

SMALL PERIODIC PERTURBATIONS OF AUTONOMOUS SELF-OSCILLATING PLANAR SYSTEMS

Mikhail Kamenskii ^{*,1} Oleg Makarenkov ^{**,2}
Paolo Nistri ^{***,3}

^{*} Voronezh State University, Russia

^{**} Voronezh State University, Russia

^{***} University of Siena, Italy

Abstract: In this paper a nonlinear planar autonomous system having a limit cycle of period T which is perturbed by a small parameter nonautonomous T -periodic nonlinear term is considered. By using the topological index of the equilibrium points of the unperturbed system and a geometric condition on the perturbation, similar to the condition of the guiding function method as introduced by M. A. Krasnoselskii - A. I. Perov, the existence of T -periodic solutions of the perturbed system close as we like to the limit cycle of the autonomous system is proved.

Keywords: limit cycle, planar systems, small amplitude perturbations, topological index.

1. INTRODUCTION

This paper represents an extension, in a sense that will be precised in the sequel, of the results obtained by Kamenskii *et al.* (2004a) for perturbed nonautonomous systems to the case when the unperturbed system is autonomous.

Consider the following system of ordinary differential equations

$$\dot{x} = \psi(x), \quad (1)$$

where $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a continuously differentiable function, and assume that (1) has a T -periodic limit cycle x_0 . Then, consider the perturbed system

$$\dot{x} = \psi(x) + \varepsilon\phi(t, x), \quad (2)$$

where $\phi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is continuous and T -periodic with respect to the first variable and $\varepsilon > 0$ is a small parameter. Denote by $U \subset \mathbb{R}^2$ the interior of the limit cycle x_0 . Kamenskii *et al.* (2004a), in the case when system (1) is nonautonomous and T -periodic, have provided sufficient conditions to guarantee the existence of a T -periodic solution x_ε to system (2), with $\psi(x)$ replaced by $\psi(t, x)$, such that $x_\varepsilon(t) \in U$, for any $t \in [0, T]$. This paper deals with the same problem when ψ does not depend on time t .

In order to present the results stated in Kamenskii's *et al.* (2004a) let us introduce some notations. Let $\Omega(\cdot, t_0, \xi)$ be the solution of (1) satisfying $x(t_0) = \xi$ and let $\eta(\cdot, s, \xi)$ be the solution of the linearized system

$$\dot{y} = \frac{\partial \psi}{\partial x}(t, \Omega(t, 0, \xi))y + \phi(t, \Omega(t, 0, \xi)), \quad (3)$$

¹ Partially supported by RFBR, grants 02-01-00189 and 02-01-00307

² Partially supported by U.S.CRDF - RF Ministry of Education grant VZ-010

³ Partially supported by the national research project MIUR: "Feedback control and optimal control" and by GNAMPA-CNR

such that $y(s) = 0$. For any $\varepsilon > 0$, define the integral operator $F_\varepsilon : C([0, T], \mathbb{R}^2) \rightarrow C([0, T], \mathbb{R}^2)$ as follows

$$F_\varepsilon(x)(t) = x(t) - x(T) - \int_0^t (\psi(x(\tau)) + \varepsilon\phi(\tau, x(\tau))) d\tau,$$

where $C([0, T], \mathbb{R}^2)$ denotes the space of continuous functions acting from $[0, T]$ to \mathbb{R}^2 .

Clearly, if $F_\varepsilon(x) = 0$ then x is a T -periodic solution of system (2).

In the sequel $\gamma(g, X)$ will denote the rotation number of a compact map $g : \bar{X} \rightarrow \bar{X}$ on the boundary of the set X (see Krasnoselskii and Zabreyko (1984)). Finally, let $W = \{x \in C([0, T], \mathbb{R}^2) : x(t) \in U, \text{ for any } t \in [0, T]\}$.

Kamenskii *et al.* (2004a) proved the following.

