Bollettino U.M.I.
Analisi Funzionale e Applicazioni
Serie VI, Vol. V – C, N. 1 - 1986

ON A GENERAL NOTION OF CONTROLLABILITY FOR NONLINEAR SYSTEMS

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SUNTO. *Si considera una nozione generale di controllabilità (detta* 1*-controllabilità) per processi di controllo non lineari. Impiegando tecniche basate sul grado topologico di una applicazione si stabiliscono condizioni sufficienti per ottenere sia la locale che la globale* 1*-controllabilità per mezzo di controlli che appartengono ad un sottospazio* U *di* $(L^\infty([0,1],R^m)$ *di dimensione finita. Alcuni esempi illustrano l'applicazione dei risultati ottenuti.*

0. INTRODUCTION.

Consider the nonlinear control process described by the differential system

$$\dot{x}(t) = f(t,x(t),u(t)) \quad \text{for almost all (a.a.)} \quad t \in [0,1] \tag{1}$$

where the function $f : [0,1] \times R^n \times R^m \to R^n$ satisfies the following Carathéodory conditions

(f_1) - for any $(p,q) \in R^n \times R^m$, the mapping $t \to f(t,p,q)$ is (Lebesgue) measurable on $[0,1]$; for a.a. $t \in [0,1]$, the mapping $(p,q) \to f(t,p,q)$ is continuous on $R^n \times R^m$.

(f_2) - for each $\rho > 0$ there exists $\gamma_\rho \in L^1([0,1],R)$ such that for a.a. $t \in [0,1]$ and every (p,q) with $|p| + |q| \le \rho$ one has $|f(t,p,q)| \le \gamma_\rho(t)$.

The function $t \to u(t)$, describing the control law, will be assumed to belong to the Banach space $L^\infty([0,1],R^m)$ of the Lebesgue measurable essentially bounded functions $u : [0,1] \to R^m$. The function $t \to x(t)$, describing the state of (1), will belong to the Banach space $AC([0,1],R^n)$ of absolutely continuous functions $x : [0,1] \to R^n$.

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In this paper we are concerned with a general type of controllability of (1). Namely, we will consider a (not necessarily linear) continuous "boundary" operator

$$l : AC([0,1],R^n) \times L^\infty([0,1],R^m) \to R^k \quad \text{with} \quad k > n ,$$

mapping bounded sets into bounded sets, and for a given $y \in R^k$ we will say that the system (1) is l-controllable at y if the nonlinear boundary value control problem

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t)) & \text{for a.a. } t \in [0,1] \\ l(x,u) = y \end{cases} \tag{2}$$

is solvable for some $u \in L^\infty([0,1],R^m)$.

The system (1) is said to be locally l-controllable at $y_o \in R^k$ if there exists a neighborhood $N(y_o)$ of y_o such that for any $y \in N(y_o)$, problem (2) is solvable for some $u \in L^\infty([0,1],R^m)$. Finally, if (1) is l-controllable for any $y \in R^k$ then it is said to be globally l-controllable.

In particular, if $l(x,u) = (x(0),x(1)) \in R^n \times R^n$, where $x(1) = x(0) + \int_o^1 f(t,x(t),u(t))dt$, then the above definitions reduce to the usual definitions of controllability in R^n, i.e. if (1) is l-controllable at $y = (x_o,x_1) \in R^n \times R^n$ then the boundary value problem

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t)) & \text{for a.a. } t \in [0,1] \\ x(0) = x_o,\ x(1) = x_1 \end{cases}$$

is solvable for some $u \in L^\infty([0,1],R^m)$, i.e. the system (1) is controllable from x_o to x_1.

The aim of this paper is to provide sufficient conditions for local and global l-controllability by means only of controls belonging to a subspace U of $L^\infty([0,1],R^m)$ of dimension $d = k - n$.

Our approach is based on Mawhin's coincidence topological degree (see [8] and the extensive references therein). In the first Section, using this tool, and an appropriate homotopy, we prove (Theorem 1) a continuation principle which shows the solvability of (2). Theorem 2 gives sufficient conditions for the global l-controllability of (1). For a linear control process in R^n with a general linear boundary operator, controllability results were obtained by Marchiò in [7], and these results have been generalized in [2] to the case of perturbed linear processes.

Topological arguments were previously used by the author in [9] to treat periodic control problems. Moreover in [4], by means of a suitable homotopy, system (1) was transformed into a completely controllable linear system (i.e. $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ for a.a. $t \in [0,1]$), then, via the Leray-Schauder continuation principle, we obtained sufficient conditions for the global controllability of (1), using only controls belonging to an n-dimensional subspace of $L^{\infty}([0,1],R^m)$.

In the second part of this paper, we deal with the problem of obtaining criteria for "a priori bounds" on the solution pairs (x,u) of the homotopy considered in Theorem 1. In fact, in order to verify this type of condition we investigate hypotheses which enable us to deduce the boundedness of the parameters on which x depends. Some well known methods to calculating the topological degree are also presented.

Finally in the third Section we give some examples which illustrate the abstract results obtained in the other Sections. In what follows $(AC)^s$, $(C)^s$ and $(L^p)^s$, $s \in N$, $1 \le p \le +\infty$, will denote the Banach spaces $AC([0,1],R^s)$, $C([0,1],R^s)$ and $L^p([0,1],R^s)$ equipped with the usual norms. (We will omit the superscript if s=1). Lastly $|\cdot|$ and $|\cdot|_X$ will denote respectively the Euclidean norm in any finite dimensional space and the norm in an infinite dimensional Banach space X.

1. l-CONTROLLABILITY RESULTS

Let U be a finite dimensional subspace of $(L^{\infty})^m$ with $\dim U = k - n = d$. Choose, for simplicity, $y = 0$ in problem (2). The following Theorem states conditions for the l-controllability at 0 of (1).

