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Linear Controllability by Piecewise Constant Controls with Assigned Switching Times

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Abstract. For a linear time-dependent control process in R^n, we prove that the complete controllability by means of the space of all the admissible controls is equivalent to the complete controllability by means of a suitable n-dimensional space of piecewise constant controls with at most n preassigned switching times. An analogous result is also established for more general controllability problems.

Key Words. Linear control processes, complete controllability, piecewise constant functions, switching times.

1. Introduction

In this paper, we are concerned with the controllability of a linear control process described by the differential system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{1}$$

where $t \in J = [0, T]$, $A(\cdot)$ and $B(\cdot)$ are time-dependent $n \times n$ and $n \times m$ matrices with coefficients in $L^1(J, \mathbb{R})$, and u belongs to the space

$$U_B = \{u \in L^1(J, \mathbb{R}^m) : B(\cdot)u(\cdot) \in L^1(J, \mathbb{R}^n)\}$$

of all the admissible controls.

Our aim is to show that, if the system (1) is completely controllable at time T by means of U_B (i.e., if, for every $x_0, x_1 \in \mathbb{R}^n$, there exists a control

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$u \in U_B$ steering x_0 to x_1 at the fixed time T), then it is still completely controllable by means of an appropriate n-dimensional space of controls consisting of piecewise constant controls with at most n preassigned switching times. In the case where the control is scalar (i.e., $m = 1$), we give a necessary and sufficient condition to reduce the switching times to their minimum number which is $n-1$.

Analogous results are established also in the case where a more general type of controllability is considered.

To be more specific, let us consider the linear control process (1), together with the boundary condition

$$Dx = y, \tag{2}$$

where

$$D: AC(J, \mathbb{R}^n) \to \mathbb{R}^q$$

is a bounded linear surjective operator defined on the Banach space $AC(J, \mathbb{R}^n)$ of absolutely continuous functions $x: J \to \mathbb{R}^n$, with norm

$$\|x\| = |x(0)| + \int_0^T |\dot{x}(t)|\, dt.$$

Following Marchiò (Ref. 1), we will say that (1) is D-controllable by means of U_B if, for every $y \in \mathbb{R}^q$, the system (1)-(2) is solvable for some $u \in U_B$. Clearly, if

$$Dx = (x(0), x(T)) \in \mathbb{R}^{2n},$$

the above notion coincides with the usual one of complete controllability. We prove that, if (1) is D-controllable by means of U_B, then it is still D-controllable by means of a p-dimensional space of piecewise constant controls with at most p preassigned switching times, where

$$p = q - n + \dim(\operatorname{Ker} D \cap \operatorname{Ker} L)$$

and

$$L: AC(J, \mathbb{R}^n) \to L^1(J, \mathbb{R}^n)$$

is defined by

$$(Lx)(t) = \dot{x}(t) - A(t)x(t).$$

Some other results establishing bounds on the number of switching times have been obtained in the bang-bang controllability context, where, actually, the considered bang-bang controls turn out to be also time-optimal controls. We refer to Refs. 2 and 3 for linear autonomous control processes in $\mathbb{R}^n$ and to Ref. 4 for nonlinear autonomous control systems of the form

$$\dot{x} = f(x) + ug(x),$$

with f and g analytic vector fields defined on an analytic manifold and $u \in [-1, 1]$. However, it should be observed that there is a substantial difference between the problem in which we are interested here and the one examined in the above papers. In fact, such a difference depends essentially on the choice of the controls to be employed: in our approach (the switching times being the same for all controls), we deal with a linear control space, while, in the bang-bang case, we may only bound the number of switching times on assigned time intervals (the number of them being dependent on the control). Thus, in particular, the control set considered in this latter case is not a linear space.

In what follows, $(AC)^k$, $k \in \mathbb{N}$, will denote the Banach space $AC(J, \mathbb{R}^k)$. The space $L^p(J, \mathbb{R}^k)$, $1 \le p \le +\infty$, will be briefly indicated by $(L^p)^k$. Finally, $(PC)^k$ will be the subspace of $(L^\infty)^k$ consisting of piecewise constant functions.

