JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 45, No. 2, FEBRUARY 1985

Topological Methods for the Global Controllability of Nonlinear Systems

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Communicated by R. Conti

Abstract. Sufficient conditions for the local and global controllability of general nonlinear systems, by means of controls belonging to a fixed finite-dimensional subspace of the space of all admissible controls, are established with the aid of topological methods, such as homotopy invariance principles. Some applications to certain classes of nonlinear control processes are given, and various known results on the controllability of perturbed linear systems are also derived as particular cases.

Key Words. Nonlinear control processes, local and global controllability, continuation principles.

1. Introduction

Consider the nonlinear control process

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad t \in J = [0, T], \tag{1}$$

where

$$f: J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

is a continuous function or, more generally, it satisfies some Carathéodory-type conditions that will be specified later.

Our aim is to apply topological methods, such as degree theory or homotopy invariance results, to obtain sufficient conditions for the global and local controllability of (1). Roughly speaking, we will show that, if (1)

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0022-3239/85/0200-0231$04.50/0

can be deformed in some admissible way to a linear system by means of a suitable homotopy, then it is globally controllable. More precisely, our method can be described as follows.

Given the one-parameter family of nonlinear control systems

$$\dot{x}(t) = F_\lambda(t, x(t), u(t)), \qquad t \in J, \tag{2}$$

which depends continuously on $\lambda \in [0, 1]$, restrict the attention to some n-dimensional subspace U of the space of admissible controls. Assume that the *starting system*

$$\dot{x}(t) = F_0(t, x(t), u(t))$$

is linear and admits a control $u_0 \in U$ which steers a point $x_0 \in \mathbb{R}^n$ to a point $x_1 \in \mathbb{R}^n$ at a fixed time T. Then, a steering control will exist also for the nonlinear system

$$\dot{x}(t) = F_1(t, x(t), u(t)),$$

provided that the solution pairs (x, u) of (2) which satisfy the boundary conditions

$$x(0) = x_0, \qquad x(T) = x_1$$

are *a priori* bounded.

This method will permit us to deduce, as particular cases, some known results about the global controllability of nonlinear systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f_1(t, x(t), u(t)), \qquad t \in J, \tag{3}$$

where the perturbing function f_1 is continuous and satisfies some less than linear growth conditions (see Refs. 1-7). In this case, in fact, an appropriate homotopy will be given by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \lambda f_1(t, x(t), u(t)), \qquad \lambda \in [0, 1], t \in J. \tag{4}$$

We will prove, however, that the sublinear assumption on the perturbing term f_1 may be relaxed in many cases, especially when the asymptotic behavior of f_1 is coherent with that of the linear part of system (3).

To get an idea of what kind of n-dimensional space of controls one could look for in order to deal with a homotopy joining a nonlinear control process to a linear one, the reader is referred to Ref. 8, where it is proved that, if a linear system such as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad t \in J,$$

is completely controllable, then it is also completely controllable by means of an appropriate n-dimensional space of piecewise constant controls with

at most n fixed switching times. It seems that such a kind of space is particularly suitable in order to establish *a priori* bounds for the solution pairs (x, u) of nonlinear processes.

In what follows, $(AC)^k$, $k \in \mathbb{N}$, will denote the Banach space of absolutely continuous functions $x: J \to \mathbb{R}^k$ with the norm

$$\|x\| = |x(0)| + \int_0^T |\dot{x}(t)| \, dt.$$

The space $L^p(J, \mathbb{R}^k)$, $1 \leq p \leq \infty$, will be briefly indicated by $(L^p)^k$.

2. Controllability of Nonlinear Systems

Let us consider the nonlinear control process

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad t \in J, \tag{5}$$

where

$$f: J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

satisfies the following Carathéodory-type conditions:

(F1) for each $(p, q) \in \mathbb{R}^n \times \mathbb{R}^m$, the mapping $t \mapsto f(t, p, q)$ is Lebesgue measurable on J; for almost all $t \in J$, the mapping $(p, q) \mapsto f(t, p, q)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m$;

(F2) for each $\rho > 0$ there exists $\gamma_\rho \in L^1$ such that for almost all $t \in J$ and every (p, q) with $(|p| + |q|) \leq \rho$, one has

$$|f(t, p, q)| \leq \gamma_\rho(t).$$

Let $x_0, x_1 \in \mathbb{R}^n$ be given. The system (5) is said to be *controllable from x_0 to x_1* if there exists a control $u \in (L^\infty)^m$ which steers a solution $x \in (AC)^n$ of (5), with $x(0) = x_0$, to the final state $x(T) = x_1$. Moreover, (5) is said to be *locally controllable* from x_0 to x_1 if there exist two neighborhoods $N(x_0)$ and $N(x_1)$ of x_0 and x_1, respectively, such that (5) is controllable from any point of $N(x_0)$ to any point of $N(x_1)$. Furthermore, if the system is controllable for any $x_0, x_1 \in \mathbb{R}^n$, then it is called *globally controllable* (or completely controllable, if it is linear).

Our purpose here is to provide conditions which enable us to obtain the controllability of the nonlinear system (5) by means only of controls which belong to an n-dimensional subspace of $(L^\infty)^m$.

In our approach, the basic idea consists, roughly, in transforming the nonlinear problem to a completely controllable linear one by means of a *good homotopy*, that is, a homotopy which preserves suitable *a priori* bounds.

More precisely, let U be an n-dimensional subspace of $(L^\infty)^m$, Ω a bounded open subset of U with closure $\bar{\Omega}$ and boundary $\partial\Omega$; and let $A(\cdot)$, $B(\cdot)$ be time-dependent $n \times n$, $n \times m$ matrices with coefficients in L^1. For given $x_0, x_1 \in \mathbb{R}^n$, consider the system

$$\dot{x}(t) = \lambda f(t, x(t), u(t)) + (1-\lambda)(A(t)x(t) + B(t)u(t)), \qquad \lambda \in [0, 1], t \in J,$$
$$x(0) = x_0, \tag{6}$$

and define

$S = \{(x, u) \in (AC)^n \times \bar{\Omega}$: for some $\lambda \in [0, 1]$, x is a solution of (6), corresponding to u, which satisfies $x(T) = x_1\}$.

We can state the following theorem.

Theorem 2.1. Let U, Ω, $A(\cdot)$, $B(\cdot)$, S be as above. Assume that the linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad t \in J,$$
$$x(0) = x_0,$$
$$x(T) = x_1$$

is solvable in $(AC)^n \times \Omega$. Also, assume that the following *a priori* bounds can be established:

(i) there exists a constant $M > 0$, independent of λ, such that

$$(\max_{t \in J} |x(t)| + \|u\|_\infty) \le M, \qquad \text{for all } (x, u) \in S;$$

(ii) the set $\{u \in \bar{\Omega}: (x, u) \in S \text{ for some } x \in (AC)^n\}$ does not intersect $\partial\Omega$.

Then, the system (5) is controllable from x_0 to x_1.

The proof of the above theorem is essentially based on the well-known Leray-Schauder continuation principle (Ref. 9). We recall it below in the form needed for our purposes.

