using multifunctions we are able to give a unified treatment which automatically includes discontinuous problems. For systems with hysteresis (continuous or discontinuous) we use our techniques to justify the method of harmonic balance under certain conditions. Our conditions, necessitated in large part by a need for semi-continuity of a certain multifunction, turn out to be quite similar, in cases when comparisons are possible, to those required by less abstract approaches.

Research of JWM supported in part by the Natural Sciences and Engineering Research Council of Canada under grant NRC-A-3053.

Research of PN and PZ supported in part by the research project M.P.I. 40%.

2. THE ABSTRACT FORMULATION

We assume that the system is modelled by a differential inclusion

$$\dot{z}(t) - Az(t) \in F[z(t)], \qquad t \in [0,\infty) \tag{1}$$

where $z(t) \in R^m$, A is a constant $m \times m$ matrix and $F : R^m \to 2^{R^m}$. For example, for a system modelled by a scalar m^{th} order autonomous equation $L_m[y] = f(y(t))$, where f may have jump discontinuities, we convert this to a system of ordinary differential equations in the usual way, setting $z(t) = (y(t),\dots,y^{m-1}(t))$, and then replace the vector $(0,0,\dots,0,f(y(t)))$ by the multifunction $F[z(t)]$, defined as $\{(0,\dots,0,f(y(t)))\}$ where F is continuous, and as $\{(0,\dots,0,v) \mid f_-(y(t)) \le v \le f_+(y(t))\}$ at the discontinuities.

Let $W^{1,2}([0,2\pi],R^m)$ be the usual Sobolev space of functions from $[0,2\pi]$ into R^m, and let Z be the subspace of functions $x(\cdot)$ which satisfy the boundary conditions $x(0) = x(2\pi)$, extended to R by 2π-periodicity. We note that $W^{1,2}([0,2\pi],R^m)$ is the set of absolutely continuous functions with derivative in $L^2((0,2\pi),R^m)$. We will denote by X the Hilbert space of functions in $L^2((0,2\pi),R^m)$ extended to R by 2π-periodicity.

For each $x(\cdot) \in X$ the associated Fourier expansion $x(t) \to a_0 + \sum_1^\infty (a_k e^{ikt} + a_{-k}e^{-ikt})$, converges in norm to $x(\cdot)$, and we can define the projection operator $P_n : x(t) \to a_0 + \sum_1^n (a_k e^{ikt} + a_{-k}e^{-ikt}) \equiv P_n[x](t)$, $a_{-k} = \bar{a}_k$. Since the period $T = 2\pi/\omega$ of the sought-after solution of (1) is unknown, we scale (1) so as to fix the period at 2π. If we define $x(t) = z(2\pi t/T)$, then $z(\cdot)$ is a T-periodic solution of (1) if and only if $x(\cdot)$ is a 2π-periodic solution of

$$\omega\dot{x}(t) - Ax(t) \in \mathcal{F}[x](t), \qquad 0 \le t < \infty, \tag{2}$$

where $\mathcal{F}$ denotes the Nemytskii operator, defined as all measurable selections $y(\cdot)$ from $\{F[x(t)] \mid 0 \le t < \infty\}$ which satisfy $y(t) = y(t+2\pi)$ for almost all

$t \in [0,2\pi]$. As usual, a solution of (2) is a function $x \in Z$ which satisfies (2) a.e. for some $\omega > 0$. We will assume that the operator $[\omega \frac{d}{dt} - A] : Z \to X$ is invertible, i.e., we avoid resonance. Then we can consider the inverse $G_\omega : X \to Z$:

$$G_\omega[y](t) = \int_0^{2\pi} G_\omega(t-s)y(s)ds ,$$

$$G_\omega(t-s) = \frac{1}{\omega}[I - e^{2\pi A/\omega}]^{-1} \begin{cases} e^{A(t-s)/\omega}, & 0 \le s < t , \\ e^{A(2\pi+t-s)/\omega}, & t \le s \le 2\pi . \end{cases} \tag{3}$$

Now the problem of finding a solution of (2) can be rewritten as a fixed point problem in the space X. In fact, set $T_\omega = i \circ G_\omega \circ F : X \to 2^X$, where i is the natural imbedding of Z into X, and consider the equation

$$0 \in [I - T_\omega][x]. \tag{4}$$

It is clear that $x \in X$ solves (4) if and only if $x \in Z$ and solves (1).