2. Controllability by Means of *n*-Dimensional Control Spaces

Consider the linear control process

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad t \in J = [0, T], \tag{3}$$

where $A(\cdot)$ and $B(\cdot)$ are time-dependent $n \times n$ and $n \times m$ matrices with L^1-coefficients, and where $u \in U_B$.

Observe that U_B, the space of controls, is a vector subspace of $(L^1)^m$ which contains $(L^\infty)^m$. Given a subspace U of U_B, the system (3) is said to be *completely controllable* by means of U if, for every $x_0, x_1 \in \mathbb{R}^n$, there exists a control $u \in U$ such that the solution x of the Cauchy problem

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \qquad t \in J, \\ x(0) &= x_0 \end{aligned} \tag{4}$$

satisfies the condition

$$x(T) = x_1. \tag{5}$$

In other words, (3) is completely controllable by U (or simply *completely controllable*, in the case $U = U_B$) iff the boundary-value problem (3)-(5) is solvable for some $u \in U$.

It is well known, and it is easy to check, that (3) is completely controllable by U if and only if (3) is *controllable from zero* [*controllable to zero*]

by U; i.e., iff, for any $x_1 \in \mathbb{R}^n$ $[x_0 \in \mathbb{R}^n]$, the boundary-value problem (3)-(5) is solvable for $x_0 = 0$ $[x_1 = 0]$ and some $u \in U$. Because of this equivalence, in what follows, we shall restrict our attention to the control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad t \in J, \qquad x(0) = 0. \tag{6}$$

Now, for any $u \in U_B$, (6) admits a unique solution

$$x = Ku \in (AC)^n,$$

that can be represented by the linear operator

$$(Ku)(t) = X(t) \int_0^t X^{-1}(s)B(s)u(s)\, ds,$$

where $X(\cdot)$ is the $n \times n$ matrix solution of

$$\dot{X}(t) = A(t)X(t)$$

which satisfies the condition

$$X(0) = I,$$

where I is the identity matrix. Thus, if by

$$C: (AC)^n \to \mathbb{R}^n$$

we denote the surjective linear operator

$$Cx = x(T),$$

the controllability from zero of (3) is nothing else but the surjectivity of the linear operator

$$CK: U_B \to \mathbb{R}^n,$$

given by

$$CK(u) = X(T) \int_0^T X^{-1}(s)B(s)u(s)\, ds.$$

It is therefore clear that (3) is controllable from zero [or, as we shall say later, that (6) is C-controllable], if and only if the restriction of CK to some n-dimensional subspace U of U_B is onto (and thus, an isomorphism). In other words, if we can steer the solution of (3) from zero to any desired point (at the fixed time T), then some n-dimensional subspace U of U_B will serve our purposes as well.

The main result of this paper (Theorem 2.1 below) shows that we can seek U in the class of the n-dimensional subspaces of the spaces which consist of piecewise constant $\mathbb{R}^m$-valued functions with at most n discontinuities at fixed switching times [i.e., at switching times $\{t_1, t_2, \ldots, t_n\}$ which depend only on the space].

For a given interval $[a, b] \subset \mathbb{R}$, let $\chi_{[a,b]}$ denote the characteristic function of $[a, b]$. Given $\tau \in J$ and $i = 1, 2, \ldots, m$, we denote by χ^i_τ the step function

$$\chi^i_\tau(t) = \chi_{[0,\tau]}(t)e_i,$$

where

$$\{e_i: i = 1, 2, \ldots, m\}$$

is the standard basis of $\mathbb{R}^m$. Observe that

$$(PC)^m = \operatorname{span} \chi,$$

where

$$\chi = \{\chi^i_\tau: \tau \in J, i = 1, 2, \ldots, m\}.$$

Theorem 2.1. If (3) is completely controllable, then it is still completely controllable by means of an n-dimensional space of the type

$$U = \operatorname{span}\{\chi^{i_1}_{t_1}, \ldots, \chi^{i_n}_{t_n}\}.$$

Proof. It is enough to show that the family $CK(\chi)$ is a system of generators for $\mathbb{R}^n$. Suppose that