Leray–Schauder Continuation Principle. Let E, F be Banach spaces, and $\mathcal{O}$ an open subset of E containing the origin. Let $L: E \to F$ be an isomorphism, and let $h: \bar{\mathcal{O}} \to F$ be continuous and compact (i.e., sending bounded subsets of $\bar{\mathcal{O}}$ into relatively compact subsets of F). Assume that the set

$$\{x \in \bar{\mathcal{O}}: Lx = \lambda h(x), \text{ for some } \lambda \in [0, 1]\}$$

is bounded and does not intersect the boundary of $\mathcal{O}$. Then, the equation

$$Lx = h(x)$$

is solvable in $\mathcal{O}$. More precisely, $L - h$ maps $\mathcal{O}$ onto a neighborhood of the origin.

The Leray-Schauder continuation principle is usually proved by means of degree arguments (Ref. 9). However, we would like to remark that it can also be derived directly from Schauder's fixed-point theorem, making use of the quite elementary theory of zero-epi maps (Refs. 10 and 11).

Proof of Theorem 2.1. Let

$$L: (AC)^n \times U \to (L^1)^n \times \mathbb{R}^n \times \mathbb{R}^n$$

be the bounded linear operator given by

$$L(x, u)(t) = (\dot{x}(t) - A(t)x(t) - B(t)u(t), x(0), x(T)).$$

Observe that the equation

$$L(x, u) = (0, x_0, x_1),$$

which, by assumptions, is solvable in $(AC)^n \times \Omega$, has in fact a unique solution because of (i) and (ii). Consequently,

$$L(x, u) = (0, 0, 0),$$

if and only if

$$(x, u) = (0, 0).$$

Moreoever, if $K: U \to \mathbb{R}^n$ is the bounded linear operator which associates to any control u the value in $t = T$ of the unique solution of the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad t \in J,$$
$$x(0) = 0,$$

one has

$$Ku = 0,$$

if and only if

$$u = 0.$$

So, K is one-to-one, and thus onto. Therefore, the above linear system is completely controllable, so that L is also onto. Hence, L is an isomorphism.

Let us consider now the Nemitskii operator

$$N: (AC)^n \times U \to (L^1)^n,$$

defined by

$$N(x, u)(t) = f(t, x(t), u(t)) - A(t)x(t) - B(t)u(t).$$

By assumptions (F1), (F2), and the Lebesgue's dominated convergence theorem, it follows easily that N is continuous. We will show that N is also compact.

Let $\{(x_n, u_n)\}_{n\in\mathbb{N}}$ be a bounded sequence of $(AC)^n \times U$. Since $(AC)^n$ is compactly imbedded in $(L^1)^n$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ has a subsequnece $\{x_{n_k}\}_{k\in\mathbb{N}}$, such that

$$x_{n_k} \to x_0, \quad \text{in } (L^1)^n,$$

$$x_{n_k}(t) \to x_0(t), \quad \text{a.e. in } J.$$

Moreover, U being finite dimensional, without loss of generality, we may also assume that

$$u_{n_k} \to u_0, \quad \text{in } (L^\infty)^m,$$

$$u_{n_k}(t) \to u_0(t), \quad \text{a.e. in } J.$$

Hence, by using again (F1), (F2), and Lebesgue's theorem, we get

$$\int_0^T |f(t, x_{n_k}(t), u_{n_k}(t)) - A(t)x_{n_k}(t) - B(t)u_{n_k}(t) - f(t, x_0(t), u_0(t)) + A(t)x_0(t) + B(t)u_0(t)|\, dt \to 0,$$

that is, $N(x_{n_k}, u_{n_k})$ converges to $N(x_0, u_0)$ in $(L^1)^n$. This proves that N is compact.

Observe that system (6), together with the boundary condition $x(T) = x_1$, can be written in the abstract form

$$L(x, u) - y = \lambda h(x, u), \qquad \lambda \in [0, 1], \tag{7}$$

where L is the isomorphism defined above,

$$y = (0, x_0, x_1),$$

and

$$h : (AC)^n \times U \to (L^1)^n \times \mathbb{R}^n \times \mathbb{R}^n$$

is the compact map given by

$$h(x, u) = (N(x, u), 0, 0).$$

By the change of variables

$$\xi = x - \tilde{x}, \qquad v = u - \tilde{u},$$

$(\tilde{x}, \tilde{u})$ being the unique solution of

$$L(x, u) = y,$$

Eq. (7) can be transformed into

$$L(\xi, v) = \lambda h(\xi + \tilde{x}, v + \tilde{u}), \qquad \lambda \in [0, 1]. \tag{8}$$

Therefore, the solvability of (7) in $(AC)^n \times \Omega$ is equivalent to the solvability of (8) in $(AC)^n \times (\Omega - \tilde{u})$. Notice that $\Omega - \tilde{u}$ is still an open set and contains the origin.

Thus, according to the Leray-Schauder continuation principle, Eq. (7) is solvable in $(AC)^n \times \Omega$ provided that the set

$$\{(x, u) \in (AC)^n \times \bar{\Omega} : L(x, u) - y = \lambda h(x, u), \text{ for some } \lambda \in [0, 1]\},$$

which is nothing but the set S in the statement of Theorem 2.1, is bounded in $(AC)^n$ with respect to x and does not intersect $(AC)^n \times \partial\Omega$. The second condition is clearly ensured by assumption (ii). For the first condition, it suffices to observe that, since by (i) there exists $M > 0$ such that

$$(\max_{t\in J} |x(t)| + \|u\|_\infty) \leq M, \qquad (x, u) \in S,$$

one has

$$|\dot{x}(t)| \leq |\lambda f(t, x(t), u(t))| + |(1-\lambda)(A(t)x(t) + B(t)u(t))|$$
$$\leq \gamma_M(t) + |A(t)x(t)| + |B(t)u(t)|, \qquad \text{a.e. in } J,$$

which implies that

$$\|x\| \leq \|\gamma_M\|_1 + C_1 (\max_{t\in J} |x(t)| + \|u\|_\infty) \leq C_2.$$

Thus,

$$L(x, u) = h(x, u)$$

has a solution in $(AC)^n \times \Omega$; that is, system (5) is controllable from x_0 to x_1. □

Remark 2.1. Assumptions (i) and (ii) imply, in particular, that there exists a unique control $\tilde{u} \in U$, which steers the solution x of the linear problem

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad t \in J,$$

$$x(0) = x_0$$

to the final state x_1. Consequently, the linear system is completely controllable with controls belonging to U. In other words, the fact that one can find a linear system which is homotopic to a nonlinear one in such a way that, during the homotopy, all the possible zeros remain uniformly bounded turns out to be sufficient to obtain a precise information even about the controllability of the linear system itself.

Remark 2.2. In many cases, it seems to be useful to choose as U an appropriate space of piecewise constant controls with fixed (i.e., depending only on the space) switching times. More precisely, if χ^i_τ, $\tau \in J$, $i = 1, 2, \ldots, m$, denotes the step function

$$\chi^i_\tau(t) = \chi_{[0,\tau]}(t) e_i,$$

where $\chi_{[0,\tau]}$ is the characteristic function of the interval $[0, \tau]$ and $\{e_i : i = 1, 2, \ldots, m\}$ is the standard basis of $\mathbb{R}^m$, we may take U to be an n-dimensional space of the form (see Ref. 8)

$$\operatorname{span}\{\chi^{i_1}_{t_1}, \ldots, \chi^{i_n}_{t_n}\}, \qquad t_l \in (0, T], \qquad i \in \{1, 2, \ldots, m\}.$$

Sometimes, U can also be taken as the n-dimensional kernel of a suitable linear operator on $(L^\infty)^m$; see, e.g., Example 2.2.

