DYNAMICAL STABILITY OF THE APPROXIMATE EQUIVALENT CONTROL IN SLIDING MANIFOLD SYSTEMS

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Abstract

We study the dynamical stability properties of a suitably defined approximation of the equivalent control in sliding manifold systems. For this, we use spectral theory to solve a problem of singularly perturbed dynamical systems, then apply the theory of twist maps. Simple examples of control systems which preserve dynamical stability properties are presented.

1 Introduction

Our goal is to develop an approach to the study of sliding manifold control systems which combines spectral theoretic methods with the theory of singular perturbation and the theory of twist maps. Sliding manifold systems are of importance in a wide range of applied problems; see the standard texts [9, 10] for an introduction to the subject.

The notion of equivalent control for nonlinear systems as developed by Bartolini and Zolezzi [1] is the basic means by which a nonlinear system is steered to a sliding manifold. We will discuss a smooth approximation in the uniform norm to the equivalent control which does not exhibit the chattering phenomenon and which has good robustness properties. We will then discuss the dynamical stability properties of our approximate equivalent control, or "a.e. control". This is an important question when one tries to realize the a.e. control in applications. We refer to the papers [3, 4], where in fact the question of dynamical stability was sidestepped in an elegant and effective way by introducing an adjustment term which depends on the initial state x_0 of the dynamical system.

Here we will meet the issue directly, since the initial state is not always available. We will use the basic theory of the twist maps to discuss simple but fairly realistic examples of control processes which illustrate the following points. (i) In certain situations, dynamical stability (in a sense which we will make precise) can be achieved. Roughly speaking, the a.e. control holds initial states near the sliding manifold in a prescribed neighborhood of the sliding manifold, without introducing adjustment terms. (ii) However, the presence of dynamical stability tends to the sensitively related to the structure of the control process itself.

2 The Approximate Equivalent Control

We consider the nonlinear control system

$$x' = f(t, x, u) \tag{1}$$

where the state vector x is an element of $\mathbf{R}^n$ and the control vector u belongs to a subset U of $\mathbf{R}^m$. We suppose that $m \leq n$.

An important class of control systems modelled by equation (1) is given by the variable structure control problems. The basic idea is the following. Let $s : \mathbf{R}^n \to \mathbf{R}^m$ be a continuously differentiable function. We define the *sliding manifold* $S \subset \mathbf{R}^n$ to be the zero-set of s:

$$S = \{x \in \mathbf{R}^n \mid s(x) = 0\}. \tag{2}$$

In a typical variable-structure problem, motions of the state vector on the manifold S will have desirable properties (of stability, for example). Therefore it is natural to pose the following problem: seek a feedback control $u = u(t, x)$, which steers initial states x_0 to the sliding manifold S and holds them there. The feedback control law u is allowed to be discontinuous along S.

Such control systems frequently have very good properties. They exhibit stable behavior, accurate tracking, robust performance, and insensitivity with respect to disturbance and variation of plant parameters. The main drawbacks of these systems are the chattering phenomenon, and the necessity of considering a generalized notion of solution of (1) (usually that of Filippov [5]). This last difficulty is due to the discontinuity of $u(x, t)$ along S.

We introduce an approach to the control problem (1)+(2) which avoids chattering and renders unnecessary the consideration of Filippov-type solutions. We introduce a small parameter ε in the following way:

$$\begin{aligned} \varepsilon u' &= (\nabla s \cdot f)(t, x, u) \doteq g(t, x, u) \\ x' &= f(t, x, u). \end{aligned} \tag{3}$$

Here ∇ = gradient. We suppose from now on that f and s are analytic, though this could be reduced to Carathéodory-type regularity assumptions in much of what we say. Let $D \subset \mathbf{R}^n$ be the domain of interest of the state variable x. We will suppose that the equation

$$0 = g(t, x, u)$$

is uniquely solvable for $x \in U$ as a function $u = u_0(t, x)$ in the set $\mathbf{R} \times D$. We suppose that u_0 is analytic in its arguments. The function u_0 is by definition the *equivalent control* relative to which we will study the stability properties of the sliding manifold S.

Next we impose the "Tychonov condition": we suppose that

$$\frac{\partial g}{\partial u}(t, x, u_0(t, x)) < 0$$

for each $(t, x) \in \mathbf{R} \times D$. We arrive now at a fundamental application of spectral theory which is crucial for further developments. Namely, using the Tychonov condition and results of invariant manifold theory [2, 6], one can show that, for small values of ε, there is a locally invariant manifold M_ε for equations (3) which is a graph:

$$M_\varepsilon = \{(t, x, u_\varepsilon(t, x)) \mid (t, x) \in \mathbf{R} \times D\}.$$

Given an integer $r > 0$, the function $(t, x) \to u_\varepsilon(t, x)$ is of class C^r for sufficiently small ε; however, it need not be analytic [2].

