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Periodic Solutions of a Control Problem Via Marginal Maps (*).

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Summary. – *We investigate the existence of periodic solutions to the control problem*

$$\dot{x} = f(t, x, u) + g(t), \quad x \in \mathbf{R}^n, \ u \in \mathbf{R}^m, \tag{1}$$

with g and f periodic in t with period 1. We form the associated quantities

$$s(t, x) = \sup_{u \in \Omega} (x, f(t, x, u)), \quad i(t, x) = \inf_{u \in \Omega} (x, f(t, x, u))$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbf{R}^n$ and Ω is a nonempty compact set in $\mathbf{R}^n$. If $u_s(t, x)$, $u_i(t, x)$ denote the (in general multivalued) controls for which $s(t, x)$, $i(t, x)$ are respectively attained, then we can form the family of marginal problems

$$\dot{x} \in \lambda(t)\, \overline{\text{co}}\, f(t, x, u_s(t, x)) + (1 - \lambda(t))\, \overline{\text{co}}\, f(t, x, u_i(t, x)) + g(t), \ \lambda(\cdot) \in L^\infty([0, 1], [0, 1]). \tag{2}$$

We give sufficient conditions for the existence of a periodic solution of certain marginal problems, stated in terms of $\liminf_{|x|\to\infty}$ and $\limsup_{|x|\to\infty}$ of $s(t, x)/|x|^2$ and $i(t, x)/|x|^2$. Finally we state the relationship between the periodic solutions of the marginal problems and those of the original problem (1).

Introduction.

In this paper we give conditions for the existence of periodic solutions to the time-dependent control problem (1). In order to get the result we, first, associate to (1) a family of multivalued problems (2). Then we prove the existence of periodic solutions for (2) giving conditions formulated in terms of the asymptotic behaviour of certain maps corresponding to the choice of controls $u(\cdot)$ as the marginal maps of the scalar product $(x, f(t, x, u))$. The convexity assumption on the multivalued map $f(t, x, \Omega)$, with $u(t) \in \Omega \subset \mathbf{R}^m$, allows us to deduce the existence of periodic solutions to (1).

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In [5] we gave sufficient conditions for the existence of periodic solutions to differential inclusions of the form $[\dot{x} - A(t)x] \in F(t, x)$. However those conditions in fact ensure that for any measurable selection $z(x, t) \in F(t, x)$, there is a periodic solution of $[\dot{x} - A(t, x)] = z(x, t)$. This type of result is useful for modelling problems with discontinuities, but not for modelling control problems. The same comments apply to the results of Nistri in [6], where he modelled *optimal* control problems. The approach in [5] is suitable for showing the existence of periodic solution in the presence of discontinuities or in the case of optimal periodic control problems, but it is not appropriate when we are looking for the existence of a control law which produces a periodic solution.

Finally, we give a local coercive-type condition for the stability of such periodic solutions.

1. – Existence of periodic solutions.

We consider the following control process

$$\dot{x} = f(t, x, u) + g(t), \quad 0 \leqslant t \leqslant 1. \tag{1}$$

We assume

i) $f: [0, 1] \times \mathbf{R}^n \times \Omega \to \mathbf{R}^n$, Ω a nonempty compact set in R^m, satisfies the Carathéodory condition, that is $t \mapsto f(t, x, u)$ is measurable, $(x, u) \mapsto f(t, x, u)$ is continuous.

ii) $|f(t, x, u)| \leqslant a(t)|x| + b(t)$, $\forall x \in \mathbf{R}^n$, $\forall u \in \Omega$, a.e. in $[0, 1]$; $a(\cdot), b(\cdot) \in L^1([0, 1], \mathbf{R}_+)$;

iii) $g \in L^1([0, 1], \mathbf{R}^n)$.

iv) $u(\cdot) \in U = \{u(\cdot) \in L^\infty([0, 1], \mathbf{R}^m) | u(t) \in \Omega \text{ a.e. in } [0, 1]\}$.

v) $f(t, x, \Omega)$ is a compact, convex set for all $x \in \mathbf{R}^n$ and for a.a. $t \in [0, 1]$.

In what follows the functions f, g, a, b, u, are extended to $\mathbf{R}$ by 1-periodicity.

For all $x \in \mathbf{R}^n$ and almost all $t \in [0, 1]$ we define $s(t, x) = \sup_{u \in \Omega} (x, f(t, x, u))$ and $i(t, x) = \inf_{u \in \Omega} (x, f(t, x, u))$, where $(\cdot, \cdot)$ denotes the scalar product in $\mathbf{R}^n$. We denote by $u_s = u_s(t, x)$ and $u_i = u_i(t, x)$ the set of values of the control variable u such that

$$s(t, x) = (x, f(t, x, u_s)), \quad i(t, x) = (x, f(t, x, u_i)).$$

We note that $(t, x) \mapsto s(t, x)$, $(t, x) \mapsto i(t, x)$ are measurable in t and continuous in x; $(t, x) \mapsto u_s(t, x)$, $(t, x) \mapsto u_i(t, x)$ are t-measurable, x-upper semicontinuous, multivalued maps with compact values, (see [1], Th. 6, pag. 53). Thus the two maps

$$(t, x) \mapsto f(t, x, u_s(t, x)) = f_s(t, x) \quad \text{and} \quad (t, x) \mapsto f(t, x, u_i(t, x)) = f_i(t, x)$$

are t-measurable, x-upper semicontinuous multivalued maps, with compact values.