Remark 2.3. In studying nonlinear processes, one has sometimes to deal with the controllability of nonlinear operators which are more general than the usually considered

$$x \mapsto x(0), \qquad x \mapsto x(T).$$

For instance, given

$$\Phi_0 : (AC)^n \to \mathbb{R}^n, \qquad \Phi : (AC)^n \to \mathbb{R}^r$$

bounded continuous nonlinear operators, one may need to investigate whether the boundary-value problem

$$\dot{x}(t) = f(t, x(t), u(t)); \qquad t \in J,$$

$$\Phi_0(x) = x_0$$

is Φ-controllable to $x_1 \in \mathbb{R}^r$, that is, whether there exists a control u and a solution x of the boundary-value problem satisfying

$$\Phi(x) = x_1.$$

It is not hard to verify that, for Φ-controllability, the analogue of Theorem 2.1 holds, where, in this case, S will be the set of the pairs

$$(x, u) \in (AC)^n \times \bar{\Omega},$$

such that, for some $\lambda \in [0, 1]$, x is a solution, corresponding to u, of the

problem

$$\dot{x}(t) = \lambda f(t, x(t), u(t)) + (1-\lambda)(A(t)x(t) + B(t)u(t)), \qquad t \in J,$$

$$\lambda \Phi_0(x) + (1-\lambda) C_0 x = x_0,$$

$$\lambda \Phi(x) + (1-\lambda) C x = x_1,$$

with $\dim U = p$ and $C_0 : (AC)^n \to \mathbb{R}^n$, $C : (AC)^n \to \mathbb{R}^r$ bounded, surjective linear operators.

It should be observed that the assumptions of Theorem 2.1 imply, besides the solvability of the equation

$$L(x, u) - h(x, u) = y,$$

also the local surjectivity of the map $L - h$ at y. Therefore, the following result holds.

Corollary 2.1. Assume that all the hypotheses of Theorem 2.1 are satisfied. Then, system (5) is locally controllable from x_0 to x_1.

We will state now a global controllability result which is obtained by slightly strengthening assumptions (i) and (ii) of Theorem 2.1.

Theorem 2.2. Let U, $A(\cdot)$, $B(\cdot)$ be as in Theorem 2.1. Assume that:

(iii) for any $R \geq 0$, there exists a positive constant M_R, independent of λ, such that

$$(\max_{t \in J} |x(t)| + \|u\|_\infty) \leq M_R$$

for all $(x, u) \in (AC)^n \times U$ which solve, for some $\lambda \in [0, 1]$, the system

$$\dot{x}(t) = \lambda f(t, x(t), u(t)) + (1-\lambda)(A(t)x(t) + B(t)u(t)), \qquad t \in J,$$

$$x(0) = x_0,$$

$$x(T) = x_1, \qquad (|x_0| + |x_1|) \leq R.$$

Then, the system (5) is globally controllable.

Proof. Choose any x_0, $x_1 \in \mathbb{R}^n$, and let

$$R = |x_0| + |x_1|.$$

Observe now that assumption (iii) implies (i) and (ii) of Theorem 2.1 by taking as Ω any open ball in U centered at the origin and with radius strictly greater than M_R. Consequently, system (5) is controllable from x_0 to x_1. □

We give now an application of the above results to the class of perturbed linear systems, i.e., those of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f_1(t, x(t), u(t)), \qquad t \in J. \tag{9}$$

The problem of the controllability of this type of systems, under hypotheses on the perturbation f_1 which make possible the use of a fixed-point method, has been examined by several authors (Refs. 1-7, 12 and 13, and references therein).

More precisely, sufficient conditions on f_1 have been given in such a way that, if the linear part of the system (9) is completely controllable, then system (9) itself is controllable. Because of the topological approach developed, these assumptions on f_1 turn out to be less than linear growth conditions and are obtained by solving a system of nonlinear integral equations. Analogous hypotheses have also been made to derive controllability of perturbed quasilinear systems (Refs. 1 and 13-17). It should be observed that our method is quite different from the fixed-point techniques used in the above papers: instead of adding an equation, we restrict the dimension of the control space. Corollary 2.2 below, which has been established by Dauer (Ref. 1), is a consequence of Theorem 2.2. Its proof will exemplify the differences between Dauer's approach and ours.

Corollary 2.2. Assume that the function f_1 satisfies (F1), (F2) and the further condition

$$\text{(C)} \qquad \lim_{(|p|+|q|)\to\infty} [|f_1(t, p, q)|/(|p|+|q|)] = 0,$$

uniformly for $t \in J$. If the linear part of system (9) is completely controllable, then system (9) is globally controllable.

Proof. By assumption, the linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad t \in J,$$

is completely controllable. Thus, we can restrict ourselves to consider controls belonging to a fixed n-dimensional subspace U of $(L^\infty)^m$. Consequently, since, as proved in Theorem 2.1, the linear operator

$$L(x, u)(t) = (\dot{x}(t) - A(t)x(t) - B(t)u(t), x(0), x(T))$$

is an isomorphism between $(AC)^n \times U$ and $(L^1)^n \times R^n \times \mathbb{R}^n$, there exists a constant $C > 0$ such that

$$C(\max_{t\in J} |x(t)| + \|u\|_\infty) \leq (\|\dot{x} - A(\cdot)x - B(\cdot)u\|_1 + |x(0)| + |x(T)|). \tag{10}$$

Now, in order to verify assumption (iii) of Theorem 2.2, observe that, for

any $\epsilon > 0$, there exists a constant $\rho = \rho(\epsilon)$ such that, for

$$(|p|+|q|) > \rho \quad \text{and} \quad \text{a.e. } t \in J,$$

we have

$$|f_1(t, p, q)| \leq \epsilon(|p|+|q|).$$

Hence, because of (F2),

$$|f_1(t, p, q)| \leq \gamma_\rho(t) + \epsilon(|p|+|q|), \qquad \text{for a.a. } t \in J. \tag{11}$$

Therefore, if $(x, u) \in (AC)^n \times U$ is any solution of the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \lambda f_1(t, x(t), u(t)), \qquad \lambda \in [0, 1],\ t \in J,$$

$$x(0) = x_0,$$

$$x(T) = x_1,$$

with

$$(|x_0| + |x_1|) \leq R,$$

from (10) and (11) we obtain

$$C(\max_{t\in J} (|x(t)| + \|u\|_\infty)) \leq (\lambda \|f_1(\cdot, x, u)\|_1 + |x_0| + |x_1|)$$

$$\leq \|\gamma_\rho\|_1 + R + \epsilon(\max_{t\in J} |x(t)| + \|u\|_\infty). \tag{12}$$

This inequality clearly implies the required *a priori* bound (iii) provided $\epsilon < C$. □

Remark 2.4. Condition (C) is implied, for instance, by the following condition:

(C′) There exist $\rho \geq 0$, $\alpha \in [0, 1)$, $\delta \in L^1$, such that

$$|f_1(t, p, q)| \leq \delta(t)(|p|+|q|)^\alpha, \quad (|p|+|q|) > \rho, \qquad \text{a.a. } t \in J.$$

When f_1 is bounded, we obtain, as particular cases, some of the results established in Refs. 1-7 for perturbed linear systems of the form (9).