$$\operatorname{span} CK(\chi) = CK((PC)^m) \neq \mathbb{R}^n.$$

Then, there exists $v \in \mathbb{R}^n$, $v \neq 0$, such that

$$v^* CK(u) = 0, \qquad \text{for all } u \in \chi,$$

that is,

$$0 = v^* X(T) \int_0^T X^{-1}(s)B(s)\chi^i_\tau(s)\, ds$$

$$= \int_0^\tau [B^*(s)(X(T)X^{-1}(s))^* v]^* e_i\, ds,$$

for all $\tau \in J$ and $i = 1, 2, \ldots, m$. Thus,

$$B^*(s)(X(T)X^{-1}(s))^* v = 0, \qquad \text{a.e. in } J.$$

Therefore,

$$v^* CK(u) = 0, \qquad \forall u \in U_B,$$

which contradicts the complete controllability of (3). ☐

Observe that, from the proof of Theorem 2.1, one can deduce that (3) is completely controllable if and only if

$$v^*X(T)X^{-1}(s)B(s)=0, \text{ a.e. in } J, \text{ implies } v=0.$$

This result and Corollary 2.1 below are well known when the controls are taken in $(L^\infty)^m$; see, e.g., Ref. 5.

Corollary 2.1. If (3) is completely controllable, then it is also completely controllable by $(PC)^m$.

We point out that, in the case where the controls are taken in $(L^p)^m$, $1<p<\infty$, the above result follows immediately from the fact that

$$CK:(L^p)^m \to \mathbb{R}^n$$

is continuous and $(PC)^m$ is dense in $(L^p)^m$. A similar argument can be used for the case $p=\infty$ using the w^*-topology of $(L^\infty)^m$. In fact, $(PC)^m$ is w^*-dense in $(L^\infty)^m$, but not dense, and

$$CK:(L^\infty)^m \to \mathbb{R}^n$$

is w^*-continuous. However, an analogous proof cannot be carried out in the case where the controls are taken in U_B: as the following example shows, the operator $CK:U_B\to\mathbb{R}^n$ may fail to be continuous and, consequently, if it occurs, it cannot be continuous in any weaker topology.

Example 2.1. Consider the scalar equation

$$\dot{x}(t)=u(t)/\sqrt{t}, \qquad t\in[0,1].$$

The sequence

$$\{u_j\}_{j\in\mathbb{N}}, \qquad u_j(t)=(1/\sqrt{t})\chi_{[1/je,1/j]}(t),$$

converges to zero in the L^1-topology of U_B. However,

$$CK(u_j)=\int_0^1 [u_j(s)/\sqrt{s}]\,ds=\int_{1/je}^{1/j}(1/s)\,ds=1,$$

for all $j\in\mathbb{N}$.

Observe that the fact of whether or not a system such as (3) can be controlled by means of $\operatorname{span}\{\chi^{i_1}_{t_1},\ldots,\chi^{i_n}_{t_n}\}$ can be checked by numerical methods. In fact, in order to obtain controllability, it is sufficient to verify that

$$\det(X^{-1}(t_1)\tilde{x}_{i_1}(t_1),\ldots,X^{-1}(t_n)\tilde{x}_{i_n}(t_n))\neq 0,$$

where $\tilde{x}_i$, $i=1,2,\ldots,m$, is the solution of the Cauchy problem

$$\dot{x}(t)=A(t)x(t)+B(t)e_i, \qquad t\in J$$
$$x(0)=0.$$

There are many situations where (3) is completely controllable by controls of the form

$$u(t)=v(t)\gamma(t),$$

where $\gamma:J\to\mathbb{R}^m$ is a fixed L^∞-function and $v\in L^\infty$ is arbitrary (see Ref. 6). This makes the case $m=1$ particularly interesting. In Theorem 2.2 below, our aim is to give a condition which makes possible, in the case $m=1$, the use of a space of piecewise constant controls with $n-1$ fixed switching times in the open interval $(0,T)$. Such a space is clearly n-dimensional.