Theorem 1. *Suppose that*

(i) *the set*

$$S = \{(x,u) \in (AC)^n \times U : (x,u) \text{ is a solution of } \dot{x}(t) = \lambda f(t,x(t),u(t)) \text{ for a.a. } t \in [0,1],\ l(x,u) = 0 \text{ for some } \lambda \in [0,1]\}$$

is bounded in $(C)^n \times (L^{\infty})^m$.

(ii) $\deg(l_o,\Omega_o,0) \neq 0$, *where* $l_o = l|_{R^n \times U}$ *and* Ω_o *is any bounded open set* *containing* $S_o = l_o^{-1}(0)$.

Then the nonlinear boundary value control problem

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t)) & \text{for a.a. } t \in [0,1] \\ l(x,u) = 0 \end{cases} \tag{2$_o$}$$

has a solution in $(AC)^n \times U$.

Proof. Define the linear operator $L : (AC)^n \times U \to (L^1)^n$ as

$$L(x,u)(t) = \dot{x}(t) \qquad \text{for a.a. } t \in [0,1] .$$

We have that $\operatorname{Ker} L = R^n \times U$ and $\operatorname{Im} L = (L^1)^n$.
Let $\Lambda : (AC)^n \times U \to (L^1)^n \times R^k$ be the operator defined by

$$\Lambda(x,u) = (L(x,u), l(x,u)) .$$

We will prove that $\deg(\Lambda,\Omega,0) \neq 0$, where Ω is any bounded open set of $(AC)^n \times U$ such that $S \subset \Omega$. Note that, by (f_2), S is a bounded set in $(AC)^n \times (L^\infty)^m$.

Let $c = (a,b) \in R^n \times R^d$ and let $A : (L^1)^n \times R^k \to (AC)^n \times U$ be the operator

$$A(y,c) = (x,J(b)) ,$$

where $x \in (AC)^n$ is such that $x(t) = a + \int_o^t y(s)\,ds$ for all $t \in [0,1]$ and $J : R^d \to U$ is an isomorphism. It is easy to see that A is an isomorphism. Let us consider

$$(\Lambda \circ A)(y,c) = (y, l(x,J(b)))$$

or equivalently

$$(\Lambda \circ A)(y,c) = (y,c) - (0, c - l(x,J(b))) .$$

Since $|\deg(\Lambda,\Omega,0)| = |\deg(\Lambda \circ A,\tilde{\Omega},0)|$ where $\tilde{\Omega} = A^{-1}(\Omega)$, from the reduction property of topological degree it follows that

$$\deg(\Lambda \circ A, \tilde{\Omega}, 0) = \deg(\Lambda \circ A, \tilde{\Omega}_o, 0)$$

where $\tilde{\Omega}_o = \tilde{\Omega} \cap \{0\} \times R^k$.

But $\Lambda \circ A\big|_{\{0\}\times R^k} = 1_o(a, J(b)) = \tilde{1}_o(a,b)$ where $\tilde{1}_o : R^n \times R^d \to R^k$ is given by $\tilde{1}_o = 1_o \circ (I \times J)$, ($I$ is the identity in R^n). Therefore

$$\deg(\Lambda \circ A, \tilde{\Omega}_o, 0) = \deg(\tilde{1}_o, \tilde{\Omega}_o, 0)$$

and $\deg(\tilde{1}_o, \tilde{\Omega}_o, 0) \neq 0$, since by hypothesis, $\deg(1_o, \Omega_o, 0) \neq 0$ where $\Omega_o = (I \times J)(\tilde{\Omega}_o)$.

Let now $\Phi : (AC)^n \times U \to (L^1)^n$ be the Nemitskii operator generated by f, that is

$$\Phi(x,u)(t) = f(t, x(t), u(t)) \qquad \text{for a.a. } t \in [0,1] .$$

By assumptions (f_1)-(f_2) and Lebesgue's dominated convergence Theorem it follows that Φ is continuous.

Let us prove that Φ is also compact. To this end let $\{(x_n, u_n)\}_{n\in N}$ be a bounded sequence of $(AC)^n \times U$. By the compact imbedding of $(AC)^n$ into $(L^1)^n$ the sequence $\{x_n\}_{n\in N}$ has a subsequence $\{x_{n_k}\}_{k\in N}$ such that

$$x_{n_k} \to x_o \qquad \text{in } (L^1)^n$$

$$x_{n_k}(t) \to x_o(t) \qquad \text{almost everywhere (a.e.) in } [0,1] .$$

Moreover, U being a finite dimensional space, we may also assume

$$u_{n_k} \to u_o \qquad \text{in } (L^\infty)^m$$

$$u_{n_k}(t) \to u_o(t) \qquad \text{a.e. in } [0,1] .$$

Hence, by assumptions (f_1)-(f_2) and Lebesgue dominated convergence Theorem we have that

$$\int_o^1 |f(s, x_{n_k}(s), u_{n_k}(s)) - f(s, x_o(s), u_o(s))| ds \to 0$$

as $k \to +\infty$, that is $\Phi(x_{n_k}, u_{n_k}) \to \Phi(x_o, u_o)$ in $(L^1)^n$. This proves the compactness of Φ. Finally, define $\Psi : (AC)^n \times U \to (L^1)^n \times R^k$ by

$$\Psi(x,u) = (\Phi(x,u),0) .$$

Obviously, Ψ has the same properties as Φ.