Let us consider now the maps $\alpha_i(\cdot)$, $\alpha_s(\cdot)$, $\beta_i(\cdot)$, $\beta_s(\cdot) \in L^1([0,1], \boldsymbol{R})$ defined by:

$$\alpha_i(t) = \liminf_{|x|\to\infty} \frac{s(t,x)}{|x|^2}, \qquad \alpha_s(t) = \limsup_{|x|\to\infty} \frac{s(t,x)}{|x|^2}$$

$$\beta_i(t) = \liminf_{|x|\to\infty} \frac{i(t,x)}{|x|^2}, \qquad \beta_s(t) = \limsup_{|x|\to\infty} \frac{i(t,x)}{|x|^2}.$$

Noting that the two maps f_s, f_i have compact values and $\overline{\mathrm{co}}\, f_s = \mathrm{co}\, f_s$ and $\overline{\mathrm{co}}\, f_i = \mathrm{co}\, f_i$, we form the marginal differential inclusion

$$\dot{x} \in \lambda(t)\, \mathrm{co}\, f_s(t,x) + \big(1-\lambda(t)\big)\, \mathrm{co}\, f_i(t,x) + g(t) \tag{2}$$

for fixed but arbitrary $\lambda(\cdot) \in L^\infty([0,1],[0,1])$. The right side of this inclusion is a t-measurable, x-upper semicontinuous multivalued map with convex, compact values.

We observe that under our assumptions, for each compact set $Q \subset \boldsymbol{R}^n$, there exist $\gamma_s(\cdot)$ and $\gamma_i(\cdot) \in L^1([0,1], \boldsymbol{R}_+)$ such that

$$|f_s(t,x)| \leqslant \gamma_s(t), \qquad x \in Q, \qquad \text{a.e. in } [0,1];$$

$$|f_i(t,x)| \leqslant \gamma_i(t), \qquad x \in Q, \qquad \text{a.e. in } [0,1].$$

Finally, we assume that at least one of the following holds:

$$\int_0^1 \alpha_i(t)\, dt > 0, \tag{$A+$}$$

or

$$\int_0^1 \beta_s(t)\, dt < 0. \tag{$A-$}$$

THEOREM 1. – *If assumptions* (i)-(iv) *and either* ($A+$) *or* ($A-$) *holds, then* (2) *has a* 1*-periodic solution for any* $\lambda(\cdot)$ *satisfying, respectively*

$$\int_0^1 \big[\lambda(t)\,\alpha_i(t) + \big(1-\lambda(t)\big)\,\beta_i(t)\big]\, dt > 0 \qquad (\textit{under } (A+)), \tag{$+$}$$

or

$$\int_0^1 \big[\lambda(t)\alpha_s(t) + \big(1-\lambda(t)\big)\,\beta_s(t)\big]\, dt < 0 \qquad (\textit{under } (A-)). \tag{$-$}$$

PROOF. – We write $L^1 = L^1([0,1], \boldsymbol{R}^n)$ and $AC = AC([0,1], \boldsymbol{R}^n)$. For a fixed $\lambda(\cdot) \in L^\infty([0,1],[0,1])$ let $\mathcal{G}_\lambda: L^1 \to 2^{L^1}$ be the map defined by

$$\mathcal{G}_\lambda[x](t) = \{y(t) \in \lambda(t)\ \mathrm{co}\, f_s(t, x(t)) + (1-\lambda(t))\ \mathrm{co}\, f_i(t, x(t)),\ y(\cdot) \in L^1\}\,.$$

From our assumptions on f and the definitions of $\mathrm{co}\, f_s$, $\mathrm{co}\, f_i$ we see that for any such $\lambda(\cdot)$, $\mathcal{G}_\lambda$ is a u.s.c. multifunction with convex closed values (see [3]). Moreover, $\mathcal{G}_\lambda: AC \to 2^{L^1}$, and the composition of $\mathcal{G}_\lambda$ with the compact imbedding i of AC into L^1 is compact (that is, $\mathcal{G}_\lambda \circ i$ maps any bounded set into a relatively compact set). For simplicity we will write $\mathcal{G}_\lambda$ in place of $\mathcal{G}_\lambda \circ i$.

We define $\Delta = AC \cap \{x(\cdot)|x(0) = x(1)\}$, and $\mathcal{D}: \Delta \to L^1$ by $\mathcal{D}: x(\cdot) \to dx/dt(\cdot)$. Clearly $\mathcal{D}$ is a linear Fredholm operator of index zero with

$$\mathrm{Ker}\,(\mathcal{D}) = \Delta \cap \{x(\cdot)|x(t) \equiv c \in \boldsymbol{R}^n \text{ on } [0,1]\}\,,$$

$$\mathrm{Im}\,(\mathcal{D}) = \left\{y(\cdot) \in L^1 : \int_0^1 y(t)\,dt = 0\right\}.$$

If $P: L^1 \to L^1$ denotes the projection operator defined by $P: y(t) \to y(t) - \int_0^1 y(s)\,ds$, we consider the system

$$\begin{cases} 0 \in (I-P)\{\mathcal{G}_\lambda(x_1 + x^*) + g\} \\ x^* \in \mathcal{D}^{-1}P\{\mathcal{G}_\lambda(x_1 + x^*) + g\} \end{cases}$$

where

$$x = (I-P)x + Px = x_1 + x^*$$

for all $x \in L^1$. Clearly the operator $((I-P)\mathcal{G}_\lambda(\cdot), \mathcal{D}^{-1}P\mathcal{G}_\lambda(\cdot)): AC \to 2^{AC}$ is u.s.c. and compact, with closed, convex values, for any λ. Therefore we can use the theory of the topological degree for such operators (see [2], [4]).