Our original inclusion (1) has now been replaced by an equivalent *family* of inclusions (4), defined on X and parametrized by ω. If we define $X_n = P_n X \equiv \{P_n x| \ x \in X\}$, $X^* = (I-P_n)X$, $x_n(t) = P_n[x](t)$, $x^*(t) = (I-P_n)[x](t)$ for $x \in X$, then we can write a finite-dimensional approximation to (4)

$$0_n \in (I - P_n T_\omega)[x_n], \quad x_n \in X_n, \quad 0_n = P_n[0]. \tag{5}$$

To compare this approximation with the exact problem, we separate (4) into finite and infinite-dimensional parts by applying P_n and $I - P_n$ respectively:

$$\text{(a)} \quad 0_n \in P_n(I - T_\omega)[x_n + x^*] , \qquad \text{(b)} \quad 0^* \in (I - P_n)(I - T_\omega)[x_n + x^*] . \tag{6}$$

We can then construct the following homotopy

$$\begin{aligned} &\text{(a)} \quad 0_n \in (I - P_n T_\omega)[x_n] + \lambda\{P_n(I - T_\omega) - (I - P_n T_\omega)P_n\}[x] , \\ &\text{(b)} \quad 0^* \in (I - P_n)(I - \lambda T_\omega)[x], \quad \text{for } x \in X , \quad 0 \le \lambda \le 1 . \end{aligned} \tag{6_λ}$$

When $\lambda = 0$, this reduces to the equation of harmonic balance (5), when $\lambda = 1$ we have an inclusion which will have a solution if and only if the original problem (1) has a solution. Unfortunately, the system (5) does not have isolated solutions

since our system (1) is autonomous, if $(a_0, a_1, \ldots, a_n)$ satisfies (5) , for some $\omega > 0$, then also $(a_0, a_1 e^{i\theta} \ldots a_n e^{i\theta})$ for arbitrary real θ satisfies (5). Therefore we have to fix the time origin; one way to do this is to add to (5) the condition that a nonzero component $a_{i_0 j_0}$ of some a_{i_0}, $1 \le i_0 \le n$, be real, $\arg a_{i_0 j_0} = 0$. This condition implies the choice of a particular solution of (5).

We do this as follows. For any $\lambda \in [0,1]$, let $V_\lambda : R_+ \times X_n \times X^* \to R \times 2^{X_n} \times 2^{X^*}$ be the operator defined by $V_\lambda = (V_\lambda^a, V_\lambda^b)$, where $V_\lambda^a : R_+ \times X_n \times X^* \to R \times X_n$ and $V_\lambda^b : R_+ \times X_n \times X^* \to X^*$ are given by

$$\begin{aligned} V_\lambda^a[\omega, x_n x^*] = (&\arg a_{i_0 j_0}(\omega, x_n), (I - P_n T_\omega)[x_n] + \\ &+ \lambda\{P_n(I - T_\omega) - (I - P_n T_\omega)P_n\}[x_n + x^*]) , \end{aligned}$$

$$V_\lambda^b[\omega, x_n, x^*] = (I - P_n)(I - \lambda T_\omega)[x_n + x^*].$$

In the sequel we will denote by (H_λ) the following equations:

$$\text{(a)} \quad 0 \in V_\lambda^a[\omega, x_n, x^*], \qquad \text{(b)} \quad 0 \in V_\lambda^b[\omega, x_n, x^*] . \tag{H_λ}$$

3. THE ABSTRACT THEOREM

We make the following assumptions. Here, $|x|_2$ denotes the $L^2((0,2\pi), R^m)$ norm.