Consider now the homotopy $H : [0,1] \times (AC)^n \times U \to (L^1)^n \times R^k$ defined by

$$H(\lambda,x,u) = \Lambda(x,u) - \lambda\Psi(x,u) , \qquad \lambda \in [0,1] .$$

By assumptions (i)-(ii) the homotopy is admissible in $[0,1] \times \Omega$ (i.e. $H(\lambda,x,u) \neq 0$ for all $(x,u) \in \partial\Omega$ and any $\lambda \in [0,1]$), hence by the homotopy invariance property of topological degree we have that $\deg(\Lambda - \Psi,\Omega,0)$ is different from zero. The solution property implies that $(2)_o$ has a solution in $\Omega \subset (AC)^n \times U$. This completes the proof.

Corollary 1. *Suppose that the Assumptions* (i)-(ii) *of Theorem* 1 *are satisfied. Then there exists* $\varepsilon > 0$ *such that for every* $y \in R^k$ *with* $|y| < \varepsilon$, *problem* (2) *has a solution in* $(AC)^n \times U$.

Furthermore, assume that

(iii) *given* $\bar{y} \in R^k$, *the set*

$$\{(x,u) \in (AC)^n \times U : (x,u) \textit{ is a solution of}$$
$$\dot{x}(t) = f(t,x(t),u(t)), l(x,u) = \mu\bar{y} \quad \textit{for some} \quad \mu \in [0,1]\}$$

is bounded in $(C)^n \times (L^\infty)^m$.

Then (2) *is solvable in* $(AC)^n \times U$ *for* $y = \bar{y}$.

Proof. The first assertion of the Corollary is an immediate consequence of the fact that $\deg(\Lambda - \Psi,\Omega,0) \neq 0$, in fact this implies the local surjectivity of the map $\Lambda - \Psi$ at the point 0.

In order to prove the second part, let us consider the compact homotopy $H : [0,1] \times (AC)^n \times U \to (L^1)^n \times R^k$ defined by

$$H(\mu,x,u) = (0,\mu\bar{y}) .$$

By Assumption (iii) the set

$$\{(x,u) \in (AC)^n \times U : (\Lambda - \Psi)(x,u) = H(\mu,x,u) \quad \text{for some} \quad \mu \in [0,1]\}$$

is bounded in $(C)^n \times (L^\infty)^m$. Therefore, by (f_2), this set is bounded in $(AC)^n \times (L^\infty)^m$, and by the homotopy invariance property of topological degree the equation

$$(\Lambda - \Psi)(x,u) = H(1,x,u)$$

is solvable in $(AC)^n \times U$.

Finally we give a Theorem which gives conditions for the global 1-controllability of (1).

Theorem 2. *Assume that Assumptions* (i)-(ii) *of Theorem* 1 *are satisfied. Moreover, assume that*

(iii) *for every* $M \geq 0$ *the set*

$$\{(x,u) \in (AC)^n \times U : (x,u) \text{ is a solution of (2) with } |y| \leq M\}$$

is bounded in $(C)^n \times (L^\infty)^m$.

Then system (1) *has a connected component* Σ *of solutions* (x,u) *such that, for any* $y \in R^k$ *there exists* $(x,u)_y \in \Sigma$ *which solves* (2).

We omit the proof of this Theorem, since it can be easily deduced from Theorem 2.1 of [5] which is based on the notions of 0-regular and 0-regularizable maps introduced by Furi-Pera in [6]. In fact it suffices to observe that, since $\deg(\Lambda-\Psi,\Omega,0) \neq 0$, it follows that $(L - \Phi)$ is an 0-regularizable map by means of the continuous boundary operator l.

2. A PRIORI BOUNDS AND COMPUTATION OF DEGREE.

Since all of the previous abstract results are based on Assumptions (i)-(ii) of Theorem 1, in this Section we will give some practical conditions which imply that (i)-(ii) hold. More precisely, in Theorem 3 we will state a result which will enable us to deduce the boundedness of the set S from that of parameters $c = (a,b) \in R^n \times R^d$, and finally will discuss some methods of computing the degree of the map l.

Let A, B be bounded open sets of R^n and R^d respectively and let $\bar{A}$, $\bar{B}$

be their respective closures. Consider the following condition:

(H) for every $c = (a,b) \in \bar{A} \times \bar{B}$ and every $\lambda \in [0,1]$ the set

$$X(c,\lambda) = \{x \in (AC)^n : \dot{x}(t) = \lambda f(t,x(t),u(t)) \quad \text{a.e. in } [0,1],$$
$$x(t) = a \text{ for some } t \in [0,1] \text{ and } u = J(b)\}$$

is bounded in $(C)^n$, i.e. there exists a constant $M = M(c,\lambda) > 0$ independent of x such that $\max_{t\in[0,1]} |x(t)| \le M$ for any $x \in X(c,\lambda)$.

Now we can prove the following

Theorem 3. *If for every pair* $(x,u) \in S$ *we have* $x(t) \in A$ *for some* $t \in [0,1]$ *and* $J^{-1}(u) \in B$, *then condition* (H) *implies Assumption* (i) *of Theorem* 1.

Proof. Let $\{(x_n,u_n)\}_{n\in N}$ be a sequence in S. Then there exist $\{\lambda_n\}_{n\in N}$, $\{t_n\}_{n\in N} \subset [0,1]$ such that, for every $n \in N$, we have

$$\dot{x}_n(t) = \lambda_n f(t,x_n(t),u_n(t)) \quad \text{a.e. in } [0,1], \quad x_n(t_n) = a_n \in A ,$$
$$b_n = J^{-1}(u_n) \in B \quad \text{and} \quad l(x_n,u_n) = 0 .$$

The sequence $\{c_n\}_{n\in N} = \{(a_n,b_n)\}_{n\in N}$ is bounded in R^k, hence, by passing to a subsequence if necessary, we have that

$$c_n \to c_o = (a_o,b_o) \in \bar{A} \times \bar{B}, \quad \lambda_n \to \lambda_o \in [0,1] \quad \text{and} \quad t_n \to t_o \in [0,1].$$

For every $n \in N$, define $g_n : [0,1] \times R^n \to R^n$ as follows

$$g_n(t,p) = \lambda_n f(t,p,u_n(t)) \quad \text{for a. a. } t \in [0,1] .$$

It is easy to see that the sequence $\{g_n\}_{n\in N}$ satisfies all the Assumptions of Theorem 2.4 of [4] which states the continuous dependence of the solutions of (1) on the parameters c_n.

