ON GLOBAL STABILITY OF HOPFIELD NEURAL NETWORKS WITH DISCONTINUOUS NEURON ACTIVATIONS

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ABSTRACT

The paper introduces a general class of neural networks where the neuron activations are modeled by discontinuous functions. The neural networks have an additive interconnecting structure and they include as particular cases the Hopfield neural networks (HNNs), and the standard Cellular Neural Networks (CNNs), in the limiting situation where the HNNs and CNNs possess neurons with infinite gain. Conditions are obtained which ensure global convergence toward the unique equilibrium point in *finite time*, where the convergence time can be easily estimated on the basis of the relevant neural network parameters. These conditions are based on the concept of Lyapunov Diagonally Stable (LDS) neuron interconnection matrices, and are applicable to general non-symmetric neural networks.

Nomenclature

$\mathbb{R}^n$: real n-space
$A = [A_{ij}] \in \mathbb{R}^{n\times n}$: square matrix
A' : transpose of A
$A^S = \frac{1}{2}(A + A')$: symmetric part of A
$\Lambda_m\{A\}$: minimum eigenvalue of the symmetric matrix A
$\alpha = \text{diag}(\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^{n\times n}$: diagonal matrix with diagonal entries α_i, $i = 1, \cdots, n$
$x = (x_1, \cdots, x_n)' \in \mathbb{R}^n$: column vector
$\|x\|_2 = \left[\sum_{i=1}^n x_i^2\right]^{1/2}$: Euclidean norm of x
$\text{int}(E)$: interior of E
∂E : boundary of E
$\text{co}(E)$: convex hall of E
$K[E] = \overline{\text{co}}(E)$: closure of the convex hull of E
$\emptyset$: empty set

1. INTRODUCTION

An overview of some common neural network models reveals that discontinuous neural networks do frequently arise in practice. Consider for example the classical Hopfield neural networks (HNNs) with graded response neurons [1]. The standard assumption is that the activations be employed in the high-gain limit where they approach a discontinuous hard comparator function. A conceptually analogous model based on the use of hard comparators is used also to describe the Discrete-Time Cellular Neural Networks [2]. Another important example concerns the class of neural networks introduced by Kennedy and Chua [3] to solve linear and nonlinear programming problems. Those networks exploit constraint neurons with a diode-like input-output activations. In order to guarantee satisfaction of the constraints, the diodes are required to possess a very high slope in the conducting region, i.e., they should approximate the discontinuous characteristic of an ideal diode.

In the recent literature, one of the most investigated dynamical issues has been to obtain conditions ensuring that a neural network possesses a unique equilibrium point which is globally attractive for the trajectories, see, e.g., [4, 5, 6], and references therein. The property of global convergence prevents a neural network from the risk of getting stuck at some local minimum of the energy function. Said another way, a globally attractive neural network is well suited to solve global optimization problems in real time.

All the quoted results on global convergence concern neural networks where the neuron activations are modeled by Lipschitz continuous functions. Thus, they leave open the issue of global convergence in the limiting ideal case of discontinuous neuron activations. As it was noted before this issue is of interest since discontinuous neural networks are frequently encountered in practice. Moreover, the analysis of the ideal discontinuous case might reveal the existence of phenomena, such as the presence of sliding modes and the global convergence in finite time, which are known to be potentially useful for the design of real-time optimization solvers [7].

In this paper we introduce and analyze a general class of neural networks which possess discontinuous neuron activations. The neural networks have an additive interconnecting structure and they include as particular cases the HNNs [1], and the Cellular Neural Networks (CNNs) [8], in the limiting situation where the HNNs and CNNs are modeled by

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neurons with infinite gain. The paper establishes conditions ensuring global convergence of the neural network in *finite time*, where the convergence time can be easily estimated on the basis of the relevant neural network parameters. These conditions, which are applicable to general non-symmetric neural networks, involve the hypothesis of Lyapunov Diagonally Stable (LDS) interconnection matrices and constraints on the location of the equilibrium point with respect to the points where the activations are discontinuous.

All the results in this paper are stated without proof. The reader is referred to [9] for a proof of the main results.