We consider the homotopy

$$(H_\mu) \qquad \begin{cases} 0 \in (I-P)\{\mathcal{G}_\lambda(x_1 + x^*) + g\}\,, \\ x^* \in \mu\mathcal{D}^{-1}P\{\mathcal{G}_\lambda(x_1 + x^*) + g\}\,, \qquad \mu \in [0,1]\,. \end{cases}$$

This homotopy is called admissible if there exists an open bounded set $\Phi \subset AC$ such that $\{x|x$ is a solution of (H_μ) for some $\mu \in [0,1]\} \cap \partial\Phi = \emptyset$. If we can show that (*a*) (H_μ) is admissible, and (*b*) $\deg((I-P)\mathcal{G}_\lambda, \Phi \cap \mathrm{Ker}\,(\mathcal{D}), 0) \neq 0$ for some fixed $\lambda(\cdot) \in L^\infty([0,1],[0,1])$, then we can conclude that for this $\lambda(\cdot)$ the inclusion (2) has a periodic solution, since topological degree associated with each of (H_0), (H_1), (2) will be the same.

Assume for definiteness that $(A+)$ holds. We begin by proving the admissibility of (H_μ). Assume the contrary, that is, that there exist sequences $\{\mu_n\}_{n\in N} \subset (0,1]$ (the case $\mu = 0$ will be considered later), and $\{x_n\}_{n\in N}$ with $\max_{t\in[0,1]} |x_n(t)| \to +\infty$ and

$x_n(0) = x_n(1)$ (observe that by virtue of hypothesis (ii)) on f the boundedness of $\{x_n\}_{n\in\boldsymbol{N}}$ in C implies the boundedness of $\{x_n\}_{n\in\boldsymbol{N}}$ in AC with $\dot{x}_n(t) = \mu_n[\lambda(t)y_{s,n}(t) + (1-\lambda(t))y_{i,n}(t) + g(t)]$ a.e. in $[0, 1]$, for some selections $y_{s,n}(t) \in \operatorname{co} f_s(t, x_n(t))$ and $y_{i,n}(t) \in \operatorname{co} f_i(t, x_n(t))$, for $n \in \boldsymbol{N}$. It is easy to see that $\min\limits_{t\in[0,1]} |x_n(t)| \to +\infty$ as $n \to +\infty$ (see [5]), and also

$$\mu_n \int_0^1 \frac{\big(x_n(t), \lambda(t) y_{s,n}(t) + (1-\lambda(t)) y_{i,n}(t) + g(t)\big)}{|x_n(t)|^2}\, dt = 0$$

for all n large enough to guarantee $\min\limits_{[0,1]} |x_n(t)| > 0$. Since $\big(x_n(t), y_{s,n}(t)\big) = s\big(t, x_n(t)\big)$ and $\big(x_n(t), y_{i,n}(t)\big) = i\big(t, x_n(t)\big)$, we get:

$$\int_0^1 \left(\lambda(t)\frac{s\big(t, x_n(t)\big)}{|x_n(t)|^2} + (1-\lambda(t))\frac{i\big(t, x_n(t)\big)}{|x_n(t)|^2} + \frac{\big(x_n(t), g(t)\big)}{|x_n(t)|^2}\right) dt = 0\,.$$

Hence

$$0 \geqslant \int_0^1 \left(\lambda(t) \liminf_{n\to\infty} \frac{s\big(t, x_n(t)\big)}{|x_n(t)|^2} + (1-\lambda(t)) \liminf_{n\to\infty} \frac{i\big(t, x_n(t)\big)}{|x_n(t)|^2}\right) dt \geqslant$$

$$\geqslant \int_0^1 \big(\lambda(t)\alpha_i(t) + (1-\lambda(t))\beta_i(t)\big)\, dt,$$

so if $\lambda(\cdot)$ is such that this last expression is greater than zero we will get a contradiction.

In the case $\mu = 0$, we have $x^* = Px = 0$ which implies $x \equiv c \in \boldsymbol{R}^n$. Assume that there exists a sequence $\{c_n\}_{n\in\boldsymbol{N}} \subset \boldsymbol{R}^n$ such that $|c_n| \to +\infty$ satisfying (H_0). Then we would have

$$\int_0^1 \big(\lambda(t) z_{s,n}(t) + (1-\lambda(t)) z_{i,n}(t) + g(t)\big)\, dt = 0$$

for some $z_{s,n}(t) \in \operatorname{co} f_s(t, c_n)$, and $z_{i,n}(t) \in \operatorname{co} f_i(t, c_n)$, and

$$\int_0^1 \left(\lambda(t)\frac{(c_n, z_{s,n}(t))}{|c_n|^2} + (1-\lambda(t))\frac{(c_n, z_{i,n}(t))}{|c_n|^2} + \frac{(c_n, g(t))}{|c_n|^2}\right) dt = 0\,.$$

The argument above shows that this is impossible if

$$\int_0^1 \big(\lambda(t)\alpha_i(t) + (1-\lambda(t))\beta_i(t)\big)\, dt > 0\,.$$

Therefore, there exists a ball $B(0, r)$ in $C([0, 1], \boldsymbol{R}^n)$ containing in its interior all the solutions of (H_μ) for $\mu \in [0, 1]$. The hypotheses on f imply the exixtence of a ball $B(0, R) \subset AC$ with the same property.