Definition The *approximate equivalent control* (a.e. control) corresponding to the problem (1)+(2) is the function u_ε just introduced.

Let us observe that, if $\varepsilon = 0$ and $u = u_0(t, x)$ in equations (3), then the manifolds s = const. are invariant in the sense that, if $x(t)$ is the solution of $x' = f(t, x, u_0(t, x))$ with $x(0) = x_0$, then $s(x(t)) = s(x_0)$ for all t. Of course the sliding manifold S is one of these manifolds. Now, this fibering of D into invariant manifolds is clearly a degenerate structure. One may hope that, if f is appropriately constructed, then "many" of the invariant manifolds will persist for $\varepsilon \neq 0$. We are going to define dynamical stability in such a way that its presence follows from such persistence.

We now turn to the definition of dynamical stability, and do the construction of a class of sliding manifold control systems for which it holds.

3 Dynamical Stability of the a.e. Control

We will discuss the dynamical stability of the a.e. control u_ε introduced in the previous section. We will consider the case when the vector field f is of period 1 in t:

$$f(t+1, x) = f(t, x).$$

Under the Tychonov assumption, one can show that the a.e. control $u_\varepsilon(t, x)$ is also of period 1 in t:

$$u_\varepsilon(t+1, x) = u_\varepsilon(t, x)$$

for all $t \in \mathbf{R}$, $x \in D$.

It is convenient to consider the time-one maps P_ε defined by equations (3). Let $x_0 \in D$, and let $x(t)$ be that solution of

$$x' = f(t, x, u_\varepsilon(t, x))$$

with $x(0) = x_0$. By the local invariance of M_ε, $(x(t), u_\varepsilon(t, x(t)))$ is a solution of (3) for all t in the existence domain of $x(\cdot)$. We define

$$P_\varepsilon(x_0) = x(1).$$

We assume that there exists $\varepsilon_0 > 0$ such that P_ε is defined for all $x_0 \in D$ $(0 \leq \varepsilon \leq \varepsilon_0)$.

Definition We say that the sliding manifold $S \subset D$ is *dynamically stable* if for every neighborhood N_1 of S there exist a positive ε_1 and a neighborhood N of S such that, if $x_0 \in N$, then the iterate $P_\varepsilon^k(x_0)$ exists and belongs to N_1 for all $k \geq 0$ and $0 \leq \varepsilon \leq \varepsilon_1$.

This notion of stability (essentially that of Lyapounov) is useful when one does not have complete information concerning the initial state x_0. When the initial state is known, then it is frequently possible to introduce an auxiliary control—which depends on x_0—which steers x_0 to S and holds it there. Our notion of dynamical stability is a property of problem (1)+(2) and makes no direct reference to a single initial state.

We now illustrate the significance of our definition by means of examples in the case when the state space has dimension $n = 2$. The examples will illustrate that, while dynamical stability can certainly be obtained, to do so requires that the controller be designed rather carefully.

Let (r, θ) be the usual polar coordinates in the plane. We write $x_1 = \theta/2\pi$, $x_2 = r$ (note the order), and define $s(x_1, x_2) = x_2 - r_0$ where r_0 is a positive constant. Thus the sliding manifold is a circle centered at the origin in $\mathbf{R}^2$. We suppose that $f_1(t, x_1, x_2) = x_2 = r$, and write $f(t, x_1, x_2) \equiv f_2(t, x_1, x_2)$. Our system (3) becomes

$$\begin{aligned} \varepsilon u' &= f(t, x_1, x_2) \\ x_1' &= x_2 \\ x_2' &= f(t, x_1, x_2). \end{aligned} \tag{4}$$

Finally we suppose that the domain D is an annulus $A = \{(\theta, r) \mid 0 \leq \theta < 2\pi,\ 0 \leq r_1 < r < r_2\}$, and that $S \subset A$.

Because of the second equation in (4) (which can be written $\theta' = 2\pi r$), the family $\{P_\varepsilon\}$ of period maps satisfies the *monotone twist condition* [8]. This just means that, if $r' > r$, then P_ε rotates (r', θ) more than it does (r, θ). Such maps have a rich theory, a small part of which will be used below.

We look f in the form

$$f(t, x_1, x_2, u) = 1 - B(t, x_1, x_2)u,$$

where B is to satisfy the conditions indicated below. First of all:

$$B(t, x_1, x_2) > 0 \tag{5}$$

$$B(t+1, x_1, x_2) = B(t, x_1, x_2) \tag{6}$$

for all $(t, x_1, x_2) \in \mathbf{R} \times A$. Condition (5) implies that the singular perturbation problem (3) satisfies the Tychonov condition.