2. NEURAL NETWORK MODEL

We consider neural networks described by the system of differential equations

$$\dot{x} = Bx + Tg(x) + I \qquad \text{(N)}$$

where $x = (x_1, \cdots, x_n)' \in \mathbb{R}^n$ is the vector of the neuron states, $B = \mathrm{diag}(-b_1, \cdots, -b_n) \in \mathbb{R}^{n\times n}$ is a diagonal matrix such that $b_i > 0$, $i = 1, \cdots, n$, model the neuron self-inhibitions, $T \in \mathbb{R}^{n\times n}$ is a matrix whose entries represent the synaptic neuron interconnections, and $I \in \mathbb{R}^n$ is a vector of constant neuron inputs.

The diagonal mapping $g(x) = (g_1(x_1), \cdots, g_n(x_n))' : \mathbb{R}^n \to \mathbb{R}^n$ has components $g_i(x_i)$ that model the nonlinear input-output activations of the neurons. In the paper, we assume that g belongs to the following class of discontinuous functions.

Definition 1 (Function Class $\mathcal{G}_D$) *We say that $g \in \mathcal{G}_D$ if and only if, for $i = 1, \cdots, n$, g_i satisfies the next assumptions:*

a) g_i is bounded on $\mathbb{R}$;

b) g_i is piecewise continuous, i.e., g_i has finite right and left limits ($g_i(\rho^+)$, $g_i(\rho^-)$, respectively) at any $\rho \in \mathbb{R}$ and it results $g_i(\rho) = g_i(\rho^+)$ if g_i is discontinuous at ρ; in particular, g_i has a finite number of discontinuities on any compact interval of $\mathbb{R}$;

c) g_i is non-decreasing on $\mathbb{R}$, i.e., for any ρ_1 and ρ_2 such that $\rho_1 > \rho_2$ and g_i is continuous at ρ_1 and ρ_2, it results $g_i(\rho_1) \geq g_i(\rho_2)$.

Class $\mathcal{G}_D$ includes a number of neuron activations of interest for the applications. For example, the standard hard comparator function $\mathrm{sgn}(\rho)$, where $\mathrm{sgn}(\rho) = 1$ if $\rho \geq 0$, $\mathrm{sgn}(\rho) = -1$ if $\rho < 0$, and the discontinuous multilevel activation functions.

We remark that in the case where $g_i(x_i) = \mathrm{sgn}(x_i)$, $i = 1, \cdots, n$, we may consider model (N) as the limit of a HNN where the maximum gain of the neuron activations tends to infinity. It is of importance to note that in the actual applications the Hopfield networks are indeed usually employed in such a 'high-gain' limit situation [1]. Moreover, also the Discrete-Time CNNs introduced in [2] exploit similar neurons with infinite gain.

Since for $g \in \mathcal{G}_D$ the right-hand side of (N) is a discontinuous function of the state x, it is needed to explain what is meant by a solution of a Cauchy problem associated to (N). A possible definition, which we shall adopt in this paper, is that of A. F. Filippov [10]. Under our assumptions this definition reduces to the next one.

Definition 2 ([10]) *A solution of (N) on an interval $[t_0, t_1]$, $t_0 \leq t_1 \leq +\infty$, which satisfies the initial condition $x(t_0) = x_0$, is an absolutely continuous function $x(t)$ defined on $[t_0, t_1]$, such that $x(t_0) = x_0$, and for almost all (a.a.) $t \in (t_0, t_1)$ it results*

$$\dot{x}(t) \in K[Bx(t) + Tg(x(t)) + I] \qquad (1)$$

where the right-hand side of (1) is the closure of the convex hull of the limit points of the function $y \to By + Tg(y) + I$, as $y \to x$.