(A1) $F : R^m \to 2^{R^m}$ in (1) is upper semi-continuous and satisfies

$$\sup_{y \in F(x)} |y| \equiv |F(x)| \le \alpha|x| + \beta \quad \text{for some } \alpha > 0, \text{ and } \beta \ge 0.$$

In addition, $0 \in F(0)$ and $F(x)$ is a nonempty convex compact set for all $x \in R^m$.

(A2) Let $A_n \subset R_+ \times X_n$ be the open set satisfying the following assumptions

(b_1) For all $(\omega, x_n) \in A_n$, there exists $r_1 = r_1(\omega, x_n) > 0$ such that

$$[\sum_{|k|>n} |\hat{G}_\omega(ik)|^2]^{1/2} \sup_{|x^*|_2 < r_1} \ \sup_{y \in F[x_n + x^*]} |(I - P_m)y|_2 < r_1 ,$$

where $\hat{G}_\omega(ik)$ is the Fourier transform of the matrix $G_\omega(t)$ evaluated at

ik, $k = \pm(n+1), \pm(n+2), \ldots$

(h_2) For all $(\omega, x_n) \in A_n$ we have $|(I - P_n T_\omega)[x_n]|_2 < \sigma(\omega, x_n)$ where

$$0 < \sigma(\omega, x_n) = \Big[\sum_{|k|=0} |\hat{G}_\omega(ik)|^2\Big]^{1/2} \sup_{|x^*|_2 < r_1} |P_n F[x_n + x^*] - P_n F[x_n]|_2$$

and equality holds on the boundary of A_n.

The term $|\hat{G}_\omega(ik)|^2$ is the sum of the squares of the entries, and the last "norm" on the right is defined by the usual convention:

$$\sup \{|z - z_n|_2 : z \in P_n F[x_n + x^*],\ z_n \in P_n F[x_n]\}.$$

Note that $(I - P_n T_\omega)[x_n]$ is set-valued. Its norm is defined by the usual convention. Let Ω_n be the connected component of A_n containing the solution $(\bar{\omega}, \bar{x}_n)$ of the harmonic balance equation (5) for which $\arg a_{i_0 j_0}(\bar{\omega}, \bar{x}_n) = 0$. Assume that

(A3) (i) $I - e^{2\pi A/\omega}$ is invertible whenever ω is such that $(\omega, x_n) \in \Omega_n$ for some x_n;

(ii) $(\omega, 0) \notin \bar{\Omega}_n$ for any $\omega \in R_+$. Moreover, the function $(\omega, x_n) \to a_{i_0 j_0}(\omega, x_n)$ has real part different from zero in Ω_n. (This implies that $\arg a_{i_0 j_0}(\omega, x_n)$ is a continuous function in Ω_n).

(iii) $\deg(V_0^a, \Omega_n, 0)$ is well defined and different from zero.

<u>Theorem</u>

Under Assumptions (A1), (A2), (A3), *the inclusion* (4) *will have a solution* (ω, x), *where* $x = x_n + x^*$, $(\omega, x_n) \in \bar{\Omega}_n$ *and* $|x^*|_2 \le r_1(\omega, x_n)$.

<u>Remark 1</u>: The conditions on F in (A1) ensure that the Nemytskii operator F maps X into 2^X and is closed-convex-valued and bounded (i.e., the image of a bounded set is a bounded set) (Lasry-Robert [11]). Moreover the imbedding i of Z into X is compact (i.e., it sends bounded sets into relatively compact sets). Therefore the operator $T_\omega[x] = i \circ G_\omega \circ F[x]$ is a closed-convex-valued upper semi-continuous; compact operator. Hence T_ω in the space X will meet the conditions for using the Schauder fixed point theorem (see Dugundji-Granas [4]) and Leray-Schauder degree theory for the set-valued compact vector field V_λ for any $\lambda \in [0,1]$ (see Cellina-Lasota [2]). In the sequel we will omit the imbedding map i in the notation of T_ω, keeping in mind that for any $x \in X$, $T_\omega[x] \subset X$ is a set of absolutely continuous, 2π-periodic functions with derivative in X.