Finally we prove that $\deg\big((I - P)\mathcal{G}_\lambda, B(0, R_1) \cap \operatorname{Ker}(\mathcal{D}), 0\big) \neq 0$, where $R_1 > R$. In fact, let us consider the homotopy

$$\nu(I - P)\{\mathcal{G}_\lambda(c\mathbb{1}) + g\} + (1 - \nu)c\mathbb{1}, \qquad \nu \in [0, 1]\,.$$

It is easy to see, using for example $(A\,+)$, that this is an admissible homotopy in $B(0, R_1) \cap \operatorname{Ker} \mathcal{D}$, when $\lambda(\cdot)$ satisfies the conditions of our Theorem. This concludes the proof under $(A\,+)$.

Under $(A\,-)$ we can give a dual argument to show that there exists a 1-periodic solution of (2) if $\lambda(\cdot)$ is such that

$$\int_0^1 \big(\lambda(t)\,\alpha_s(t) + (1 - \lambda(t))\,\beta_s(t)\big)\,dt < 0\,.$$

In this case the homotopy is:

$$\nu(I - P)\{\mathcal{G}_\lambda(c\mathbb{1}) + g\} - (1 - \nu)c\mathbb{1}\,, \qquad \nu \in [0, 1]\,. \qquad \square$$

REMARK 1. – Note that under $(A\,+)$, the condition $(+)$ is satisfied for $\lambda(t) = 1$; dually, under $(A\,-)$, the condition $(-)$ is satisfield for $\lambda(t) = 0$. For each of these cases the marginal equation (2) reduces to the «relaxed controls» equation associated with (1). In general, if $\lambda(t) \in \{0, 1\}$ a.e. on $[0, 1]$, then (2) can be written a.e. as

$$\dot{x} \in \operatorname{co} f\big(t, x, \lambda(t)\,u_s(t, x) + (1 - \lambda(t))\,u_i(t, x)\big) + g(t)\,.$$

REMARK 2. – Since $\alpha_i(t) \geqslant \beta_i(t)$ a.e. in $[0, 1]$ we see that if $\int_0^1 \alpha_i(t)\,dt \leqslant 0$ then $\int_0^1 \big(\lambda(t)\,\alpha_i(t) + (1 - \lambda(t))\,\beta_i(t)\big)\,dt \leqslant 0$ for any $\lambda(\cdot) \in L^\infty([0, 1], [0, 1])$. Moreover, since $\alpha_s(t) \geqslant \beta_s(t)$ a.e. in $[0, 1]$, if $\int_0^1 \beta_s(t)\,dt \geqslant 0$ then $\int_0^1 \big(\lambda(t)\,\alpha_s(t) + (1 - \lambda(t))\,\beta_s(t)\big)\,dt \geqslant 0$.

Therefore, we cannot weaken the hypotheses $(A\,+)$ and $(A\,-)$. This implies that among the functions $\lambda(\cdot)$ for which the system (2) has a 1-periodic solution there is always one solution with either $\lambda(t) = 1$ or $\lambda(t) = 0$ a.e. in $[0, 1]$.

REMARK 3. – In [6] one of us (Nistri) used the following assumptions:

$$\int_0^1 \alpha_s(t)\,dt < 0 \qquad \text{or} \qquad \int_0^1 \beta_i(t)\,dt > 0\,,$$

$f(t, x, \Omega) = \{f(t, x, u), u \in \Omega\}$ convex a.e. in $[0, 1]$, for $x \in \boldsymbol{R}^n$. Under these assumptions he proved that (1) has a 1-periodic solution for *any* control function $u \in U$. Observe that our conditions $(A+)$ and $(A-)$ in Theorem 1 imply that either $\int_0^1 \alpha_s(t)\,dt > 0$ or $\int_0^1 \beta_i(t)\,dt < 0$.

REMARK 4. – Note that if $\int_0^1 \alpha_s(t)\,dt \leqslant 0$ and $\int_0^1 \beta_i(t)\,dt \geqslant 0$, then since $\alpha_s(t) \geqslant \beta_i(t)$ a.e. in $[0, 1]$, $\int_0^1 \alpha_s(t)\,dt = \int_0^1 \beta_i(t)\,dt = 0$ and $\alpha_s(t) = \beta_i(t)$ a.e. in $[0, 1]$. Therefore in this case $\int_0^1 \big(\lambda(t)\,\alpha_s(t) + (1 - \lambda(t))\,\beta_i(t)\big)\,dt = 0$ for any $\lambda(\cdot)$ and our approach fails.