Therefore the sequence $\{x_n\}_{n\in N} \subset (AC)^n$ satisfying

$$\dot{x}_n(t) = g_n(t,x_n(t)) \quad \text{a. e. in } [0,1] ,$$
$$x_n(t_n) = a_n ,$$

for every $n \in N$, has a subsequence converging uniformly in $[0,1]$ to a function $x_o(.,u_o,\lambda_o)$ which is a solution of the following Cauchy problem

$$\dot{x}(t) = \lambda_o f(t,x(t),u_o(t)) \qquad \text{for a.a. } t \in [0,1].$$

$$x(t_o) = a_o$$

where $u_o = J(b_o)$. Hence $x_o(.,u_o,\lambda_o) \in X(c_o,\lambda_o)$ and by the Assumptions on f and l we have also $l(x_o,u_o) = 0$.

This means that S is compact in $(C)^n \times U$ and so, by (f_2) is bounded in $(AC)^n \times U$. This completes the proof.

Remark 1. Condition (H) is satisfied, if for every $\rho > 0$ there exists a function $\omega_\rho : [0,1] \times R \to R$ satisfying the following conditions

(a) $|f(t,p,q)| \le \omega_\rho(t,|p|)$ for all $|q| \le \rho$, all $p \in R^n$ and a.a. $t \in [0,1]$.

(b) $\omega_\rho(t,r)$ is measurable in t for every r and is continuous in r for a.a. $t \in [0,1]$.

(c) for every $K > 0$ there exists $\gamma_K \in L^1$ such that $|\omega_\rho(t,r)| \le \gamma_K(t)$ for $|r| \le K$ and for a.a. $t \in [0,1]$.

(d) $\omega_\rho(t,r)$ is nondecreasing with respect to the second variable.

(e) for every $(t_o,a) \in [0,1] \times \bar{A}$ the maximal solution of the Cauchy problem

$$\begin{cases} \dot{\eta}(t) = \omega_\rho(t,\eta(t)) & \text{a.e. in } [0,1], \\ \eta(t_o) = a \end{cases}$$

is defined in all of $[0,1]$.

The proof of this fact can be obtained by appropriate modifications of the proof of Corollary 2.3 of [4].

The existence of such a function ω_ρ is assured, for instance, if for any $\rho > 0$ there exist two functions α_ρ, $\beta_\rho \in L^1$ such that

$$|f(t,p,q)| \le \alpha_\rho(t) + \beta_\rho(t)|p|$$

for all $|q| \le \rho$, all $p \in R^n$ and a.a. $t \in [0,1]$.

Obviously, condition (H) is also satisfied if for any $c \in \bar{A} \times \bar{B}$ and

$\lambda \in [0,1]$ the set $X(c,\lambda)$ is a singleton.

We now collect some well known basic results, used in part in the next examples, to imply condition ii), i.e. that $\deg(\tilde{1}_o,\tilde{\Omega}_o,0)$ is different from zero. Recall that the map $\tilde{1}_o : R^n \times R^d \to R^k$ has been defined in Section 1 and $\tilde{\Omega}_o$ is a bounded open set of $R^n \times R^d$ containing the set $\tilde{1}_o^{-1}(0)$ (which is bounded by Assumption (i)).

Proposition 1. *Denote by* $(.,.)$ *the inner product in* R^k *corresponding to the Euclidean norm. Assume that there exists a constant* $r > 0$ *such that* $(\tilde{1}_o(c),c) > 0$ *for any* $c \in R^k$ *with* $|c| > r$.
Then $\deg(\tilde{1}_o,B_{r_1},0) \neq 0$ *for any* $r_1 \geq r$, *where* $B_{r_1} = \{x \in R^k : |x| \leq r_1\}$.

Proof. The homotopy $\tilde{1}_{o,\lambda} : [0,1] \times R^k \to R^k$ defined by

$$\tilde{1}_{o,\lambda}(c) = (1-\lambda)c + \lambda\tilde{1}_o(c) \qquad \lambda \in [0,1]$$

is admissible in $[0,1] \times B_{r_1}$ whenever $r_1 \geq r$. In fact, if we suppose the contrary then there exists $c \in \partial B_{r_1}$ such that $\tilde{1}_{o,\lambda}(c) = 0$ for some $r_1 \geq r$ and $\lambda \in [0,1]$. Hence

$$0 = (1-\lambda)|c|^2 + \lambda(\tilde{1}_o(c),c) \ ,$$

contradicting the fact that

$$0 < (1-\lambda)|c|^2 + \lambda(\tilde{1}_o(c),c)$$

for every $c \in R^k$ with $|c| > r$ and every $\lambda \in [0,1]$.
It is clear that we can choose $\tilde{\Omega}_o = B_{r_1}$ for r_1 sufficiently large.

Proposition 2. *Assume that* $\tilde{1}_o(c) = -\tilde{1}_o(-c)$ *for any* $c \in R^k$ *with* $|c| \geq r$, *for some* $r > 0$. *Then* $\deg(\tilde{1}_o,B_{r_1},0) \neq 0$ *for any* $r_1 \geq r$.

Proof. The assertion follows immediately from Borsuk's Theorem (see e.g. [3], Theorem 4.1).