Theorem 1. Assume, that

$$\eta(T, s, \xi) - \eta(0, s, \xi) \neq 0, \quad (4)$$

for any $s \in [0, T]$, and any $\xi \in \partial U$. Then, for $\varepsilon > 0$ sufficiently small, we have that

$$\gamma(F_\varepsilon, W) = \gamma(\eta(T, 0, \cdot), U). \quad (5)$$

A direct consequence of theorem 1 is that if (4) is satisfied and the rotation number of the vector field $\eta(T, 0, \cdot)$ is different from zero on the boundary of the set U then system (2) has a T -periodic solution inside U . This result is significant only in the case when system (1) is nonautonomous, i.e. $\psi = \psi(t, x)$. In fact, if (1) is autonomous the presence of a limit cycle for (1) implies, by the Poincaré-Bendixson's theorem, the existence of an equilibrium point inside U which has in many cases topological index equals to 1. On the other hand, if system (1) has an equilibrium point with nonzero topological index then, by well known result (see, for instance, Krasnoselskii and Zabreyko, 1984; Mawhin, 1993), system (2) has a T -periodic solution for any $\varepsilon > 0$ sufficiently small which converges to this point as $\varepsilon \rightarrow 0$. Therefore, for $\varepsilon > 0$ small enough, there exists a T -periodic solution in U and so theorem 1 together with the condition $\gamma(\eta(T, 0, \cdot), U) \neq 0$, do not provide any further information on the existence of T -periodic solutions in U . Aim of this paper is to give conditions to guarantee the existence of a T -periodic solution to system (2) which converges as $\varepsilon \rightarrow 0$ to the limit cycle of (1).

This problem is a classical problem of the theory of nonlinear oscillations and it was solved by implicit function method by Malkin (1949) and Loud (1959). The advantage of the proposed topological

approach is that it requires less regularity assumptions on the functions ψ and ϕ . In fact, Malkin and Loud assumed that ϕ is continuously differentiable with respect to all variables and ψ is twice continuously differentiable, while we only assume that ϕ is a continuous function and ψ is a continuously differentiable function. The same generalization, in the case when ψ is linear is provided by Buica and Llibre. To weaken the regularity assumptions on the functions ϕ and ψ is of relevant importance for dealing with the problem of synchronization of self-oscillating autonomous systems perturbed by external periodic stimulus in presence of noise. This problem is considered, for instance, in chapter 9 of the book by Pikovsky *et al.* (2002), see also Blekhnman (1981). Moreover, when ψ is a linear function, it was shown by Makarenkov (2003) that it is possible, by means of the topological methods employed in the proof of theorem 1, to give an explicit estimation of the parameter $\varepsilon > 0$ for which condition (5) holds. Further advantages of the topological approach with respect to others methods have been discussed by Cronin (1964), Krasnoselskii and Zabreyko (1984) and Mawhin (1993). Similar results, when ψ is a linear function, were obtained by Baillieul (1993) and Schneider (2001).

The paper is organized as follows. In section 2, under condition (4) and the assumption that $\gamma(\eta(T, 0, \cdot), U)$ is different from the integer defined by (8), it is shown that theorem 1 ensures, for $\varepsilon > 0$ sufficiently small, the existence of a T -periodic solution $x_\varepsilon(t)$ to system (2) such that $x_\varepsilon(t)$ converges to ∂U as $\varepsilon \rightarrow 0$ for any $t \in [0, T]$. Moreover, in theorem 3 a simple method to verify condition (4) and to calculate $\gamma(\eta(T, 0, \cdot), U)$ is presented. The paper ends by discussing, in section 3, how the presented results can be improved and extended.

2. THE MAIN RESULT

Assume that there are no limit cycles of system (1) belonging to U and that the number of equilibrium points a_1, \dots, a_k of (1) in U is finite. Denote by $\text{ind}(a_i, \psi)$ the topological index of the vector field ψ at the point a_i , (see, for instance, Krasnoselskii (1966)).

The following result holds.