<u>Remark 2</u>: The assumption (A3)(i) is not essential, since any given system can be "pole-shifted" to an equivalent system satisfying (A3)(i).

<u>Remark 3</u>: Assumption (A3)(ii) guarantees that the trivial solutions is excluded when we apply a fixed point theorem or degree theory and that $\arg a_{i_0 j_0}$ is a continuous function in Ω_n. Let $\Omega = \{(\omega, x_n, x^*) \in R_+ \times X_n \times X : (\omega, x_n) \in \Omega_n,\ |x^*|_2 < r_1(\omega, x_n)\}$. If $0 \notin V_\lambda(\partial\Omega)$ for any $\lambda \in [0,1)$ (i.e. the homotopy (H_λ) is admissible in $\bar{\Omega}$) then assumption (A3)(iii) and the homotopy invariance property of the topological degree guarantee that $\deg(V_\lambda, \Omega, 0) = \deg(V_0^a, \Omega_n, 0) \neq 0$ for any $\lambda \in [0,1)$. Hence the conclusion of the Theorem will follow from the solution property of topological degree.

We remark, however, that the equations (6_λ), and (4) are not entirely equivalent, since they involve algebraic operations on sets (e.g. $A - A \neq \{0\}$ in general). On the other hand, it is easy to show that any one of them has the relevant degree nonzero if and only if the others have the same property: we refer the reader to [14] for details.

<u>Remark 4</u>: Assumption (A2)(h_1) is the "low-pass filter" assumption (Mees and Bergen [17]), i.e., it implies that the linear part of the system as represented by G_ω attenuates high frequencies.

<u>Remark 5</u>: The region Ω_n, in the scalar case, can often be determined by graphical and numerical techniques, using the fact that equality holds in (A2)(h_2) for $(\omega, x_n) \in \partial\Omega_n$.

<u>Remark 6</u>: The determination of the solution $(\bar{\omega}, \bar{x}_n)$ to $V_0^a = 0$ and the verification of (A3)(iii) is a finite-dimensional "algebraic" problem. If $x_n(t) \sim a_0 + \sum_{|k|=1}^{n} a_k e^{ikt}$, the a_k's unknown, then in many cases we can explicitly compute the Fourier expansion of $G_\omega F[x_n] \sim c_0 + \sum_{|k| \ge 1} c_k e^{ikt}$, i.e., we can explicitly determine $c_k(\omega, a_0, a_1, a_{-1}, \ldots, a_n, a_{-n})$, keeping in mind that c_k is set-valued. Then $V_0^a = 0$ reduces to

$$0 \in a_j - c_j(\omega, a_0, a_1, \ldots, a_n, a_{-n}) \qquad j = 0, \pm 1, \ldots, \pm n,$$

with the condition that $\arg a_{i_0 j_0}(\bar{\omega}, \bar{x}_n) = 0$.

<u>Proof of Theorem 1</u>: First, for each $(\omega, x_n) \in \tilde{\Omega}_n$ and each $0 \le \lambda \le 1$ we show that we can apply the Schauder fixed point theorem for set-valued maps in the ball $B(0,r_1)$ to get a solution $x^*_\lambda(\omega,x_n)$ of $H_\lambda(b)$. Thus $H_\lambda(a)$ becomes an equation in ω and x_n, when we replace x^* by $x^*_\lambda(\omega,x_n)$. We then show that the original system (4) has a solution by showing that H_λ represents an admissible homotopy in $\bar{\Omega} = \{(\omega,x_n,x) \in R'_+ \times X_n \times X^* : (\omega,x_n) \in \Omega_n, |x^*|_2 \le r_1(\omega,x_n)\}$.