Under the stated assumptions on g, i.e., $g \in \mathcal{G}_D$, by accounting for the properties of the convex hull, it results $K[Bx(t) + Tg(x(t)) + I] = B\ x(t) + TK[g(x(t))] + I$, where $K[g(x)] = (K[g_1(x_1)], \cdots, K[g_n(x_n)])'$, and for $i = 1, \cdots, n$

$$K[g_i(x_i)] = [g_i(x_i^-), g_i(x_i^+)]. \qquad (2)$$

Therefore, the multi-valued map $x \multimap TK[g(x)]$ has compact convex values. Moreover, it is upper semicontinuous (see [10, Lemma 1, p. 67]) and hence it is measurable. By the measurable selection Theorem (see [11, Theorem 8.1.3, p. 308]) we have that if $x = x(t)$ is a solution of (N) on $[t_0, t_1]$, there exists a measurable function $\gamma(t) \in K[g(x(t))]$, $t \in [t_0, t_1]$, such that for a.a. $t \in [t_0, t_1]$ it results

$$\dot{x}(t) = Bx(t) + T\ \gamma(t) + I. \qquad (3)$$

Note that the vector function $\gamma(t)$ actually represents the neuron outputs on $[t_0, t_1]$, which are therefore only measurable functions of time.

Property 1 *Suppose that $g \in \mathcal{G}_D$. Then, for any $x_0 \in \mathbb{R}^n$ there is at least a local solution $x(t)$ of (N) such that $x(0) = x_0$. Furthermore, any solution is bounded and hence defined on $[0, +\infty)$.*

By an equilibrium point (EP) $e \in \mathbb{R}^n$ of (N), we mean a constant solution of (N), $x(t) = e$, $t \in [0, +\infty)$. Furthermore, if e is an EP of (N), we say that $\eta(e) \in \mathbb{R}^n$ is an output equilibrium point (OEP) of (N) corresponding to e, if it results $\eta \in K[g(e)]$ and $0 = B\ e + T\eta + I$.

The goal of this paper is to find conditions on the neuron interconnection matrix T, which ensure that (N) has a unique EP e, and a unique corresponding OEP η. Moreover, we are interested in the case where the neural network is globally convergent to e and η in *finite time*, according to the next definition.

Definition 3 *We say that (N) is globally convergent in finite time, if the following conditions holds:*

1) (N) has a unique EP e and a unique corresponding OEP η;

2) for each $x_0 \in \mathbb{R}^n$, any trajectory $x(t)$ of (N), $t \in [0, +\infty)$, with $x(0) = x_0$, is such that it results $x(t) = e$, for $t > \tilde{t}$, for some $\tilde{t} > 0$. Moreover, $\gamma(t) \in K[g(x(t))]$, $t \in [0, +\infty)$, which satisfies (3), is such that $\gamma(t) = \eta$ for $t > \tilde{t}$.

The notion of global convergence in finite time is stronger than the standard concept of global attractivity, since it implies that the state $x(t)$ and output $\gamma(t)$ become exactly equal to their limits after some finite time has elapsed. Convergence in finite time represents a peculiar feature of systems described by vector fields which are discontinuous in the state [12, 7]. We recall that on the contrary for systems described by smooth vector fields there can be only asymptotic convergence of trajectories towards the limit.

3. MAIN RESULTS

To state the main results in this paper, we need to introduce the following classes of matrices.

Definition 4 ([13]) *Matrix $A \in \mathbb{R}^{n\times n}$ is said to belong to the class P if and only if all the principal minors of A are positive.*

Definition 5 ([14]) *Matrix $A \in \mathbb{R}^{n\times n}$ is said to be Lyapunov Diagonally Stable (LDS) if and only if there exists a diagonal and positive definite matrix $\alpha \in \mathbb{R}^{n\times n}$ such that the symmetric part of αA, $[\alpha A]^S = 1/2(\alpha A + A'\alpha)$, is positive definite.*

There are large classes of symmetric and non-symmetric matrices of practical interest that satisfy the LDS condition [4, 14].

The next result concerning existence and uniqueness of the equilibrium point of (N) holds.

Theorem 1 *Suppose that $-T \in P$. Then, for any neuron activations $g \in \mathcal{G}_\mathcal{D}$, any diagonal matrix B, and any input vector $I \in \mathbb{R}^n$, (N) has a unique EP e and a unique corresponding OEP η.*

Now, let us consider the set

$$\mathcal{D}_g = \{x \in \mathbb{R}^n : g_i(x_i) \text{ is discontinuous at } x_i \text{ for all } i = 1, \cdots, n\}.$$

Below we define a set of inputs I that plays a crucial role in the study of global convergence of (N) in finite time.