Observe also that assumption (C) can be replaced by the weaker one (Ref. 14):

(C″) There exist $\rho \geq 0$ and $\delta \in L^1$, such that

$$|f_1(t, p, q)| \leq \delta(t)(|p|+|q|), \qquad (|p|+|q|) > \rho, \qquad \text{a.a. } t \in J,$$

$$\|\delta\|_1 < C,$$

C being the constant of Ineq. (10) above.

As already pointed out, in the previous theorems we are always concerned with the assumptions involving *a priori* bounds type conditions. It is not hard to convince oneself that, usually, in order to verify this sort of conditions, most of the difficulty arises in proving that x's are uniformly bounded. For this reason, it is of some interest to investigate which kind of hypothesis enables to deduce the boundedness of the x's from the boundedness of the controls. In this direction, we have the following theorem.

Theorem 2.3. Assume that, for each $u \in \bar{\Omega}$ and $\lambda \in [0, 1]$, the solutions $x(\cdot, u, \lambda)$ of the initial-value problem (6) are equibounded in J, i.e., there exists a constant $C \equiv C(u, \lambda)$, independent of x, such that

$$\max_{t \in J} |x(t)| \leq C.$$

Then, assumption (i) of Theorem 2.1 is satisfied. The same conclusion holds, in particular, when, for any $u \in \bar{\Omega}$ and $\lambda \in [0, 1]$, system (6) has a unique solution x defined on the whole interval J.

In order to prove the above theorem, we need to know how the solutions of problem (6) depend on the control u and the parameter λ. A classical result concerning the dependence of the solutions on parameters and initial conditions in the case where the right-hand side of the initial-value problem is a continuous function of its arguments and, for any value of the parameter, the problem has a unique solution, can be found, for instance, in Ref. 18. We will give here a generalized version of this result for Carathéodory right-hand sides and nonuniqueness of the solutions.

Theorem 2.4. Let

$$\{g_n\}_{n \in \mathbb{N}}, \quad g_n : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n,$$

be a sequence of functions such that:

(i) for a.a. $t \in [a, b]$ and all $n \in \mathbb{N}$, the function $p \mapsto g_n(t, p)$ is continuous;

(ii) for all $p \in \mathbb{R}^n$ and all $n \in \mathbb{N}$, the function $t \mapsto g_n(t, p)$ is Lebesgue measurable on $[a, b]$;

(iii) for each $\rho > 0$, there exists $\gamma_\rho \in L^1([a, b])$ such that, for a.a. $t \in [a, b]$, for all p with $|p| \leq \rho$, for any $n \in \mathbb{N}$,

$$|g_n(t, p)| \leq \gamma_\rho(t).$$

Let $\{(t_n, a_n)\}_{n \in \mathbb{N}}$ be a sequence in $[a, b] \times \mathbb{R}^n$ converging to a point $(\bar{t}, \bar{a})$; and let x_n be any noncontinuable, absolutely continuous solution of the initial-value problem

$$\dot{x}(t) = g_n(t, x(t)), \qquad t \in [a, b],$$
$$x(t_n) = a_n.$$

Assume, in addition, that there exists a function $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$, such that:

(iv) for a.a. $t \in [a, b]$, the sequence $\{p \mapsto g_n(t, p)\}_{n \in \mathbb{N}}$ converges uniformly to $p \mapsto g(t, p)$ on any compact subset of $\mathbb{R}^n$;

(v) any absolutely continuous solution of the initial problem

$$\dot{x}(t) = g(t, x(t)), \qquad t \in [a, b], \qquad x(\bar{t}) = \bar{a} \tag{13}$$

is defined on $[a, b]$; moreover there exists a constant $C > 0$ such that

$$\max_{t \in [a, b]} |x(t)| \leq C,$$

for all solutions x of (13).

Then, there exists $\bar{n} \in \mathbb{N}$ such that, for any $n \geq \bar{n}$, x_n is defined in $[a, b]$ and the sequence $\{x_n\}_{n \geq \bar{n}}$ has a subsequence converging uniformly in $[a, b]$ to a solution of (13).

Proof. Set

$$\rho = C + 1,$$

and define

$$F : [a, b] \to \mathbb{R}$$

by

$$F(t) = \int_{\bar{t}}^{t} \gamma_\rho(s) \, ds.$$

Let $\delta > 0$ be such that, if

$$t, t' \in [a, b] \quad \text{and} \quad |t - t'| < 2\delta,$$

then

$$|F(t) - F(t')| < \tfrac{1}{2}.$$

We will show that, for n large enough, x_n is defined in $[\bar{t} - \delta, \bar{t} + \delta] \cap [a, b]$ and $|x_n(t)| < \rho$ for any t in this interval.

Suppose that, for any $n \in \mathbb{N}$, there exists $\tau_n \in \mathbb{R}$ such that

$$|\tau_n| < 2\delta, \qquad |x_n(t_n + \tau_n)| = \rho,$$
$$|x_n(t_n + t)| < \rho, \qquad |t| < |\tau_n|.$$

Hence,

$$1 \le |\rho - |\bar{a}|| \le |\rho - |a_n|| + ||a_n| - |\bar{a}||$$
$$\le |x_n(t_n + \tau_n) - x_n(t_n)| + |a_n - \bar{a}|$$
$$= \left|\int_{t_n}^{t_n+\tau_n} g_n(t, x_n(t))\, dt\right| + |a_n - \bar{a}|$$
$$\le |F(t_n + \tau_n) - F(t_n)| + |a_n - \bar{a}| < 1, \qquad \text{if } |a_n - \bar{a}| < \tfrac{1}{2},$$

a contradiction. Thus, for n large enough, x_n is defined in $[t_n - 2\delta, t_n + 2\delta] \cap [a, b]$, so that for $|t_n - t| < \delta$, x_n is defined in $[\bar{t} - \delta, \bar{t} + \delta] \cap [a, b]$. Moreover, the functions x_n are equicontinuous in this interval, since it results that

$$|x_n(t) - x_n(t')| \le |F(t) - F(t')|, \qquad \text{for any } t, t' \in [\bar{t} - \delta, \bar{t} + \delta] \cap [a, b].$$

Hence, by Ascoli's theorem, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ which converges uniformly to a continuous function $\bar{x}$ in $[\bar{t} - \delta, \bar{t} + \delta] \cap [a, b]$. We have

$$x_{n_k}(t) = a_{n_k} + \int_{\bar{t}}^{t} g_{n_k}(s, x_{n_k}(s))\, ds - \int_{\bar{t}}^{t_{n_k}} g_{n_k}(s, x_{n_k}(s))\, ds.$$

By (iii) and (iv), for a.a. s,

$$|g_{n_k}(s, x_{n_k}(s))| \le \gamma_\rho(s),$$

and $g_{n_k}((s), x_{n_k}(s))$ converges to $g(s, \bar{x}(s))$ as $k \to \infty$. Therefore, passing to the limit, we obtain

$$\bar{x}(t) = \bar{a} + \int_{\bar{t}}^{t} g(s, \bar{x}(s))\, ds.$$

This proves the existence of an interval $[\bar{t}, \tau]$ having the property below:

(P) The solutions x_n are defined in $[\bar{t}, \tau]$ for n large enough, and $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence converging uniformly to a solution of (13) in this interval.