We now consider the case when both of the following limits exist.

vi) $$\lim_{|x| \to \infty} \frac{s(t, x)}{|x|^2} = \alpha(t) \text{ a.e. in } [0, 1].$$

vii) $$\lim_{|x| \to \infty} \frac{i(t, x)}{|x|^2} = \beta(t) \text{ a.e. in } [0, 1].$$

COROLLARY 1. – *Under assumptions* (i)-(iv), (vi)-(vii) *and either*

$(B+)$ $$\int_0^1 \alpha(t)\,dt > 0\,,$$

or

$(B-)$ $$\int_0^1 \beta(t)\,dt < 0\,,$$

(2) *has a* 1-*periodic solution for those* $\lambda(\cdot)$ *satisfying*

$$\int_0^1 \big(\lambda(t)\,\alpha(t) + (1 - \lambda(t))\,\beta(t)\big)\,dt \neq 0\,.$$

PROOF. – Suppose we have a sequence $\{x_n\}_{n \in \boldsymbol{N}} \subset AC$ of 1-periodic solutions of (2) with $\max_{t \in [0,1]} |x_n(t)| \to +\infty$ (hence $\min_{t \in [0,1]} |x_n(t)| \to +\infty$ as $n \to +\infty$). Then

$$\int_0^1 \left[\lambda(t)\,\frac{s(t, x_n(t))}{|x_n(t)|^2} + (1 - \lambda(t))\,\frac{i(t, x_n(t))}{|x_n(t)|^2} + \frac{(x_n(t), g(t))}{|x_n(t)|^2}\right] dt = 0$$

for all $n \in \boldsymbol{N}$.

On the other hand it follows from (ii) that

$$\frac{|s(t, x_n(t))|}{|x_n(t)|^2} \quad \text{and} \quad \frac{|i(t, x_n(t))|}{|x_n(t)|^2}$$

can be bounded by the same function belonging to $L^1([0,1], \boldsymbol{R}_+)$ for n sufficiently large. Therefore, we can use Lebesgue's Dominated Convergence Theorem to obtain

$$\int_0^1 [\lambda(t)\,\alpha(t) + (1-\lambda(t)(\beta(t)]\,dt = 0\,.$$

So if for some $\lambda \in L^\infty([0,1],[0,1])$ this equality fails, then we can parallel the arguments in the proof of Theorem 1 to obtain the conclusion of Corollary. □

Observe that under $(B+)$ (dually, $(B-)$) we can choose $\lambda(t)=1$ (dually, $\lambda(t)=0$) a.e. in $[0,1]$.

REMARK 5. – Consider the following perturbation of (1)

$$\dot{x} = f(t,x,u) + g(t) + f_0(t,x)$$

and the quantities

$$\gamma_1(t) = \liminf_{|x|\to+\infty} \frac{(x, f_0(t,x))}{|x_{\bar{z}}|}$$

and

$$\gamma_2(t) = \limsup_{|x|\to+\infty} \frac{(x, f_0(t,x))}{|x|^2}\,.$$

Theorem 1 and Corollary 1 remain true if the following conditions are satisfied:

$$(A+)' \quad \int_0^1 (\alpha_i(t) + \gamma_1(t))\,dt > 0 \qquad \text{or} \qquad (A-)' \quad \int_0^1 (\beta_s(t) + \gamma_2(t))\,dt < 0\,,$$

respectively

$$(B+)' \quad \int_0^1 (\alpha(t) + \gamma_1(t))\,dt > 0 \qquad \text{or} \qquad (B-)' \quad \int_0^1 (\beta(t) + \gamma_2(t))\,dt < 0.$$

THEOREM 2. – *Under assumptions of Theorem* 1 *and* (v), *the* 1-*periodic solutions of* (2) *are* 1-*periodic solutions to* (1) *corresponding to some control* $u(\cdot)\in U$.

The proof is a consequence of a result of ROXIN ([7], Theorem 1). □

2. – Stability of periodic solutions.

In the previous section we gave results on the existence of a periodic solution of a control problem (1). Now we want to impose hypotheses on the vector field associated with the control problem in order to ensure the stability of such periodic solutions. In the context of differential inclusions a related result on the exponential stability of stationary solutions was given in [8].

DEFINITION. – A periodic solution $x_p = x_p(t)$ corresponding to the control $u_p \in U$ for the system (1) is said to be locally uniformly asymptotically stable if there exist $\mu, \eta > 0$ such that for any $t_1 \geqslant 0$, any solution x of (1) corresponding to a control $u \in U$ with $\|u - u_p\|_{L^\infty} < \eta$ and $|x(t_1) - x_p(t_1)| < \mu$ satisfies $|x(t) - x_p(t)| \to 0$ as $t \to +\infty$.

Let us suppose that the following property holds:

$$(S) \qquad \exists M, \mu, \eta \in \boldsymbol{R}^+, \quad z \in L^1([0, \infty), \boldsymbol{R}^+)$$

s.t. $\forall x, y \in B(0, M)$, $\forall u, v \in \Omega$, with $|x - y| < \mu$ and $|u - v| < \eta$ we have

$$\frac{(x - y, f(t, x, u) - f(t, y, v))}{|x - y|^2} < -z(t) < 0 \text{ a.e. in } [0, 1].$$

Then we can establish the following stability result.