Definition 6 *Suppose that $-T \in$ LDS and $g \in \mathcal{G}_\mathcal{D}$. Let $\mathcal{I}_\mathcal{F}$ be the set of vectors $I \in \mathbb{R}^n$ such that it results $e \in \mathcal{D}_g$, and for any $i = 1, \cdots, n$*

$$\eta_i \in \mathrm{int}(K[g_i(e_i)]) = (g_i(e_i^-), g_i(e_i^+)). \qquad (4)$$

The set $\mathcal{I}_\mathcal{F}$ simply corresponds to vectors $I \in \mathbb{R}^n$ such that, for all $i = 1, \cdots, n$, the component g_i of g is discontinuous at e_i, where $e = (e_1, \cdots, e_n)'$, and condition (4) is satisfied. This set $\mathcal{I}_\mathcal{F}$ can be explicitly characterized, as it is shown in the next result.

Property 2 *Suppose that $-T \in$ LDS and $g \in \mathcal{G}_\mathcal{D}$. Moreover, assume that $\mathcal{D}_g \neq \emptyset$. Then, $\mathcal{I}_\mathcal{F}$ is the non-empty open set*

$$\mathcal{I}_\mathcal{F} = \bigcup_{x^d \in \mathcal{D}_g} \tilde{\mathcal{I}}_\mathcal{F}(x^d)$$

where for $x^d \in \mathcal{D}_g$, $\tilde{\mathcal{I}}_\mathcal{F}(x^d)$ is the set of vectors $I \in \mathbb{R}^n$ such that

$$I = -Bx^d - T\eta \text{ with } \eta \in \mathrm{int}\,(K[g(x^d)])$$

and it coincides with the interior of the parallelotope in $\mathbb{R}^n$ defined by the 2^n vertexes

$$v^i = -Bx^d - Ty^i; \quad y^i = (g(x_1^{d\pm}), \cdots, g(x_n^{d\pm}))'.$$

We note that in the particular interesting case where the neurons activations are modeled by $g_i(x_i) = \mathrm{sgn}(x_i)$, $i = 1, \cdots, n$, as in the Hopfield model, we have $\mathcal{D}_g = \{0\}$ and $\mathcal{I}_\mathcal{F} = \tilde{\mathcal{I}}_\mathcal{F}(0)$, where $\tilde{\mathcal{I}}_\mathcal{F}(0)$ is the interior of the parallelotope in $\mathbb{R}^n$ defined by the 2^n vertexes

$$v^{Hi} = -Ty^{Hi}; \quad y^{Hi} = (\pm 1, \cdots, \pm 1)'.$$

Example 1. Let us consider the neural network

$$\begin{cases} \dot{x}_1 &= -x_1 - \frac{1}{4}\mathrm{sgn}(x_1) - \mathrm{sgn}(x_2) + I_1 \\ \dot{x}_2 &= -x_2 + \mathrm{sgn}(x_1) - \frac{1}{4}\mathrm{sgn}(x_2) + I_2 \end{cases} \qquad (5)$$

where $I = (I_1, I_2)'$ is a generic input. By choosing $\alpha = \mathrm{diag}(1, 1)$, we get $[\alpha(-T)]^S = \mathrm{diag}(1/4, 1/4)$, i.e., $-T \in$ LDS. It turns out that $\mathcal{I}_\mathcal{F} = \tilde{\mathcal{I}}_\mathcal{F}(0)$ is the interior of the parallelotope in $\mathbb{R}^2$ with vertexes $(-5/4, 3/4)'$, $(5/4, -3/4)'$, $(3/4, 5/4)'$, and $(-3/4, -5/4)'$. ■

The main result in this paper is as follows.