We turn to the first stage. It is clear that for a given (ω,x_n) in Ω_n, $H_\lambda(b)$ represents a fixed-point problem $x^* \in M_\lambda[x^*]$ for the map $M_\lambda : x^* \to \lambda(I-P_n)T_\omega[x^*+x_n]$ on $(I-P_n)X$. If $|x^*|_2 < r_1$, then (recalling that for a multifunction $H[z], |H[z]| = \sup_{y\in H(z)} |y|$):

$$|M_\lambda[x^*]|_2 \equiv |\lambda(I-P_n)T_\omega[x^*+x_n]| \le |G_\omega(I-P_n)F[x^*+x_n]|_2$$

$$\le \{\sum_{|k|>n} |\hat{G}_\omega(ik)|^2\}^{1/2} \sup_{|x^*|_2\le r_1} \sup_{y\in F[x^*+x_n]} |(I-P_n)[y]|_2 < r_1$$

by (A2)(h_1), so $M_\lambda : B(0,r_1) \to 2^{B(0,r_1)}$ in $(I-P_n)X$. Now the Schauder fixed-point theorem for multivalued maps (Dugundji-Granas [4], p. 96) is valid by Assumption (A1) for M_λ for any $\lambda \in [0,1]$ (Lasry-Robert [11], p. 60). Therefore, it has a fixed point $x^*_\lambda(\omega,x_n)$ for each $(\omega,x_n) \in \Omega_n$, $0 \le \lambda \le 1$.

Under our assumptions the topological degree of the compact-convex-valued vector field $V_\lambda : (\omega,x) \to V_\lambda[(\omega,x)]$ on Ω is defined for $\lambda \in [0,1]$ ([2],[12],[13]). We can now show that $0 \notin V_\lambda[(\omega,x)]$ for $(\omega,x) \in \partial\Omega$, $0 \le \lambda \le 1$, thus $0 \ne \deg(V^a_0,\Omega_n,0_n) = \deg(V_\lambda,\Omega,0)$ for $\lambda \in [0,1)$, which implies the existence of a nontrivial $2\pi/\omega$ periodic solution of (1). Suppose there existed $(\omega,x_n,x^*) \in \partial\Omega$ such that $0 \in V_\lambda[\omega,x_n,x^*]$ for some $\lambda \in [0,1)$ where x^* stands for $x^*_\lambda(\omega,x_n)$. Then, we would have selections $y_n \in F[x_n]$ and $(y,\bar{y}) \in F[x]$ such that at least one of the following inequalities holds as equality:

$$\text{(a)}\quad |x_n - P_nG_\omega y_n|_2 \le \sigma(\omega,x_n), \qquad \text{(b)}\quad |x^*|_2 \le r_1(\omega,x_n) \tag{7}$$

and for which

$$\text{(a)}\quad 0_n = (x_n - P_nG_\omega y_n) + \lambda[(x_n - P_nG_\omega y) - (x_n - P_nG_\omega y_n)], \tag{8a}$$

$$\text{(b)}\quad 0^* = (I-P_n)(x - \lambda G_\omega \bar{y}) \tag{8b}$$

Equations (8a)-(8b) imply that $0 \ge |x_n - P_nG_\omega y_n|_2 - \lambda|P_nG_\omega(y-y_n)|_2$, $0 \ge |x^*|_2 - \lambda|(I-P_n)G_\omega\bar{y}|_2$ respectively. From the inequalities we obtain

$$\text{(a)}\quad 0 \ge |x_n - P_nG_\omega y_n|_2 - \lambda\sigma(\omega,x_n), \qquad \text{(b)}\quad 0 \ge |x^*|_2 - \lambda r_1(\omega,x_n) \tag{9}$$

where the estimates on the right hand side of (9a) and (9b) are obtained by the usual Fourier expansion techniques, i.e.

$$|P_nG_\omega(y-y_n)|_2 \le [\sum_{|k|=0}^{n} |\hat{G}_\omega(ik)|^2]^{1/2}|y-y_n|_2 ,$$

$$|(I-P_n)G_\omega\bar{y}|_2 \le [\sum_{|k|\ge n+1} |\hat{G}_\omega(ik)|^2]^{1/2}|\bar{y}|_2 ,$$

and by using (A2)(h_2) and (A2)(h_1) respectively. But $\lambda \in [0,1)$, hence from (9a)-(9b) we obtain $0 > |x_n - P_nG_\omega y_n|_2 - \sigma(\omega,x_n)$, $0 > |x^*|_2 - r_1(\omega,x_n)$. Therefore, neither (7a) nor (7b) can reach equality, contradicting the fact that $(\omega,x_n,x^*) \in \partial\Omega$ for some $\lambda \in [0,1)$.