THEOREM 3. – *Let the system* (1) *satisfy hypotheses* (i)-(v), (S) *and* $(A+)$ *or* $(A-)$ *(resp.* $(B+)$ *or* $(B-)$*). Then any periodic solution of the system,* $x_p(\cdot)$*, such that* $|x_p(t)| < M$*,* $\forall t \in [0, 1]$ *is locally uniformly asymptotically (exponentially) stable.*

PROOF. – Let $x_p(\cdot)$ be a periodic solution of (1) such that $|x_p(t)| < M$ for all $t \in [0, 1]$, and let $u_p(\cdot)$ be the corresponding control. Let $x(\cdot)$ be any solution corresponding to a control $u(\cdot)$, such that $|x(t_1) - x_p(t_1)| < \mu$, and $|u(t) - u_p(t)| < \eta$, $\forall t \geqslant t_1$. Then we will have $|x(t) - x_p(t)| < \mu$ on some maximal interval $[t_1, T)$ with $|x(T) - x_p(T)| = \mu$ and

$$\frac{\big(x(t) - x_p(t), f(t, x(t), u(t)) - f(t, x_p(t), u_p(t))\big)}{|x(t) - x_p(t)|^2} < -z(t) < 0 .$$

If we denote $d(t) = |x(t) - x_p(t)|^2$ we have a.e. in $[t_1, T]$

$$\frac{1}{2}\frac{\dot{d}(t)}{d(t)} = \frac{\big(x(t) - x_p(t), f(t, x(t), u(t)) - f(t, x_p(t), u_p(t))\big)}{|x(t) - x_p(t)|^2} < -z(t) < 0 ,$$

that is $d/dt \log d(t) < -2z(t) < 0$ and finally $d(t) \leqslant d(t_1) \exp\left[-2\int_{t_1}^{t} z(s)\, ds\right]$, $t_1 \leqslant t \leqslant T$. Thus

$$|x(t) - x_p(t)| \leqslant |x(t_1) - x_p(t_1)| \exp\left[-\int_{t_1}^{t} z(s)\, ds\right] \quad \text{on } [t_1, T] .$$

If we had $T < \infty$, then we would have

$$\mu = |x(T) - x_p(T)| \leqslant |x(t_1) - x_p(t_1)| \exp\left[-\int_{t_1}^{T} z(s)\, ds\right] < \mu$$

a contradiction. ☐

3. – Examples.

EXAMPLE 1. – Consider the system

$$\dot{x} = h(t, x) + u(t)\, b(t, x) + g(t)\,, \qquad x \in \mathbf{R}^n\,,$$

with $u(\cdot)$ measurable, $u(t) \in [-1, 1]$, and g, b, h extended by 1-periodicity from $[0, 1]$ to $\mathbf{R}$. Suppose

$$(*) \qquad \big(x, b(t, x)\big) = o(|x|^2) \text{ and } \big(x, h(t, x)\big) = \varphi(t)|x|^2 + o(|x|^2), \text{ as } |x| \to \infty,$$

uniformly for $t \in [0, 1]$, with $\int_0^1 \varphi(t)\, dt \neq 0$. Then Assumptions (vi) and (vii) are valid with $\alpha(t) = \beta(t) = \varphi(t)$. Assuming h and b satisfy the Carathéodory condition and that $|h|$ and $|b|$ are each bounded by a linear function of $|x|$ with L^1 coefficients, then the Corollary to Theorem 1 implies that there exists a 1-periodic solution of (2) for any $\lambda(\cdot) \in L^\infty([0, 1], [0, 1])$. Note that since our problem is linear in u,

$$\mathrm{co}\, f_s(t, x) = f_s(t, x) = f\big(t, x, u_s(t, x)\big)\,, \qquad \mathrm{co}\, f_i(t, x) = f\big(t, x, u_i(t, x)\big)\,.$$

As remarked earlier, when $\lambda(t) \in \{0, 1\}$ a.e. then

$$\lambda(t) f_s(t, x) + \big(1 - \lambda(t)\big) f_i(t, x) = f\Big(t, x, \lambda(t)\, u_s(t, x) + \big(1 - \lambda(t)\big)\, u_i(t, x)\Big)\,.$$

Thus for each such $\lambda(\cdot)$ there exists a periodic solution of (1).

One can weaken the hypotheses in this example by replacing $(*)$ with

$$(**) \quad \psi(t)|x|^2 + o(|x|^2) = \inf_{u \in [-1,1]} (x, h + ub) \le \sup_{u \in [-1,1]} (x, h + ub) = \varphi(t)|x|^2 + o(|x|^2)$$

and requiring either $\int_0^1 \varphi(t)\, dt > 0$ or $\int_0^1 \psi(t)\, dt < 0$. In this case $\alpha(t) = \varphi(t)$, $\beta(t) = \psi(t)$, so the Corollary again applies.

Finally, we can treat the non-isotropic case by replacing $\psi(t)$, $\varphi(t)$ respectively in $(**)$ with $\psi(t, x/|x|)$, $\varphi(t, x/|x|)$. In this case,

$$\alpha_i(t) = \inf_{w \in S} \varphi(t, w)\,, \qquad \alpha_s(t) = \sup_{w \in S} \varphi(t, w)\,,$$

$$\beta_i(t) = \inf_{w \in S} \psi(t, w)\,, \qquad \beta_s(t) = \sup_{w \in S} \psi(t, w)\,,$$

where S is the unit sphere in $\mathbf{R}^n$. Theorem 1 states that (2) has a periodic solution if either

$$\int_0^1 \inf_{w \in S} \varphi(t, w)\, dt > 0 \qquad \text{or} \qquad \int_0^1 \sup_{w \in S} \psi(t, w)\, dt < 0\,.$$

To examine stability, we write out the condition (S):

$$(x-y,\, h(t,x)-h(t,y)) + (x-y,\, u(t)\,b(t,x)-v(t)\,b(t,y)) \leqslant -z(t) < 0 \,.$$