Put

$$u_0(t, x_1, x_2) = \frac{1}{B(t, x_1, x_2)},$$

so that $f(t, x_1, x_2, u_0) \equiv 0$. Applying the theory and remarks given previously, we obtain a 1-parameter family $\{P_\varepsilon\}$ of monotone twist maps from A into $\mathbf{R}^2$. In terms of the polar variables (θ, r), one has

$$P_0\begin{pmatrix} \theta \\ r \end{pmatrix} = \begin{pmatrix} \theta + 2\pi r \\ r \end{pmatrix}.$$

Now, if the maps P_ε are *area-preserving*, then the well-known invariant curve theorem of [8] applies. From this one can deduce dynamical stability of S, as we will explain later. In our situation, it seems unnatural to require that the maps P_ε be strictly area-preserving. However, we are going to determine u_0 is such a way that the P_ε are area-preserving to order ε^2: that is, if $Q \subset A$ is a Borel set, then

$$m(P_\varepsilon(Q)) = m(Q) + O(\varepsilon^2) \quad (m = \text{Lebesgue measure}).$$

We will then indicate how to use the theory of monotone twist maps to study the dynamical stability of S.

Let us consider the differential equation (4) on the invariant manifold M_ε:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= f(t, x_1, x_2, u_\varepsilon(t, x_1, x_2)). \end{aligned} \tag{7}$$

We wish to arrange f so that the time-one maps P_ε are area-preserving to order ε^2. For this, consider the gradient $G = \varepsilon G_1 + \varepsilon^2 G_2 + \ldots$ of the vector field $\begin{pmatrix} x_2 \\ f \end{pmatrix}$. One finds that

$$G_1 = \frac{\partial^2 u_0}{\partial t \partial x_2} + \frac{\partial u_0}{\partial x_1} + x_2 \frac{\partial^2 u_0}{\partial x_1 \partial x_2}.$$

One then determines u_0 so that $G_1 = 0$ and so that (5) and (6) hold: for example, one can put

$$u_0(t, x_1, x_2) = C + \frac{a \cos 2\pi(t + x_1 + \varphi)}{1 + x_2},$$

where a is an amplitude, φ a phase, and C a sufficiently large positive constant. One can now verify that the maps P_ε are indeed area-preserving to order ε^2.

To verify the dynamical stability of the sliding manifold S, one proves the following

Theorem *Let $A \subset \mathbf{R}^2$ be an annulus centered at $r = 0$. Let $P_\varepsilon : A \to \mathbf{R}^2$ be a one-parameter family of analytic maps defined for $0 \le \varepsilon \le \varepsilon_0$ and of class C^2 in ε. Suppose that:*

(i) $P_0\begin{pmatrix} \theta \\ r \end{pmatrix} = \begin{pmatrix} \theta + 2\pi r \\ r \end{pmatrix}$;

(ii) *there is a constant $c_0 \ge 0$ such that, for each Borel set $Q \subset A$ and each $0 \le \varepsilon \le \varepsilon_0$:*

$$m(P_\varepsilon(Q)) \le m(Q)[1 + c(Q, \varepsilon)\varepsilon^2]$$

where $|c(Q, \varepsilon)| \le c_0$;

(iii) *if $P_\varepsilon(\theta, r) = \begin{pmatrix} \psi_\varepsilon(\theta, r) \\ \rho_\varepsilon(\theta, r) \end{pmatrix}$ and if $\rho_\varepsilon(\theta, r) = r + \varepsilon\rho_1(\theta, r) + \ldots$, then $\rho_1(\theta, r)$ does not reduce to a function of r alone in A.*

Then there is a closed set $A_\varepsilon \subset A$, consisting of simple closed curves C enclosing the origin, such that $P_\varepsilon(C) = C$ (set equality) for each curve C, and such that $m(A_\varepsilon) \to m(A)$ as $\varepsilon \to 0$.

The proof of this theorem is given in [7]. It is clearly a stability theorem. For, if N_1 is a neighborhood of S, then for sufficiently small ε there are P_ε-invariant curves C_1 and C_2 so that S is in the annulus A_* bounded by C_1 and C_2, and so that $A_* \subset N_1$. Hence A_* is invariant under P_ε, and we can put $N = A_*$ in the definition of dynamical stability.

We note that our theorem cannot be directly applied to the period maps P_ε, because they are only C^r in (x_1, x_2) and not analytic. However this is not an essential problem, and a method to get around it is given in [7].

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