Theorem 2. Assume, that the condition (4) of theorem 1 is satisfied. Assume, that

$$\det\|\psi'(a_i)\| \neq 0, \quad i \in \overline{1, k} \quad (6)$$

and

$$\text{ind}(a_1, \psi) + \dots + \text{ind}(a_k, \psi) \neq \gamma(\eta(T, 0, \cdot), U). \quad (7)$$

Then, for $\varepsilon > 0$ sufficiently small, system (2) has a T -periodic solution $x_\varepsilon(t)$ which tends, for any $t \in [0, T]$, to $\partial U = \{\xi \in \mathbb{R}^2 : \xi = x_0(t), t \in [0, T]\}$ as $\varepsilon \rightarrow 0$.

Proof. Under condition (6) by Poincaré theorem (1892) one has that for every $i \in \overline{1, k}$ there exists an unique family $\{x_{i,\varepsilon}\}_{\varepsilon \in (0, \varepsilon_0)}$, with ε_0 sufficiently small, such that $x_{i,\varepsilon}$ is the solution of (2) for every $\varepsilon \in (0, \varepsilon_0)$ and $x_{i,\varepsilon}(t) \rightarrow a_i$ as $\varepsilon \rightarrow 0$. Assume ε_0 to be small enough to guarantee that for $\varepsilon \in (0, \varepsilon_0)$ the conclusion of theorem 1 is verified, that is $\gamma(F_\varepsilon, W) = \gamma(\eta(T, 0, \cdot), U)$. Then, from Krasnoselskii (1966) it is known that the number $\varepsilon_0 > 0$ can be chosen in such a way that for every $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta > 0$ such that $\text{ind}(a_i, \psi) = \gamma(F_\varepsilon, W_\delta(x_{i,\varepsilon}))$, $i \in \overline{1, k}$, where $W_\delta(x)$ denotes the δ -neighbourhood of the point $x \in C([0, T], \mathbb{R}^2)$. Therefore

$$\begin{aligned} & \text{ind}(a_1, \psi) + \dots + \text{ind}(a_k, \psi) = \\ & = \gamma(F_\varepsilon, W_\delta(x_{1,\varepsilon})) + \dots + \gamma(F_\varepsilon, W_\delta(x_{k,\varepsilon})). \end{aligned} \quad (8)$$

From (5), (7) and (8), for any $\varepsilon \in (0, \varepsilon_0)$, one obtains that

$$\begin{aligned} \gamma(F_\varepsilon, W) \neq & \gamma(F_\varepsilon, W_\delta(x_{1,\varepsilon})) + \dots + \\ & + \gamma(F_\varepsilon, W_\delta(x_{k,\varepsilon})), \end{aligned}$$

and so for every $\varepsilon \in (0, \varepsilon_0)$ there exists a solution x_ε of system (2) belonging to the set $W \setminus (W_\delta(x_{1,\varepsilon}) \cup \dots \cup W_\delta(x_{k,\varepsilon}))$. Assume, that x_ε does not converge to ∂U as $\varepsilon \rightarrow 0$. Therefore there exist $i_0 \in \overline{1, k}$ and a subsequence $\{x_{\varepsilon_p}\}_{p \in \mathbb{N}}$ such that $x_{\varepsilon_p} \rightarrow a_{i_0}$ as $p \rightarrow \infty$. This contradicts the uniqueness of the family $\{x_{i_0,\varepsilon}\}_{\varepsilon \in (0, \varepsilon_0)}$ and so the theorem is proved.

Although the topological degree (or rotation number) of the vector field in (7) can be calculated by using the methods from Krasnoselskii and Zabreyko (1984), it will be provided here an alternative method to verify this condition which turns out to be of simple application.