With a sufficient degree of smoothness this will be true for $|x-y|$ and $|u-v|$ small if

$$(x-y,\, [J_h(t,x^*) + uJ_b(t,x^{**})](x-y)) + (x-y,\, (u-v)\,b(t,y)) \leqslant -z(t)\,,$$

where J_h, J_b are the respective Jacobian matrices of h and b, with respect to x, and x^*, x^{**} lie on the line in $\boldsymbol{R}^n$ between x and y. This last inequality will hold for $(u-v)$, $(x-y)$ sufficiently small if there is an $\varepsilon > 0$ such that for all x in a tube in $\boldsymbol{R}^n$ containing $x_p(\cdot)$ we have

$$\sup_{u\in[-1,1]}\ \sup_{|v-x|<\varepsilon} \lambda_{\max}\left\{\frac{1}{2}\left[\frac{\partial h_i}{\partial x_j}+\frac{\partial h_j}{\partial x_i}\right]\Bigg|_{(t,x)} + \frac{u}{2}\left[\frac{\partial x_i}{\partial b_j}+\frac{\partial b_j}{\partial x_i}\right]\Bigg|_{(t,v)}\right\} \leqslant -\frac{3}{2}\,z(t) < 0\,,$$

where $\lambda_{\max}$ denotes the maximum eigenvalue of the quantity in brackets. In this case we can restrict $|u-v|$ so that the term $(x-y,\, (u-v)\,b(t,y))$ is less than $\frac{1}{2}z(t)$.

EXAMPLE 2. – Consider the same system as in Example 1. Note that $u_i(t,x) = -\operatorname{sgn}(x, b(t,x)) = -u_s(t,x)$. Assume that

$$\alpha_s(t) = \limsup_{|x|\to\infty} \frac{(x, h(t,x)) + |(x, b(t,x))|}{|x|^2} \leqslant 0 \qquad \text{for } t\in \mathcal{S}\subset[0,1]\,,$$

and that

$$\beta_s(t) = \limsup_{|x|\to\infty} \frac{(x, h(t,x)) - |(x, b(t,x))|}{|x|^2} < 0 \qquad \text{for } t\in\mathcal{T}\supset\mathcal{S}^c\,, \qquad |\mathcal{T}|>0\,,$$

where $\mathcal{S}^c$ denotes the complement of $\mathcal{S}$ with respect to $[0,1]$. Then

$$\int_0^1 \beta_s(t)\,dt = \int_{\mathcal{T}} \beta_s(t)\,dt + \int_{\mathcal{T}^c} \beta_s(t)\,dt < 0$$

since on $\mathcal{T}^c$, $\beta_s(t) \leqslant \alpha_s(t) \leqslant 0$. Thus $(A-)$ holds and Theorem 1 applies. Since the system is linear in u, the marginal equation (2) again reduces to (1) when $u(\cdot)$ is restricted to convex combinations of $u_i(t,x)$ and $u_s(t,x)$. The condition $(-)$ $\int_0^1 [\lambda(t)\,\alpha_s(t) + (1-\lambda(t))\,\beta_s(t)]\,dt < 0$ will be satisfied also for $\lambda = \chi_{\mathcal{S}}$, the characteristic function of $\mathcal{S}$.

The dual example requires

$$\alpha_i = \liminf_{|x|\to\infty} \frac{(x,h)+|(x,b)|}{|x|^2} > 0 \ \text{ on } \mathcal{T}\supset\mathcal{S}^c \text{ and } \beta_i = \liminf_{|x|\to\infty}\frac{(x,h)-|(x,b)|}{|x|^2} \geqslant 0 \ \text{ on } \mathcal{S}\,.$$

As a special case, consider the system $\dot{x} = \varphi(t)x + u(t)\psi(t)x + g(t)$ with φ, u, ψ scalar valued. Then $(x, h) = \varphi(t)|x|^2$, $(x, b) = \psi(t)|x|^2$. If there exists a set $\mathcal{S}$ with $|\mathcal{S}^c| > 0$ such that $\varphi(t) \leqslant -|\psi(t)|$ on $\mathcal{S}$, $\varphi(t) < |\psi(t)|$ on $\mathcal{T} \supset \mathcal{S}^c$, then there exists a periodic solution. (The dual: $\varphi(t) > -|\psi(t)|$ on $\mathcal{T} \supset \mathcal{S}^c$ and $\varphi(t) \geqslant |\psi(t)|$ on $\mathcal{S}$).

EXAMPLE 3. – Consider the system $\dot{x} = h(t, x) + B(t, x)u$ with $u(t) \in R^m$, $-1 \leqslant u_k(t) \leqslant 1$ $(k = 1, \dots, m)$, B an $n \times m$ matrix with columns $B^{(1)}, B^{(2)}, \dots, B^{(m)}$. We assume h and the entries of B satisfy the Carathéodory condition and a linear growth condition in $|x|$ with L^1 coefficients. We see immediately that the k-th component of u_s and u_i is given by

$$[u_s(t, x)]_k = -[u_i(t, x)]_k = \operatorname{sgn} \sum_{p=1}^{n} x_p b_{kp} ,$$

so

$$\alpha_i(t) = \liminf_{|x| \to \infty} \frac{(x, h) + \sum_{j=1}^{m} |(B^j, x)|}{|x|^2} ,$$

$$\alpha_i(t) = \limsup_{|x| \to \infty} \frac{(x, h) + \sum_{j=1}^{m} |(B^j, x)|}{|x^2|} ,$$

and β_i, β_s are respectively the above expressions with $+$ replaced by $-$.