Proposition 3. *Denote by* $\tilde{1}_o^i$ *the* i-th *component of the vector* $\tilde{1}_o$. *Assume that for any* $i = 1,2,\ldots,k$ *there exist positive constants* α_i, β_i *such that*

$$\liminf_{c_i \to +\infty} \tilde{1}_o^i(c_1,\ldots,c_i,\ldots,c_k) = \alpha_i > 0$$

and

$$\limsup_{c_i \to -\infty} \tilde{l}_o^i(c_1,\dots,c_i,\dots,c_k) = -\beta_i < 0$$

uniformly with respect to $c_j \in R$ *with* $j \neq i$. *Then there exists a constant* $r > 0$ *such that* $\deg(\tilde{l}_o, B_{r_1}, 0) \neq 0$ *for any* $r_1 \geq r$.

Proof. From our assumptions it follows that there exists a constant $r > 0$ such that, for any $i = 1,2,\dots,k$, we have

$$\tilde{l}_o^i(c) > 0 \quad \text{if} \quad c_i = r \quad \text{and} \quad |c_j| \leq r, \ j \neq i\,,$$

$$\tilde{l}_o^i(c) < 0 \quad \text{if} \quad c_i = -r \quad \text{and} \quad |c_j| \leq r, \ j \neq i\,.$$

Then using the Poincaré-Miranda Theorem (see e.g. [1], Proposition 3)) it is immediate that the homotopy considered in the proof of Proposition 1 is admissible in $[0,1] \times B_{r_1}$, whenever $r_1 \geq r$.

Other methods to compute the degree are presented, for example, in [8].

3. EXAMPLES

The following examples illustrate how the abstract results given above can be applied under suitable conditions on the boundary operator l and on the dynamics f. Moreover, these examples point out the role of an appropriate choice for the finite dimensional space U of controls in order to guarantee the validity of the Assumptions in our abstract results. Obviously, such a choice will depend, in general, on the conditions imposed on l and f.

Example 1. Let $f:[0,1] \times R \times R \to R$ be a function satisfying the Assumptions (f_1)-(f_2) specified in the Introduction. We shall consider the problem of the l-global controllability of the scalar equation

$$\dot{x}(t) = f(t,x(t),u(t)) \quad \text{for} \quad \text{a.a.} \quad t \in [0,1]$$

in the case where $l : AC \times U \to R^2$ has the following form $l = (l_1,l_2) \in R \times R$ with

$$l_1(x,u) = g_1(x(0)) + h_1(x(1)) + \int_0^1 k_1(t,x(t),u(t))dt$$

$$l_2(x,u) = g_2(x(0)) + h_2(x(1)) + \int_0^1 k_2(t,x(t),u(t))dt \; .$$

Assume that

(c_1) $g_1, g_2, h_1, h_2 : R \to R$ are continuous functions such that

$$\lim_{p\to\pm\infty} g_1(p) + h_1(p) = \pm\infty$$

and

$$|g_2(p)| + |h_2(p)| \le M \quad \text{for all} \quad p \in R \; ,$$

for some constant $M \ge 0$.

(c_2) the functions $k_1, k_2 : [0,1] \times R \times R \to R$ satisfy (f_1)-(f_2). Furthermore assume that for every $r > 0$ there exist $\rho > 0$, $\eta_1, \eta_2 \in R$ such that

$$k_1(t,p,q) \ge \eta_1 \quad \text{for} \quad p \ge \rho \; , \quad |q| \le r, \quad \text{a.e. in} \quad [0,1] \; ,$$

$$k_1(t,p,q) \le \eta_2 \quad \text{for} \quad p \le -\rho \; , \quad |q| \le r, \quad \text{a.e. in} \quad [0,1] \; .$$

Finally, suppose that there exist $\varepsilon, \delta > 0$; $\alpha, \beta \in L^1$ with $\alpha(t) \ge 0$ for a.a. $t \in [0,1]$ and $\beta(t) \ge \beta > 0$ for a.a. $t \in [0,1]$, such that

$$\liminf_{|q|\to+\infty} \frac{q\, k_2(t,p,q)}{|q|^{1+\delta}} \ge \alpha(t)\, |p|^{\varepsilon} + \beta(t)$$

for all $p \in R$ and for a.a. $t \in [0,1]$.

Define the 1-dimensional space $U \subset L^\infty$ as follows

$$U = \{b\, v(t),\; b \in R\}$$

where $v \in L^\infty$ and $\operatorname*{ess\,inf}_{t\in[0,1]} v(t) > 0$.
We can now show that for any pair $(x,u) \in S$ we have $x(t) \in A$ for some $t \in [0,1]$ and $J^{-1}(u) \in B$, where $A,B \subset R$ are open bounded sets. Hence, if condition (H) holds,then via Theorem 3, Assumption (i) of Theorem 1 is satisfied.

To prove this by contradiction, assume that there exists a sequence $\{(x_n,u_n)\}_{n\in N} \subset S$ with corresponding $\{\lambda_n\}_{n\in N} \subset [0,1]$ such that

$$\min_{t\in[0,1]} |x_n(t)| + |J^{-1}(u_n)| \to +\infty \quad \text{as} \quad n \to +\infty .$$