4. <u>SYSTEMS WITH HYSTERESIS</u>

For a system with hysteresis we consider the case of an m^{th}-order scalar differential equation. In a typical relay - hysteresis model, the nonlinear term in the original scalar equation (1) has discontinuous jumps when $y(\cdot)$ crosses certain threshhold values. The hysteresis nonlinearity has a <u>memory</u>, which makes the $F[x(\cdot)]$ a <u>functional</u> multifunction, mapping an "input" $x(\cdot)$ from say $X \equiv C[0,2\pi]$ (with sup norm $\|x\|$) into a selection of functions from $L^\infty(0,2\pi)$. For thorough discussions of the modelling of hysteresis, we refer the reader to [3],[5],[6],[7],[8],[9],[10],[15],[19],[21],[22].

We can only briefly describe our theorem. We assume that (A2) and (A3) hold. The use of $C([0,2\pi],R)$ in place of $L^2(0,2\pi)$ introduces the factor $\sqrt{2\pi}$ into the left side of (A2)(h_1) and into the definition of σ in (A2)(h_2) (note that $\|x\|\sqrt{2\pi} \ge |x|_2$). In place of (A1) we assume that if the m^{th}-component of $x(t)$, $x_m(t) \equiv y(t)$, then $F[x(t)] = (0,\dots,f(y(t)))$, where for some given threshholds $0 < \alpha < \beta$,

$$f(y(t)) = \begin{cases} h_0(y(t)) & \text{if } y(t) \geq \beta; \\ h_c(y(t)) & \text{if } y(t) < \alpha; \\ h_0(y(t)) & \text{if } y(t) \in (\alpha,\beta) \text{ and } \tau(t) = \beta; \\ h_c(y(t)) & \text{if } y(t) \in (\alpha,\beta) \text{ and } \tau(t) = \alpha; \end{cases}$$

where $\tau(t) = \sup\{s | s \leq t,\ y(t) = \alpha \text{ or } \beta\}$ with $h_0(u)$ and $h_c(u)$ both continuous and satisfying $|h(u)| \leq A|u| + B$ for some $A > 0$, $B \geq 0$. We assume that there is a solution $(\bar{\omega}, \bar{x}_n)$ of the harmonic balance equation such that if $\bar{y}(t) \equiv (\bar{x}_n(t))_m$, the m^{th} component of $\bar{x}_n(t)$, then

(a) $\inf_{[0,2\pi]} \bar{y}(t) < \alpha < \beta < \sup_{[0,2\pi]} \bar{y}(t)$, (b) $\forall t_i$ such that $\bar{y}(t_i) = \alpha$ or β, we have $\bar{y}'(t_i) \neq 0$.

Finally, we must assume that there is a closed ball $clB(\bar{x}_n, \varepsilon_0)$ in X such that if we define $\Omega = \{(\omega, x_n, x^*) | (\omega, x_n) \in \Omega_n, \|x^*\| > r_1\}$ then

(c) $\Omega \subset R_+ \times clB(\bar{x}_n, \varepsilon_0)$, (d) $x \in clB(\bar{x}_n, \varepsilon_0) \Rightarrow y(t) \equiv x_m(t)$ satisfies (a) and and (b) above.

These assumptions postulate the existence of what Braverman et al. [1] call a "motion of the third type"; a similar assumption is made in [20].

Under the above assumptions we can define an appropriate set-valued multifunctional $F[x]$ from $C([0,2\pi],R)$ to $L^\infty(0,2\pi)$ and use Theorem 1 to assert the existence of a nontrivial periodic solution of (1). The multifunctional is not in general well-behaved on all of $C([0,2\pi],R)$, but it is well-behaved on clB.

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