Theorem 1 will apply, with $\lambda = \chi_{\mathcal{S}}$ if there exists a set $\mathcal{S} \subset [0, 1]$ such that $\int_{\mathcal{S}} \beta_i(t)\,dt \geqslant 0$ and $\int_{\mathcal{S}^c} \alpha_i(t)\,dt > 0$. $\left(\text{The dual: } \int_{\mathcal{S}} \alpha_s(t)\,dt \leqslant 0 \text{ and } \int_{\mathcal{S}^c} \beta_s(t)\,dt < 0\right)$. For example, we could require

$$(x, h(t, x)) = \varphi(t)|x|^2 + o(|x|^2) , \qquad \sum_{j=1}^{m} (B^j(t, x), x) = o(|x|^2) ,$$

with $\int_0^1 \varphi(t)\,dt > 0$. A slightly more complicated example would be

$$(x, h(t, x)) = \varphi(t)|x|^2 + o(|x|^2) , \qquad (B^j(t, x), x) = \psi^j(t)|x|^2 + o(|x|^2) ,$$

so that

$$\alpha_s(t) = \alpha_i(t) = \varphi(t) + \sum_{j=1}^{m} |\psi^j(t)| , \qquad \beta_s(t) = \beta_i(t) = \varphi(t) - \sum_{j=1}^{m} |\psi^j(t)| .$$

There will exist a periodic solution of (2) (hence of (1) since the system is linear in u) if there is a set $\mathcal{S}$ such that

$$\int_{\mathcal{S}} \left[\varphi(t) - \sum_{j=1}^{m} |\psi^j(t)|\right] dt \geqslant 0 \quad \text{and} \quad \int_{\mathcal{S}^c} \left[\varphi(t) + \sum_{j=1}^{m} |\psi^j(t)|\right] dt > 0 .$$

The condition (S) for stability of a given periodic solution $x_p(\cdot)$ becomes

$$(x-y, h(t,x)-h(t,y)+B(t,x)u-B(t,y)v)\leqslant 0$$

for $|u-v|$ and $|x-y|$ small, in some tube in $\boldsymbol{R}^n$ containing $x_p(\cdot)$. Writing

$$B(t,x)u-B(t,y)v=[B(t,x)-B(t,y)]u+B(t,y)(u-v)$$

and assuming sufficients smoothness, we can use the mean value theorem to write (S) as

$$(3)\qquad (x-y, J_h(t,x^*)(x-y))+\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k=1}^{n}(x-y)_i\frac{\partial b_{ij}}{\partial x_k}(t,x^{**})(x-y)_k u_j + \\ +(x-y, B(t,y)(u-v))\leqslant 0\,.$$

In the second term, the double sum $\sum_{i,k}(x-y)_i(\partial b_{ij}/\partial x_k)(x-y)_k$ is the quadratic form $(x-y, J_{B^j}(x-y))$, so for fixed j this double sum is bounded between the largest and smallest eigenvalues $\lambda^{(j)}_{\max}, \lambda^{(j)}_{\min}$ of the symmetrized coefficient matrix

$$\Delta B(j)=\frac{1}{2}\left[\frac{\partial b_{ij}}{\partial x_k}+\frac{\partial b_{kj}}{\partial x_i}\right]^n_{i,k=1},$$

$j=1,\dots,m$. Because $u_\lambda(t)\in[-1,1]$, we can bound the second term in (3) between

$$\pm\left\{\sum_{j=1}^{m}\max\left(|\lambda l^{(j)}_{\max}|, |\lambda^{(j)}_{\min}|\right)\right\}|x-y|^2\,.$$

If $\Lambda_{\min}, \Lambda_{\max}$ are the smallest and largest eigenvalues of $[\frac{1}{2}(\partial h_i/\partial x_j)+\frac{1}{2}(\partial h_j/\partial x_i)]$, then we need

$$\Lambda_{\max}(t,x^*)+\sum_{j=1}^{m}\max\left(|\lambda^{(j)}_{\max}|, |\lambda^{(j)}_{\min}|\right)\Big|_{(t,x^{**})}\leqslant\beta(t)<0$$

for some $\beta(\cdot)\in L^\infty([0,1],\boldsymbol{R})$, for all (t,x^*) in some tube about x_p and all x^{**} in some sufficiently small ball $B(x^*,\delta)$. Then $|u-v|$ can be restricted so as to satisfy (S).

For the case when $B(t,x)=[b_{11}(t,x), b_{12}(t,x)]$ $(n=1,\ m=2)$ we have

$$\Delta B(1)=\frac{\partial b_{11}}{\partial x_1},\qquad \Delta B(2)=\frac{\partial b_{12}}{\partial x_1},$$

$$\lambda^{(1)}_{\min}=\frac{\partial b_{11}}{\partial x_1},\qquad \lambda^{(2)}_{\min}=\frac{\partial b_{12}}{\partial x_1}$$

and our requirement is

$$\Lambda_{\max}(t,x^*)+\max_{|x^{**}-x^*|\leqslant\delta}\left\{\left(\left|\frac{\partial b_{11}}{\partial x_1}\right|+\left|\frac{\partial b_{12}}{\partial x_1}\right|\right)\Big|_{(t,x^{**})}\right\}\leqslant\beta(t)<0\,.$$

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