Let $|b_n| = |J^{-1}(u_n)| \to +\infty$ as $n \to +\infty$. Because of the choice of control space U we have $u_n(t) \to +\infty$ or $-\infty$ for a.a. $t \in [0,1]$ as $n \to +\infty$, depending on whether $b_n \to +\infty$ or $b_n \to -\infty$ as $n \to +\infty$.
On the other hand we have

$$0 = l_2(x_n,u_n) = g_2(x_n(0)) + h_2(x_n(1)) + \int_o^1 k_2(t,x_n(t),u_n(t))dt$$

for all $n \in N$, hence

$$0 = \frac{b_n[g_2(x_n(0)) + h_2(x_n(1))]}{|b_n|^{1+\delta}} + \int_o^1 \frac{b_n k_2(t,x_n(t),u_n(t))}{|b_n|^{1+\delta}} dt$$

where

$$\int_o^1 \frac{b_n k_2(t,x_n(t),u_n(t))}{|b_n|^{1+\delta}} dt = \int_o^1 \frac{v^\delta(t)J(b_n)(t)k_2(t,x_n(t),J(b_n)(t))}{|J(b_n)(t)|^{1+\delta}} dt$$

It is not hard to see that, under our hypotheses and the choice of U, the inferior limit, as $n \to +\infty$, of the right hand side of the above equation is greater than zero, which is a contradiction.
Therefore there exists $C > 0$ such that $|u_n|_{L^\infty} \le C$ for all $n \in N$, thus we must have $\min_{t\in[0,1]} |x_n(t)| \to +\infty$ as $n \to +\infty$.
In particular $|x_n(0)| \to +\infty$ and $|x_n(1)| \to +\infty$ as $n \to +\infty$ and for n sufficiently large we have $x_n(0)\cdot x_n(1) > 0$. It is easy to see that the Assumptions made on the functions g_1, h_1, k_1 imply that $\lim_{n\to+\infty} |l_1(x_n,u_n)| = +\infty$, a contradiction. Therefore, if condition (H) holds then S is bounded in $C \times L^\infty$. Moreover the preceding arguments show that also Assumption (iii) of Theorem 2 is satisfied.

Finally, in order to prove that Assumption (ii) of Theorem 1 is verified, we define the homotopy $G : [0,1] \times R \times R \to R^2$ as follows

$$G(\lambda,a,b) = (\tilde{l}_{o,1}(a,\lambda b),\ \tilde{l}_{o,2}(\lambda a,b))$$

(Recall that $\tilde{l}_o = l_o \circ (I \times J)$ where $l_o = l|_{\operatorname{Ker} L}$).

By means of the preceding procedure we can prove that the set

$$T = \{(a,b) \in R \times R : G(\lambda,a,b) = 0 \ \text{ for some } \ \lambda \in [0,1]\}$$

is bounded. Therefore G is an admissible homotopy and so we have

$$\deg(\tilde{l}_o,\tilde{\Omega}_o,0) = \deg(G(1,a,b),\tilde{\Omega}_o,0) = \deg(G(0,a,b),\tilde{\Omega}_o,0)$$

where $\tilde{\Omega}_o$ is an open bounded set containing T. On the other hand $\deg(G(0,a,b),\tilde{\Omega}_o,0)$ is different from zero, since we have

$$\lim_{a\to\pm\infty} \tilde{l}_{o,1}(a,0) = \pm\infty \quad \text{and} \quad \lim_{b\to\pm\infty} \tilde{l}_{o,2}(0,b) = \pm\infty$$

and so Proposition 3 applies.

Remark 2. If we take in the previous example $g_1 \equiv I$, $h_1 \equiv k_1 \equiv g_2 \equiv h_2 \equiv 0$ and $k_2 \equiv f$ then we obtain the global controllability of (1) with $m = n = 1$. In fact, it can immediately be seen that such a choice is consistent with the Assumptions (c_1)-(c_2) and that the surjectivity of $l(x,u) = (x(0), \int_o^1 f(t,x(t),u(t))dt)$ implies the global controllability.

In this case, we can in fact weaken the condition imposed on k_2 $(\equiv f)$. Indeed, we can prove the global controllability of the scalar equation under the following less restrictive Assumption

(c_2') Suppose that for any $\sigma \in R$ we have

$$\liminf_{q\to+\infty} f(t,p,q) = +\infty$$

uniformly for $p > \sigma$ and for a.a. $t \in [0,1]$, and

$$\limsup_{q\to-\infty} f(t,p,q) = -\infty$$

uniformly for $p < \sigma$ and for a.a. $t \in [0,1]$.

Observe that such a condition is satisfied when f is linear in p and q.

The proof of the global controllability is based on the fact that from the boundedness of the initial conditions $x(0)$ of the solutions $x(t)$ of (1) it follows that if $\{u_n\}_{n\in N} \subset U$ is a sequence such that $u_n(t) \to +\infty$ (or $-\infty$) for a.a. $t \in [0,1]$ then the corresponding solutions x_n are bounded from below (from above) for n large enough. Therefore any sequence $\{(x_n,u_n)\}_{n\in N}$ in S or in the set defined in Assumption (iii) of Theorem 2 has the preceding property. Finally, to verify Assumption (ii) of Theorem 1 we proceed as in example 1.

Remark 3. If we consider the general case when the state $x = x(t)$ is an n-vector and the control parameter $u = u(t)$ is represented by an m-vector and the boundary operator $l = (l_1, l_2) : (AC)^n \times U \to R^n \times R^n$ has the form described in example 1, then the l-controllability results obtained in the mentioned example hold for the system (1), simply by reformulating in a suitable way the Assumptions made there.

In order to do this, first of all we define the n-dimensional space $U \subset (L^\infty)^m$ in the following way.

Let $v_1, v_2, \ldots, v_n \subset L^\infty$ be n functions such that, for any n-vector $\xi = (\xi_i)_{i=1}^n$ with $|\xi| = 1$, the set

$$\{t \in [0,1] : \sum_{i=1}^n \xi_i v_i(t) = 0\}$$

has zero measure.