Suppose that $[\bar{t}, b]$ does not satisfy the above property, and let

$$t^* = \sup\{\tau : [\bar{t}, \tau] \text{ satisfies } (P)\}.$$

Let

$$\tilde{t} = \max\{t^* - \tfrac{1}{2}\delta, \bar{t} + \delta\}.$$

Then, there exists a subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ which converges uniformly to a solution $\tilde{x}$ of (13) in $[\bar{t}, \tilde{t}]$. Set

$$a'_k = x_{n_k}(\tilde{t}), \qquad a' = \tilde{x}(\tilde{t}).$$

Clearly, $\{(\tilde{t}, a'_k)\}_{k\in\mathbb{N}}$ is a sequence of points converging to $(\tilde{t}, a')$. Hence, the same argument used above shows that, for $k \in \mathbb{N}$ large enough, x_{n_k} is defined in $[\tilde{t}, \tilde{t} + \delta]$, and thus in $[\bar{t}, \tilde{t} + \delta]$, and $\{x_{n_k}\}_{k\in\mathbb{N}}$ has a subsequence, again denoted by $\{x_{n_k}\}_{k\in\mathbb{N}}$, which converges to a solution $\hat{x}$ of (13) in $[\tilde{t}, \tilde{t} + \delta]$. Therefore, the function

$$\xi(t) = \begin{cases} \tilde{x}(t), & t \in [\bar{t}, \tilde{t}], \\ \hat{x}(t), & t \in [\tilde{t}, \tilde{t} + \delta], \end{cases}$$

is a solution of (13), and $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges to ξ in $[\bar{t}, \tilde{t} + \delta]$, contradicting the definition of t^*. Thus, the interval $[\bar{t}, b]$ satisfies (P). Clearly, the same holds also for $[a, \bar{t}]$. So, the assertion is proved. □

We are now in a position to give the proof of Theorem 2.3.

Proof of Theorem 2.3. We will, actually, prove that the set S considered in Theorem 2.1 is compact in the $(C^0)^n \times (L^\infty)^m$ topology. To show this, let $\{(x_n, u_n)\}_{n\in\mathbb{N}}$ be any sequence in S. For each $n \in \mathbb{N}$, there exists $\lambda_n \in [0, 1]$ such that x_n is a solution of (6) corresponding to the control u_n and to λ_n.

Without loss of generality, we may assume that

$$u_n \to u_0 \in \bar{\Omega} \text{ in } (L^\infty)^m, \qquad \lambda_n \to \lambda_0 \in [0, 1].$$

Define

$$g_n : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, \qquad n \in \mathbb{N} \cup \{0\},$$

by

$$g_n(t, p) := \lambda_n f(t, p, u_n(t)) + (1 - \lambda_n)(A(t)p + B(t)u_n(t)).$$

It is not hard to see that the g_n's satisfy all the assumptions of Theorem 2.4. Therefore, the sequence $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence converging uniformly to a solution $\xi_0(\cdot, u_0, \lambda_0)$ of problem (6). In particular,

$$\xi_0(T) = x_1,$$

so that (ξ_0, u_0) belongs to S. Hence, S is compact and, thus, bounded in the $(C^0)^n \times (L^\infty)^m$ topology. □

We will illustrate below a simple situation to which Theorem 2.3 applies. A function $\omega : J \times \mathbb{R} \to \mathbb{R}$ will be called a *Carathéodory function* if $\omega(t, r)$ is measurable in t for all r and is continuous in r for a.a. $t \in J$, and if, for any $R > 0$, there exists $\gamma_R \in L^1$ such that

$$|\omega(t, r)| \le \gamma_R(t), \quad |r| \le R, \qquad \text{a.a. } t \in J.$$

Corollary 2.3. Assume that, for any $\lambda \in [0, 1]$ and $\rho > 0$, there exists a Carathéodory function $\omega_\rho^\lambda : J \times \mathbb{R} \to \mathbb{R}$, nondecreasing with respect to the second variable, such that

$$|\lambda f(t,p,q)+(1-\lambda)(A(t)p+B(t)q)|\le\omega_\rho^\lambda(t,|p|),$$
$$|q|\le\rho,\qquad \text{a.a. } t\in J,\ p\in\mathbb{R}^n.$$

Assume, in addition, that the maximal solution of the initial-value problem

$$\dot\eta(t)=\omega_\rho^\lambda(t,\eta(t)),\qquad t\in J,$$
$$\eta(0)=|x_0|$$

is continuable up to T. Then, for any $u\in\bar\Omega$ and $\lambda\in[0,1]$, the solutions of (6) are equibounded in J.

Proof. Let $\lambda\in[0,1]$ be fixed, and let ρ be such that

$$\|u\|_\infty\le\rho,\qquad u\in\bar\Omega.$$

For any $\epsilon>0$, let η_ϵ be a solution of

$$\dot\eta(t)=\omega_\rho^\lambda(t,\eta(t)),\qquad t\in J,$$
$$\eta(0)=\epsilon+|x_0|.$$

We will show first that, for each $(u,\lambda)\in\bar\Omega\times[0,1]$, if $x(\cdot,u,\lambda)$ is a solution of problem (6) corresponding to (u,λ), then

$$|x(t)|<\eta_\epsilon(t),$$

for any t for which η_ϵ is defined.

Choose any solution x. Since

$$x(0)=x_0\quad\text{and}\quad\eta_\epsilon(0)>|x_0|,$$

there exists $\tau>0$ such that

$$|x(t)|<\eta_\epsilon(t),\qquad 0\le t<\tau.$$

Assume that there exists $\bar t\ge\tau$ such that

$$|x(\bar t)|=\eta_\epsilon(\bar t),\quad |x(t)|<\eta_\epsilon(t),\qquad 0\le t<\bar t.$$

We have

$$|x(\bar t)|=\left|x_0+\int_0^{\bar t}[\lambda f(t,x(t),u(t))+(1-\lambda)(A(t)x(t)+B(t)u(t))]\,dt\right|$$
$$<\epsilon+|x_0|+\int_0^{\bar t}\omega_\rho^\lambda(t,|x(t)|)\,dt$$
$$\le\epsilon+|x_0|+\int_0^{\bar t}\omega_\rho^\lambda(t,\eta_\epsilon(t))\,dt=\eta_\epsilon(\bar t).$$

Thus,

$$|x(\bar t)|<\eta_\epsilon(\bar t),$$

a contradiction.