First, some notations are needed. Recall that for the vector $v \in \mathbb{R}^2$, $[v]_i$ will denote its i -th component, $i = 1, 2$, and for $a, b \in \mathbb{R}^2$ let

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} [a]_1 & [a]_2 \\ [b]_1 & [b]_2 \end{pmatrix}, \quad (a \ b) = \begin{pmatrix} [a]_1 & [b]_1 \\ [a]_2 & [b]_2 \end{pmatrix}.$$

Consider the linearization of system (2) around the limit cycle x_0 and denote by $y = y(t)$ the solution of the linearized system linearly independent with $\dot{x}_0(t)$. Finally, introduce the map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as follows

$$\begin{aligned} F(s, \theta) = & \int_{s-T}^s (\dot{x}_0(\tau + \theta) \ y(\tau + \theta))^{-1} \cdot \\ & \cdot \phi(\tau, x_0(\tau + \theta)) d\tau. \end{aligned}$$

The following result provides a practical method both to calculate $\gamma(\eta(T, 0, \cdot), U)$ and to verify condition (4).

Theorem 3. Let $f : [0, T] \rightarrow \mathbb{R}^2$ be a function satisfying the following condition

$$\left\langle F(s, \theta), \begin{pmatrix} \dot{x}_0(\theta) \\ y(\theta) \end{pmatrix} f(\theta) \right\rangle \neq 0 \quad (9)$$

for every $s, \theta \in [0, T]$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product in \mathbb{R}^2 . Then the condition (4) is satisfied and

$$\gamma(\eta(T, 0, \cdot), U) = \gamma(f(\cdot), [0, T]), \quad (10)$$

where $\gamma(f(\cdot), [0, T])$ denotes the rotation number of the vector $f(\theta)$ as θ varies from 0 to T .

Proof. A result in Kamenskii *et al.* (2004b) states that

$$\begin{aligned} & \eta(T, s, x_0(\theta)) - \eta(0, s, x_0(\theta)) = \\ & = \int_{s-T}^s \Omega'_{(3)}(0, \tau, x_0(\tau + \theta)) \phi(\tau, x_0(\tau + \theta)) d\tau \end{aligned}$$

where $\Omega'_{(3)}$ denotes the derivative of the function Ω with respect to the third argument. Since

$$\begin{aligned} & \Omega'_{(3)}(0, \tau, \Omega(\tau, 0, x_0(\theta))) = \left(\Omega'_{(3)}(\tau, 0, x_0(\theta)) \right)^{-1} = \\ & = (\dot{x}_0(\theta) \ y(\theta)) (\dot{x}_0(\tau + \theta) \ y(\tau + \theta))^{-1} \end{aligned}$$

one has that

$$\begin{aligned} & \langle \eta(T, s, x_0(\theta)) - \eta(0, s, x_0(\theta)), f(\theta) \rangle = \\ & = \left\langle F(s, \theta), \begin{pmatrix} \dot{x}_0(\theta) \\ y(\theta) \end{pmatrix} f(\theta) \right\rangle \end{aligned}$$

for every $s, \theta \in [0, T]$ and so the assertion of the theorem holds true.

Remark 1. More general conditions of (9) to calculate the rotation number of a vector field were obtained by Krasnoselskii and Perov (1958), by means of the guiding function method.

Remark 2. The first coordinate of the function $F(s, \theta)$, which is constant with respect to s and T -periodic with respect to θ , was used by Loud (1959) and Malkin (1949) to show the existence of T -periodic solutions to (2) in the case when $F(s, \theta)$ is continuously differentiable. Indeed, Loud and Malkin have proved that if $[F(s, \theta_0)]_1 = 0$ and $[F'_{(2)}(s, \theta_0)]_1 \neq 0$ then system (2) has a T -periodic solution with the initial value which tends to $x_0(\theta_0)$ as $\varepsilon \rightarrow 0$.