Moreover, let $e_1, e_2, \ldots, e_n \subset R^m$ be vectors of norm one. Put

$$U = \{u \in (L^\infty)^m : u = \sum_{i=1}^n b_i v_i e_i\}$$

where $b = (b_i)_{i=1}^n \in R^n$. Clearly in this case the isomorphism $J : R^n \to U$ is given by $J(b)(t) = \sum_{i=1}^n b_i v_i(t) e_i$ for a.a. $t \in [0,1]$. It is not hard to show that the following property (P) holds

(P) let $\{u_n\}_{n \in N} \subset U$ be a sequence such that $|u_n|_{(L^\infty)^m} \to +\infty$ as $n \to +\infty$. Then there exists a subsequence $\{u_{n_k}\}_{k \in N}$ of $\{u_n\}_{n \in N}$ such that $|u_{n_k}(t)| \to +\infty$ as $k \to +\infty$ for a.a. $t \in [0,1]$.

With this choice of the control space U, assume that

$(\bar{c}_1)$ $g_1, g_2, h_1, h_2 : R^n \to R^n$ are continuous functions satisfying the following properties:

there exists $\gamma > 0$ such that

$$\liminf_{|p| \to +\infty} \frac{(p, (g_1(p) + h_1(p)))}{|p|^{1+\gamma}} \geq \mu$$

for some $\mu > 0$, and

$$|g_2(p)| + |h_2(p)| \le M \quad \text{for all} \quad p \in R^n ,$$

for some constant $M \ge 0$.

($\bar{c}_2$) the functions $k_1, k_2 : [0,1] \times R^n \times R^n \to R^n$ satisfy (f_1)-(f_2). Furthermore assume that for every $r > 0$ there exists $C \ge 0$ such that

$$|k_1(t,p,q)| \le C \quad \text{for} \quad p \in R^n, \quad |q| \le r, \quad \text{a.e. in} \quad [0,1] .$$

Finally, suppose that there exist $\varepsilon, \delta > 0$, $\alpha, \beta \in L^1$ with $\alpha(t) \ge 0$ for a.a. $t \in [0,1]$ and $\int_0^1 \beta(t)dt > 0$ such that

$$\liminf_{|b| \to +\infty} \frac{(b, k_2(t,p,J(b)(t)))}{|b|^{1+\delta}} \ge \alpha(t)|p|^{\varepsilon} + \beta(t)$$

for all $p \in R^n$ and for a.a. $t \in [0,1]$.

Now, if the condition (H) holds, we can show the boundedness of both the set S and the set defined in Assumption (iii) of Theorem 2 with the same procedure as in example 1. Lastly, to show that (ii) is verified, consider as before the admissible homotopy

$$G(\lambda,a,b) = (\tilde{l}_{o,1}(a,\lambda b) , \tilde{l}_{o,2}(\lambda a,b))$$

and note that, by our Assumptions, we have

$$(a, \tilde{l}_{o,1}(a,0)) > 0 \quad \text{for all} \quad a \in R^n \quad \text{with} \quad |a| \quad \text{sufficiently large,}$$

and

$$(b, \tilde{l}_{o,2}(0,b)) > 0 \quad \text{for all} \quad b \in R^n \quad \text{with} \quad |b| \quad \text{sufficiently large.}$$

Hence, Proposition 1 applies to the map $G(0,a,b)$.

Observe that the following are true

F_1. Conditions $(\bar{c}_1)$-$(\bar{c}_2)$ are also consistent in the case where we consider the global controllability of (1).

F_2. We can replace the Assumptions on g_1, h_1 and k_2 by the following

- there exists $\gamma > 0$ such that

$$\limsup_{|p| \to +\infty} \frac{(p, g_1(p) + h_1(p))}{|p|^{1+\gamma}} \leq -\mu$$

  for some $\mu > 0$.

- there exist $\varepsilon, \delta > 0$; $\alpha, \beta \in L^1$ with $\alpha(t) \leq 0$ for a.a. $t \in [0,1]$ and $\int_0^1 \beta(t)dt < 0$ such that

$$\limsup_{|b| \to +\infty} \frac{(b, k_2(t,p,J(b)(t))}{|b|^{1+\delta}} \leq \alpha(t)|p|^{\varepsilon} + \beta(t)$$

  for all $p \in R^n$ and for a.a. $t \in [0,1]$.

We can do this because in this case, the a priori bounds still hold and the conclusion of Proposition 1 is still true; in fact in the proof it suffices to consider the homotopy joining $\tilde{l}_o$ with $-I$ (instead with I).

F_3. If $m \geq n$, we could choose an n-dimensional space $U \subset (L^\infty)^m$ with the property (P), taking as vectors $e_1, e_2, \ldots, e_n$ the canonical basis of R^n and assuming that $\operatorname{ess\,inf}_{t \in [0,1]} |v_i(t)| > 0$ for any $i = 1,2,\ldots,n$ where $v_1, v_2, \ldots, v_n \subset L^\infty$ are linearly independent. It is clear that in this case we could choose $J = I$.

Example 2. We shall now consider a periodic control problem for system (1). We consider the nonlinear boundary value problem

$$\begin{cases} \dot{x} = f(t,x(t),u(t)) & \text{for a.a. } t \in [0,1] \\ \int_0^1 f(t,x(t),u(t))dt = 0 & \\ \int_0^1 k(t,u(t))dt = 0 & \end{cases}$$

where $f : [0,1] \times R^n \times R^m \to R^n$, $k : [0,1] \times R^m \to R^d$, with $d \geq 1$, are functions satisfying the Carathéodory condition (f_1). Assume we have defined a control space $U \subset (L^\infty)^m$, with $\dim U = d$, satisfying property (P) and suppose that the following conditions hold

(d_1) $\quad |f(t,p,q)| \le \alpha(t)|p| + \beta(t)|q|$

where $\alpha,\beta \in L^1$ and $\alpha(t) \ge 0$, $\beta(t) \ge 0$ for a.a. $t \in [0,1]$.