On the other hand, from the assumption and Theorem 2.4, it is easily seen that, for ϵ sufficiently small, η_ϵ is defined for all $t\in J$. Consequently, for such ϵ's, any solution x of (6) satisfies

$$|x(t)|<\eta_\epsilon(t),\qquad \text{for all } t\in J.$$

This proves the assertion. □

Remark 2.5. The assumptions of Corollary 2.3 are all satisfied (for instance) when, for any $\rho>0$, there exist functions $\alpha_\rho,\beta_\rho\in L^1$ such that

$$|f(t,p,q)|\le\alpha_\rho(t)+\beta_\rho(t)|p|,$$
$$|q|\le\rho,\qquad \text{a.a. } t\in J,\ p\in\mathbb{R}^n.$$

We close the paper with some examples which illustrate the abstract results given above.

Example 2.1. Consider the second-order differential equation

$$\ddot x(t)=f(t,x(t),u(t)),\qquad t\in[0,1],\tag{14}$$

with $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ continuous and satisfying the following conditions:

(C1) for any $\rho\in\mathbb{R}$, $f(t,p,q)\to+\infty$, as $q\to+\infty$, uniformly for $p\ge\rho$ and $t\in[0,1]$; $f(t,p,q)\to-\infty$, as $q\to-\infty$, uniformly for $p\le\rho$ and $t\in[0,1]$;

(C2) for any $M_0>0$ and $\rho\in\mathbb{R}$, there exist M_1, M_2 such that

$$f(t,p,q)\ge M_1,\qquad p\ge\rho,\ |q|\le M_0,\ t\in[0,1];$$
$$f(t,p,q)\le M_2,\qquad p\le\rho,\ |q|\le M_0,\ t\in[0,1].$$

Our aim is to show that Eq. (14), which is clearly equivalent to the system

$$\dot x(t)=y(t),\qquad t\in[0,1],$$
$$\dot y(t)=f(t,x(t),u(t)),\qquad t\in[0,1],$$

is globally controllable at the time $T=1$ by means of piecewise constant controls of the form

$$u(t)=a\chi_{[0,\frac12]}(t)+b\chi_{(\frac12,1]}(t),\qquad a,b\in\mathbb{R}.$$

To this end, we will prove that (14) can be well transformed into the linear equation

$$\ddot x(t)=x(t)+u(t),\qquad t\in[0,1].$$

More precisely, according to Theorem 2.2, we need to show that, for any

$R \geq 0$, all the possible solutions (x, u) of the system

$$\ddot{x}(t) = \lambda f(t, x(t), u(t)) + (1-\lambda)(x(t)+u(t)), \qquad \lambda \in [0,1],\ t \in [0,1],$$
$$x(0) = \alpha_0, \qquad \dot{x}(0) = \beta_0, \tag{15}$$
$$x(1) = \alpha_1, \qquad \dot{x}(1) = \beta_1,$$

with

$$(|\alpha_0| + |\alpha_1| + |\beta_0| + |\beta_1|) \leq R,$$

are bounded in $C^1([0,1]) \times L^\infty([0,1])$, independently of λ.

Observe that, as regards the x's, it suffices to prove their boundedness in the C^0-norm, since, by the continuity of f, this will imply the boundedness of the $\ddot{x}$'s in L^∞ and, thus, that of the $\dot{x}$'s.

So, assume the existence of a sequence of solutions $\{(x_n, u_n)\}_{n \in \mathbb{N}}$, with

$$u_n = a_n \chi_{[0,\frac{1}{2}]} + b_n \chi_{[\frac{1}{2},1]}, \qquad a_n, b_n \in \mathbb{R},$$

such that

$$(\max_{t \in [0,1]} |x_n(t)| + \|u_n\|_\infty) \to +\infty.$$

Suppose first that there exists $M_0 > 0$ such that

$$(|a_n| + |b_n|) \leq M_0.$$

Thus,

$$\max_{t \in [0,1]} |x_n(t)| \to +\infty.$$

Without loss of generality, we may also assume that there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$ contained in $(0, \frac{1}{2}]$ such that

$$x_n(t_n) = \max_{t \in [0,1]} |x_n(t)|.$$

Consequently, for n large enough, we can find $\tau_n \in [0, t_n)$ such that

$$x_n(\tau_n) = R, \quad x_n(t) \geq R, \qquad \tau_n \leq t < t_n.$$

By Lagrange's theorem, we get

$$x_n(t_n) - x_n(\tau_n) = \dot{x}_n(\bar{t}_n)(t_n - \tau_n), \qquad \bar{t}_n \in (\tau_n, t_n).$$

Hence,

$$\dot{x}_n(\bar{t}_n) \to +\infty.$$

Moreover, using again Lagrange's theorem, we can write

$$\dot{x}_n(\bar{t}_n) = \dot{x}_n(\bar{t}_n) - \dot{x}_n(t_n) = \ddot{x}(\theta_n)(\bar{t}_n - t_n), \qquad \theta_n \in (\bar{t}_n, t_n).$$

Thus,

$$\ddot{x}_n(\theta_n) \to -\infty \quad \text{and} \quad x_n(\theta_n) \geq R,$$

contradicting Assumption (C2).

Therefore, it must be

$$\|u_n\|_\infty \to +\infty.$$

So, we may assume

$$a_n \to +\infty.$$

By (C1), there exists $n_0 \in \mathbb{N}$ such that, for

$$n \geq n_0, \qquad p \geq -\tfrac{3}{2}R, \qquad t \in [0, \tfrac{1}{2}],$$

one has

$$\lambda_n f(t, p, a_n) + (1-\lambda_n)(p + a_n) > 0.$$

Now, if x_n is any solution of (15) corresponding to (u_n, λ_n), then, for $n \geq n_0$,

$$x_n(t) > -\tfrac{3}{2}R, \qquad \text{in } [0, \tfrac{1}{2}].$$

Suppose the contrary. Then, there exists

$$\tau_n \leq \tfrac{1}{2}$$

such that

$$x_n(t) > -\tfrac{3}{2}R, \qquad 0 \leq t < \tau_n,$$
$$x_n(\tau_n) = -\tfrac{3}{2}R.$$

In fact, recall that

$$|x_n(0)| \leq R.$$

But

$$x_n(\tau_n) = x_n(0) + \dot{x}_n(0)\tau_n + \int_0^{\tau_n} (\tau_n - s)[\lambda_n f(s, x_n(s), a_n) + (1-\lambda_n)(x_n(s) + a_n)]\, ds > -R - R\tau_n \geq -\tfrac{3}{2}R,$$

a contradiction. Thus, by (C1),

$$f(t, x_n(t), a_n) \to +\infty,$$

uniformly with respect to $t \in [0, \frac{1}{2}]$, so that

$$\dot{x}_n(\tfrac{1}{2}) \to +\infty, \qquad x_n(\tfrac{1}{2}) \to +\infty.$$

Consider now the behavior of the x_n's in the interval $[\frac{1}{2}, 1]$. An argument similar to the one used in the first part of the proof implies the existence of $\theta_n \in (\frac{1}{2}, 1)$, such that

$$x_n(\theta_n) \to +\infty \quad \text{and} \quad \ddot{x}(\theta_n) \to -\infty.$$

This leads to a contradiction, whether $b_n \to +\infty$ or $\{b_n\}$ is bounded. If $b_n \to -\infty$, then, as previously, we would obtain

$$x_n(\tfrac{1}{2}) \to -\infty,$$

which contradicts the continuity of the x_n's. Hence, the controls, and thus the corresponding λ's, are bounded independently of λ, as required.