3. CONCLUSION

In this paper results ensuring the existence of T -periodic solutions of a self-oscillating autonomous systems, perturbed by a small amplitude T -periodic nonlinear term, which are close to the limit cycle of the autonomous system are provided. Indeed, it is proved that these solutions converge to the limit cycle when the amplitude of the perturbation tends to zero. It is also pointed out the importance of the investigation of the relationship between the results by Loud (1959) and Malkin (1949) and the results of the present paper. In particular, if it is possible, under conditions $[F(s, \theta_0)]_1 = 0$ and $[F'_{(2)}(s, \theta_0)]_1 \neq 0$, to prove theorem 2. Observe that theorem 2 is applicable even in the case when $[F(s, \theta_0)]_1 = 0$ and either $F(s, \theta)$ is not differentiable at $\theta = \theta_0$ or $[F'_{(2)}(s, \theta_0)]_1 = 0$. Furthermore, another problem to be addressed in future research is the extension of theorem 2 to the case when the function ψ' and ϕ are discontinuous. In fact, some control problems of mechanical systems lead to this situation which is considered by Kolovskii (1961) and Crema *et al.* (2000).

REFERENCES

- Baillieul, J. (1993). Stable average motions of mechanical systems subject to periodic forcing, in Dynamics and Control of Mechanical Systems: The Falling Cat and Related Problems, *Fields Inst. Commun.* (M. J. Enos, Ed.) **1**, 1-23. AMS, Providence, RI.
- Blekhman, I. I. (1981) *Synchronization in Nature and Technology*. Nauka Press, Moscow. (English translation by ASME Press, USA, N.Y., 1988).
- Buica, A. and J. Llibre. Averaging methods for finding periodic orbits via Brouwer degree. To appear in *Bull. Sci. Math.*
- Crema, J. and J. L. Boldrini (2000). Periodic solutions of quasilinear equations with discontinuous perturbations. *Southwest J. Pure Appl. Math.*, **1**, 55-73.
- Cronin, J. (1964). *Fixed points and topological degree in nonlinear analysis*. **11**. Mathematical surveys, AMS, Providence, RI.
- Kamenskii, M., O. Yu. Makarenkov and P. Nistri (2004a). Small parameter perturbations of nonlinear periodic systems. *Nonlinearity*, **17**, 193-205.
- Kamenskii, M., O. Yu. Makarenkov and P. Nistri (2004b). Periodic solutions for a class of singularly perturbed systems. *Dyn. Contin. Discrete Impuls. Systems, Ser. A: Math. Anal.*, **11**, 41-55.
- Kolovskii, M. Z. (1961). An application of the small-parameter method for determining discontinuous periodic solutions. Analytic methods in the theory of non-linear vibrations. *Proc. Internat. Sympos. Non-linear Vibrations*, Vol. 1, 264-276. Izdat. Akad. Nauk Ukrain. SSR, Kiev.
- Krasnoselskii, M. A. (1966). *The shift operator along the trajectories of differential equations*. Nauka, Moscow. (in Russian)
- Krasnoselskii, M. A. and A. I. Perov (1958). On a principle of existence of bounded, periodic and almost-periodic solutions to the systems of ordinary differential equations. *Dokl. Akad. Nauk*, **123**, N 2, 235-238. (in Russian)
- Krasnoselskii, M. A. and P. P. Zabreyko (1984). *Geometrical Methods of Nonlinear Analysis*. Berlin: Springer.
- Loud, W. S. (1959). Periodic solutions of perturbed autonomous systems. *Ann. of Math*, **70**, 490-529.
- Makarenkov, O. Yu. (2003). On the modified averaging method in the study of RC-amplifier periodic mode near resonance. *Proceedings of Voronezh State University, series: physics, mathematics*, **1**, 157-160.
- Malkin, I. G. (1949). On the theory of Poincaré's periodic solutions. *Appl. math. mech.*, Vol. XIII, issue 6. (in Russian)
- Mawhin, J. (1993). Topological degree and boundary value problems for nonlinear differential equations. *Lecture Notes in Math.*, 1537, Springer, Berlin.
- Pikovsky, A., M. Rosenblum and J. Kurths. (2002). *Synchronization – A universal concept in nonlinear sciences*. Cambridge University Press.
- Poincaré, A. (1892). *Les méthodes nouvelles de la mécanique céleste*, Vol. 2, Chap. III, Paris.
- Schneider, K. R. (2001). Vibrational control of singularly perturbed systems. In: *Lecture Notes in Control and Information Science* **259**, 397-408, Springer Verlag, London.