Theorem 2 *Suppose that $-T \in$ LDS and $g \in \mathcal{G}_\mathcal{D}$. Furthermore, suppose that $\mathcal{I}_\mathcal{F} \neq \emptyset$ and $I \in \mathcal{I}_\mathcal{F}$. Then, (N) is globally convergent in finite time. Moreover, the convergence time of a trajectory $x(t)$ of (N), which is defined as*

$$t_c = \inf\{\tilde{t} > 0 : x(t) = e \text{ and } \gamma(t) = \eta, \text{ for } t > \tilde{t}\}$$

satisfies

$$t_c \le t_e = \frac{1}{\lambda_m \delta^2} \sum_{i=1}^{n} \alpha_i \int_0^{x_i(0)-e_i} G_i(\rho)\, d\rho \qquad (6)$$

where

$$\begin{aligned} G(z) &= g(z+e) - \eta & (7)\\ \lambda_m &= \Lambda_m\{[\alpha(-T)]^S\} > 0 & (8)\\ \delta &= \min_{i \in \{1,\cdots,n\}} \{\min\{\delta_i^+, \delta_i^-\}\} > 0 & (9) \end{aligned}$$

and for $i \in \{1, \cdots, n\}$

$$\begin{aligned} g_i(e_i^+) - \eta_i &= \delta_i^+ > 0 & (10)\\ \eta_i - g_i(e_i^-) &= \delta_i^- > 0. & (11) \end{aligned}$$

Note that the inequalities (10) and (11) are certainly satisfied, since for $I \in \mathcal{I}_{\mathcal{F}}$ condition (4) holds.

To the authors knowledge, Theorem 2 is the first general result on global convergence in finite time for the additive neural network model (N). This result can be thought of as a generalization to the discontinuous case of previous results on global stability which involve the concept of LDS matrices [4].

It is pointed out that from expression (6) the estimated convergence time t_e is influenced by the neuron interconnection matrix T, through the eigenvalue λ_m, and by the location of the OEP η within $K[g(e)]$, through parameter δ (cf. (10) and (11)). Moreover, t_e depends on the distance of the initial condition $x(0)$ from the EP e, which corresponds to the integral term at the right-hand side of (6).

We observe that since functions G_i are nondecreasing and bounded on $\mathbb{R}$, i.e., $|G_i(\rho)| \le M$, for some $M > 0$, and any $i = 1, \cdots, n$, and $0 \in K[G_i(0)]$, $i = 1, \cdots, n$, we can obtain from (6) the explicit estimate

$$t_c \le t_e \le \frac{M}{\lambda_m \delta^2} \sum_{i=1}^{n} \alpha_i |x_i(0) - e_i|.$$

In particular, when the neurons activations are modeled by $g_i(x_i) = \mathrm{sgn}(x_i)$, $i = 1, \cdots, n$, as in the Hopfield model, by taking into account that $\int_0^{x_i} \mathrm{sgn}(\rho)\, d\rho = |x_i|$, (6) reduces to the simple formula

$$t_c \le t_e = \frac{1}{\lambda_m \delta^2} \sum_{i=1}^{n} \alpha_i |x_i(0) - e_i|. \qquad (12)$$

Example 2. Let us consider the second-order neural network in Example 1, and suppose that $I_1 = I_2 = 0$. The neural network has a unique EP $e = 0$ and a unique corresponding OEP $\eta = 0$. By choosing $\alpha = \mathrm{diag}(1, 1)$, we have $[\alpha(-T)]^S = \mathrm{diag}(1/4, 1/4)$, i.e., $-T \in$ LDS. Moreover, from Example 1, it is seen that $I = 0 \in \mathcal{I}_{\mathcal{F}}$. Therefore, the conditions of Theorem 2 hold and the neural network is globally convergent in finite time.

Let us now estimate using (6) the convergence time for a neural network solution starting at $x_0 = (2, 2)'$. It results $\lambda_m = 1/4$ and $\delta = 1$, hence from (12) we get $t_c \le t_e = 4\{|x_1(0)| + |x_2(0)|\}$. Therefore, we obtain $t_c \le t_e = 16$. This evaluation is confirmed by numerical simulations with MATLAB (not reported here due to space limitations.) ■

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