Observe finally that, in Eq. (14), if we add to the nonlinearity f any bounded continuous function $f_1 : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, then the perturbed equation

$$\ddot{x}(t) = f(t, x(t), u(t)) + f_1(t, x(t), u(t)), \qquad t \in [0, 1],$$

is still globally controllable, since also the function $f + f_1$ satisfies (C1) and (C2).

Example 2.2. Consider the system

$$\begin{aligned} \dot{x}(t) &= f(t, y(t)), && t \in [0, 1], \\ \dot{y}(t) &= u(t) + g(t, x(t), y(t), u(t)), && t \in [0, 1], \end{aligned} \tag{16}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $g : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous and such that:

(ξ1) $f(t, y) \to +\infty$, as $y \to +\infty$, uniformly in t,
$f(t, y) \to -\infty$, as $y \to -\infty$, uniformly in t;

(ξ2) there exists $C > 0$ such that

$$|g(t, x, y, u)| \le C, \qquad (t, x, y, u) \in [0, 1] \times \mathbb{R}^3.$$

We shall prove that (16) is globally controllable by means of controls u of the form

$$u(t) = a + bt, \qquad a, b \in \mathbb{R}.$$

To this purpose, consider the system

$$\begin{aligned} &\dot{x}(t) = \lambda f(t, y(t)) + (1 - \lambda) y(t), && t \in [0, 1], \\ &\dot{y}(t) = u(t) + \lambda g(t, x(t), y(t), u(t)), && t \in [0, 1], \\ &\ddot{u}(t) = 0, \\ &x(0) = \alpha_0, \quad y(0) = \beta_0, \\ &x(1) = \alpha_1, \quad y(1) = \beta_1, \\ &(|\alpha_0| + |\beta_0| + |\alpha_1| + |\beta_1|) \le R. \end{aligned} \tag{17}$$

As in Example 2.1, we need to show that, for any $R \ge 0$, there exists $M_R > 0$ such that all the solutions (x, y, u) of (17) satisfy

$$(\max_{t \in [0,1]} (|x(t)| + |y(t)|) + \max_{t \in [0,1]} |u(t)|) \le M_R.$$

Clearly, it suffices to consider the case where there exists a sequence

$$\{u_n\}_{n \in \mathbb{N}}, \qquad u_n(t) = a_n + b_n t, \qquad a_n, b_n \in \mathbb{R},$$

such that

$$(|a_n| + |b_n|) \to +\infty.$$

We have

$$\begin{aligned} 2R \ge |y_n(1) - y_n(0)| &= \left| \int_0^1 (u_n(t) + \lambda_n g(t, x_n(t), y_n(t), u_n(t)))\, dt \right| \\ &\ge \left| \int_0^1 u_n(t)\, dt \right| - \int_0^1 |g(t, x_n(t), y_n(t), u_n(t))|\, dt, \end{aligned}$$

so that

$$|a_n + \tfrac{1}{2} b_n| = \left| \int_0^1 u_n(t)\, dt \right| \le C + 2R. \tag{18}$$

Set

$$c_n = a_n + \tfrac{1}{2} b_n,$$

and observe that y_n can be written in the form

$$\begin{aligned} y_n(t) &= y_n(0) + \int_0^t (c_n - \tfrac{1}{2} b_n + b_n s + \lambda_n g(s, x_n(s), y_n(s), u_n(s)))\, ds \\ &= \sigma_n(t) - b_n v(t), \end{aligned}$$

where

$$|\sigma_n(t)| \le 2C + 3R$$

and

$$v(t) = \tfrac{1}{2} t(1 - t).$$

Thus,

$$\begin{aligned} x_n(1) - x_n(0) = \int_0^1 \dot{x}_n(t)\, dt &= \lambda_n \int_0^1 f(t, \sigma_n(t) - b_n v(t))\, dt \\ &\quad + (1 - \lambda_n) \int_0^1 (\sigma_n(t) - b_n v(t))\, dt. \end{aligned} \tag{19}$$

Clearly, (18) implies that

$$|a_n| \to +\infty,$$

if and only if

$$|b_n| \to +\infty.$$

We may suppose

$$b_n \to +\infty.$$

Then, for $t \neq 0$ and $t \neq 1$, we have

$$\sigma_n(t) - b_n v(t) \to -\infty,$$

so that, by (ξ1),

$$f(t, \sigma_n(t) - b_n v(t)) \to -\infty.$$

Consequently, since the sequence $\{f(t, y_n(t))\}_{n\in N}$ is uniformly bounded from above, we obtain

$$\lim_{n\to\infty} \int_0^1 (\lambda_n f(t, y_n(t)) + (1-\lambda_n) y_n(t))\, dt = -\infty,$$

which contradicts (19), being

$$|x_n(1) - x_n(0)| \leq 2R.$$

Therefore, the required *a priori* bounds are established, and Theorem 2.2 applies to system (16).

In our last example, we will be concerned with the C-controllability of a two-point boundary-value problem.

Example 2.3. Let the boundary-value problem

$$\ddot{x}(t) = f(t, x(t), u(t)), \qquad t \in [0, 1],$$
$$x(0) = x(1) = 0 \tag{20}$$

be given, with $f: [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous and satisfying assumptions (C1) and (C2) of Example 2.1.

Our aim is to show that, given $a \in \mathbb{R}$, there exists a constant control $u \in \mathbb{R}$ such that the system

$$\ddot{x}(t) = f(t, x(t), u), \qquad t \in [0, 1],$$
$$x(0) = x(1) = 0, \tag{21}$$
$$\int_0^1 x(t)\, dt = a$$

has a solution $x \in C^2([0, 1])$. In other words, if we define $C: C^2[0, 1] \to \mathbb{R}$ by

$$Cx = \int_0^1 x(t)\, dt,$$

according to the terminology introduced in Remark 2.3, we will show that problem (20) is C-controllable to $a \in \mathbb{R}$, by means of constant controls.