(d_2) there exists $\rho_o > 0$ such that

$$\int_o^1 (b,k(t,J(b)(t))dt > 0$$

for all $b \in R^d$ with $|b| > \rho_o$.

(d_3) Let $r_o < 0$ be such that $|J(b)(t)| \le r_o$ for $|b| \le \rho_o$ and for a.a. $t \in [0,1]$.

Define

$$I_{r_o}(t,p) = \inf_{|q|\le r_o} \frac{(p,f(t,p,q))}{|p|^2}$$

for any $p \in R$ and a.a. $t \in [0,1]$. Assume that

$$\int_o^1 \liminf_{|p|\to+\infty} I_{r_o}(t,p)dt > 0 \quad .$$

Under these Assumptions it can be shown that the set S of Theorem 1 is bounded in $(C)^n \times U$. In fact, Assumption (d_2) implies the boundedness of the controls belonging to the set S and using this fact together with (d_1) and (d_3) we obtain, by the same arguments used in the proof of Theorem 1 of [9], the boundedness of the corresponding periodic solutions. For sake of completeness we sketch the procedure below.

Let x be a 1-periodic solution of the following system

$$\dot{x}(t) = \lambda f(t,x(t),u(t))$$

for a.a. $t \in [0,1]$, for some $u = J(b) \in U$ with $|b| \le \rho_o$ and $\lambda \in [0,1]$. By (d_1) we have

$$\frac{d}{dt}\frac{|x(t)|^2}{2} = \lambda(x(t),f(t,x(t),u(t)) \le \alpha(t)(|x(t)|^2+1) - \alpha(t) + \mu(t)|x(t)|$$

where $\mu(t) = \beta(t)|u(t)|$ so $\mu \in L^1$.

Dividing by $|x(t)|^2+1$ and integrating on the interval $[\tau,t]$, with $\tau \in R$ and $t \in [\tau,\tau+1]$, produces

$$1/2\ [\log\ (|x(t)|^2+1)-\log(|x(\tau)|^2+1)] < \int_0^1 [\alpha(t)\ +\ \frac{\mu(t)|x(t)|}{|x(t)|^2+1}]\,dt = M < +\infty$$

By the 1-periodicity of x one has

$$\log \max_{t\in[0,1]} (|x(t)|^2+1) < \log \min_{t\in[0,1]} (|x(t)|^2+1) + 2\,M$$

Let $\{(x_n,u_n)\}_{n\in N} \subset S_+ = \{(x,u) \in S :\ \lambda \in (0,1]\}$, so $|b_n| = |J^{-1}(u_n)| \le \rho_o$; if we suppose that $\max_{t\in[0,1]} |x_n(t)| \to +\infty$ as $n \to +\infty$ then $\min_{t\in[0,1]} |x_n(t)| \to +\infty$ as $n \to +\infty$. Since the corresponding $\{\lambda_n\}_{n\in N}$ is contained in $(0,1]$ we have for n sufficiently large

$$\int_0^1 \frac{(x_n(t),f(t,x_n(t),u_n(t))}{|x_n(t)|^2}\,dt = 0$$

Hence

$$0 \ge \liminf_{n\to+\infty} \int_0^1 I_{r_o}(t,x_n(t))\,dt \tag{3}$$

But $|x_n(t)| \to +\infty$ as $n \to +\infty$ for all $t \in [0,1]$, hence

$$\liminf_{n\to+\infty} I_{r_o}(t,x_n(t)) \ge \liminf_{|p|\to+\infty} I_{r_o}(t,p)$$

and using Fatou's Lemma in (3) we see that (d_3) is violated.

If $\lambda = 0$, let $\{(c_n,u_n)\}_{n\in N} \subset S_o$, so $|b_n| = |J^{-1}(u_n)| \le \rho_o$, and suppose that $|c_n| \to +\infty$. From the fact that

$$\int_0^1 f(t,c_n,u_n(t))\,dt = 0$$

for all $n \in N$, we obtain

$$\int_0^1 \frac{(c_n,f(t,c_n,u_n(t)))}{|c_n|^2}\,dt = 0$$

for all $n \in N$. This fact, as proved above, contradicts (d_3).

Finally, it is easy to see that the homotopy $Q:[0,1]\times R^n \times R^d \to R^{n+d}$ defined as follows

$$Q(\lambda,a,b) = \left(\int_0^1 f(t,a,J(\lambda b)(t))dt,\ \lambda\int_0^1 k(t,J(b)(t))dt + (1-\lambda)b\right)$$

is admissible. Therefore

$$\deg\,(Q(1,a,b),V,0) = \deg\,(Q(0,a,b),V,0)$$

where V is an open bounded set containing the set

$$T = \{(a,b) \in R\times R : Q(\lambda,a,b) = 0 \ \text{ for some } \ \lambda \in [0,1]\}$$

But $\deg\,(Q(0,a,b),V,0) \neq 0$, since $Q(0,a,b) = \left(\int_0^1 f(t,a,0)dt, b\right)$ and $\left(a, \int_0^1 f(t,a,0)dt\right) + (b,b) > 0$ if $|a| + |b|$ is sufficiently large.

Remark 4. If $f(t,p,q) = A(t)p + B(t)q$, then condition (d_3) reduces to

$$\int_0^1 \inf_{|p|=1} (p,A(t)p)dt > 0$$

which implies that the only periodic solution of the system $\dot{x}(t) = A(t)x(t)$ is the zero one. Therefore the problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(0) = x(1) \end{cases}$$

is solvable for any control u belonging to a bounded subset of U.

Lastly, we mention that we could modify the assumptions on f and k as illustrated in F_2 of Remark 3.

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Ricevuto il 5/11/1986

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