To this end, we need to construct a homotopy joining, in a good manner, f with a linear map. Sometimes, as can also be seen in the previous examples, a suitable way of choosing the linear map turns out to be that of looking for a linear operator which satisfies the same hypotheses made on the nonlinearity f. Having in mind this, consider the system

$$\ddot{x}(t) = \lambda f(t, x(t), u) + (1-\lambda) u, \qquad t \in [0, 1],$$
$$x(0) = x(1) = 0, \tag{22}$$
$$\int_0^1 x(t) = a.$$

An argument analogous to the one used in the first part of the Example 2.1 shows that, if the controls u in (22) are uniformly bounded, then so are the corresponding solutions x. Suppose, therefore, that the controls are unbounded and assume

$$u_n \to +\infty.$$

For $a \geq 0$, any solution x_n of (22) corresponding to (u_n, λ_n) is such that

$$x_n(t_n) = \max_{t\in[0,1]} x_n(t) \geq 0,$$

so that, as in Example 2.1,

$$\ddot{x}_n(t_n) > 0,$$

for n large enough. This is a contradiction, since

$$\ddot{x}_n(t_n) \leq 0.$$

For $a < 0$, let $K > 0$ be such that the unique solution η of the boundary-value problem

$$\ddot{\eta}(t) = K, \qquad \eta(0) = \eta(1) = 0,$$

satisfies

$$\int_0^1 \eta(t) < a.$$

Let x_n be a solution of (22). Suppose that

$$x_n(t) \leq \eta(t), \qquad t \in [0, 1].$$

Then,

$$\int_0^1 x_n(t)\, dt < a,$$

which is impossible. Thus, there exists an interval

$$I_n = [t_{1_n}, t_{2_n}],$$

such that

$$x_n(t) > \eta(t), \qquad t \in (t_{1_n}, t_{2_n}),$$

and

$$x_n(t_{1_n}) = \eta(t_{1_n}), \qquad x_n(t_{2_n}) = \eta(t_{2_n}).$$

Now, let M be such that

$$\eta(t) \geq M, \qquad t \in [0, 1].$$

By Assumption (C1), there exists $\bar{n} \in \mathbb{N}$ such that, for $t \in I_{\bar{n}}$,

$$\lambda_{\bar{n}} f(t, x_{\bar{n}}(t), u_{\bar{n}}) + (1 - \lambda_{\bar{n}}) u_{\bar{n}}$$
$$\geq \lambda_{\bar{n}} \inf_{\substack{p \geq M \\ t \in [0,1]}} f(t, p, u_{\bar{n}}) + (1 - \lambda_{\bar{n}}) u_{\bar{n}} \geq K.$$

Therefore, the function

$$v(t) = x_{\bar{n}}(t) - \eta(t)$$

solves in $I_{\bar{n}}$ the differential inequality

$$\ddot{v}(t) \geq 0, \qquad v(t_{1_{\bar{n}}}) = v(t_{2_{\bar{n}}}) = 0.$$

This implies that

$$v(t) \leq 0, \qquad \text{in } I_{\bar{n}},$$

which contradicts the fact that $x_{\bar{n}}$ is strictly greater than η in $(t_{1_{\bar{n}}}, t_{2_{\bar{n}}})$. Hence, (22) is, for our purposes, an appropriate homotopy, and so the C-controllability of (20) follows now from a straightforward application of Theorem 2.1 (see also Remark 2.3).

Observe that, to obtain the previous local controllability result, it is, in fact, sufficient to relax Assumption (C2) into the following assumption:

(C2') for any $M_0 > 0$, there exist M_1, M_2 such that, for $|q| \leq M_0$,

$$f(t, p, q) \geq M_1, \qquad p \geq 0, \ t \in [0, 1],$$
$$f(t, p, q) \leq M_2, \qquad p \leq 0, \ t \in [0, 1].$$

Similar weaker conditions can be considered also in Examples 2.1 and 2.2 in the case where a problem of local controllablity instead of global controllability, is discussed.

3. Conclusions

The controllability conditions obtained above require the choice of a suitable linear system into which the nonlinear process has to be transformed. Therefore, it may be of some interest to examine closely what kind of linear systems turn out to be more appropriate in order to be able to deduce the *a priori* bounds needed. For instance, in many cases, a well-behaving homotopy can be constructed, provided one can find matrices $A(\cdot)$ and $B(\cdot)$ such that the linear operator $(x, u) \mapsto A(\cdot)x + B(\cdot)u$ possesses asymptotic properties analogous to those of the given nonlinearity.

We also point out that, in deriving our sufficient conditions, the choice of a finite-dimensional control space is actually needed. A deeper investigation on the possible choices of this subspace seems in fact to require a further study.

References

1. Dauer, J. P., *Nonlinear Perturbations of Quasilinear Control Systems*, Journal of Mathematical Analysis and Applications, Vol. 54, pp. 717-725, 1976.
2. Lukes, D. L., *Global Controllability of Nonlinear Systems*, SIAM Journal on Control, Vol. 10, pp. 112-126, 1972.
3. Lukes, D. L., *Erratum: Global Controllability of Nonlinear Systems*, SIAM Journal on Control, Vol. 11, p. 186, 1973.
4. Mirza, K. B., and Womack, B. F., *On the Controllability of a Class of Nonlinear Systems*, IEEE Transactions on Automatic Control, Vol. AC-17, pp. 531-535, 1972.
5. Muntean, I., *Comments on "On the Controllability of a Class of Nonlinear Systems,"* IEEE Transactions on Automatic Control, Vol. AC-19, pp. 459-460, 1974.
6. Mirza, K. B., and Womack, B. F., *On the Controllability of a Class of Nonlinear Systems: Authors' Reply*, IEEE Transactions on Automatic Control, Vol. AC-19, p. 460, 1974.

7. YAMAMOTO, Y., and SUGIURA, I., *On Controllability for Nonlinear Control Systems*, Journal of SICE in Japan, Vol. 9, 1973.
8. FURI, M., NISTRI, P., PERA, M. P., and ZEZZA, P. L., *Linear Controllability by Piecewise Constant Controls with Assigned Switching Times*, Journal of Optimization Theory and Applications, Vol. 45, pp. 219–229, 1985.
9. LERAY, J., and SCHAUDER, J., *Topologie et Equations Fonctionnelles*, Annales Scientifiques de l'École Normale Supérieure, Vol. 51, pp. 45–78, 1934.
10. FURI, M., MARTELLI, M., and VIGNOLI, A., *On the Solvability of Nonlinear Operator Equations in Normed Spaces*, Annali di Matematica Pura e Applicata, Vol. 124, pp. 321–343, 1980.
11. FURI, M., and PERA, M. P., *On the Existence of an Unbounded Connected Set of Solutions for Nonlinear Equations in Banach Spaces*, Atti dell'Accademia Nazionale dei Lincei, Rendiconti della Classe di Scienze Matematiche, Fisiche e Naturali, Vol. 67, pp. 31–38, 1979.
12. DAVISON, E. J., and KUNZE, E. G., *Some Sufficient Conditions for the Global and Local Controllability of Nonlinear Time-Varying Systems*, SIAM Journal on Control, Vol. 8, pp. 489–497, 1970.
13. YAMAMOTO, Y., *Controllability of Nonlinear Systems*, Journal of Optimization Theory and Applications, Vol. 22, pp. 41–49, 1977.
14. [illegible], G., *Global Controllability and Bang-Bang Steering of Certain Nonlinear Systems*, SIAM Journal on Control, Vol. 11, pp. 607–619, 1973.
15. DAVISON, E. J., and BALL, D., *On the Global controllability of Perturbed Controllable Systems*, IEEE Transactions on Automatic Control, Vol. AC-17, pp. 825–826, 1972.
16. KLAMKA, J., *On the Global Controllability of Perturbed Nonlinear Systems*, IEEE Transactions on Automatic Control, Vol. AC-20, pp. 170–172, 1975.
17. WEI, K. C., *A Class of Controllable Nonlinear Systems*, IEEE Transactions on Automatic Control, Vol. AC-21, pp. 787–789, 1976.
18. COPPEL, W. A., *Stability and Asymptotic Behavior of Differential Equations*, Heath, Boston, Massachusetts, 1965.