Theorem 2.2. Let $m=1$, and assume that system (3) is completely controllable. Then, (3) is completely controllable by means of a space of piecewise constant functions possessing at most $n-1$ discontinuities at fixed switching times if and only if $\tilde{x}(T)\neq 0$, where $\tilde{x}$ is the solution of the Cauchy problem (6) which corresponds to the constant control $\chi_T=\chi_{[0,T]}$.

Proof. As in the proof of Theorem 2.1, the family of vectors

$$V=\{CK(\chi_\tau):\chi_\tau=\chi_{[0,\tau]},\ 0<\tau\le 1\}$$

is a system of generators of $\mathbb{R}^n$. The assertion now follows from the fact that the condition

$$\tilde{x}(T)=CK(\chi_T)\neq 0$$

is equivalent to the existence of a basis for $\mathbb{R}^n$ containing $\tilde{x}(T)$ and contained in V. □

Let us now consider system (3), together with a bounded linear surjective operator

$$D:(AC)^n\to\mathbb{R}^q.$$

Following Marchiò (Ref. 1), we shall say that (3) is D-controllable by means of $U\subset U_B$ (or simply D-controllable when $U=U_B$) if, for any $y\in\mathbb{R}^q$, the system

$$\begin{aligned}\dot{x}(t)&=A(t)x(t)+B(t)u(t), \qquad t\in J,\\ Dx&=y\end{aligned} \tag{7}$$

is solvable for some $u\in U$. Notice that, when

$$D:(AC)^n\to\mathbb{R}^{2n}$$

is the operator

$$x \mapsto (x(0), x(T)),$$

the notion of D-controllability coincides with that of complete controllability.

We will show that Theorems 2.1 and 2.2 can be extended to D-controllable systems. Let us first assume that the boundary-value problem

$$\dot{x}(t) = A(t)x(t), \qquad t \in J,$$
$$Dx = 0$$

admits only the trivial solution. Such a condition is obviously satisfied when

$$Dx = (x(0), x(T)).$$

Consider the bounded, linear, surjective differential operator

$$L: (AC)^n \to (L^1)^n,$$

given by

$$(Lx)(t) = \dot{x}(t) - A(t)x(t).$$

By our assumption, the restriction

$$D|\text{Ker } L: \text{Ker } L \to \mathbb{R}^q$$

is one-to-one, and this implies that

$$q \geq \dim \text{Ker } L = n.$$

Therefore, without any loss of generality, we may in fact suppose that

$$q = n + p > n,$$

since otherwise $D|\text{Ker } L$ would be onto and, thus, (3) would be D-controllable by means of $u(t) \equiv 0$.

It is clear now that, under the assumption that $D|\text{Ker } L$ is one to one, n of the q coordinate functionals associated to D must be linearly independent on Ker L. This allows us to split D into the direct sum of two operators

$$C_0 \oplus C: (AC)^n \to \mathbb{R}^n \oplus \mathbb{R}^p,$$

with

$$C_0|\text{Ker } L: \text{Ker } L \to \mathbb{R}^n$$

an isomorphism [as it is, for example, the operator $C_0x = x(0)$]. Now, as in the case of

$$Dx = (x(0), x(1)),$$

the fact that $C_0|\text{Ker } L$ is onto implies that (3) is D-controllable by U iff the boundary-value problem

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad t \in J,$$
$$C_0x = 0 \tag{8}$$

is *C-controllable* by U; that is, iff, for any $z \in \mathbb{R}^p$, there exists a control $u \in U$ such that the unique solution x of (8) which corresponds to u satisfies the condition $Cx = z$.

Because of the above considerations, we shall restrict our attention to the problem of the C-controllability of the system (8). As before, let

$$K: U_B \to (AC)^n$$

be the linear operator which associates, to each $u \in U_B$, the unique solution of (8). The C-controllability of (8) is now equivalent to the surjectivity of the composition

$$CK: U_B \to \mathbb{R}^p.$$

Thus, (8) is C-controllable if and only if it is C-controllable by means of some p-dimensional subspace of U_B. It is therefore clear that, in order to extend Theorems 2.1 and 2.2 to this more general situation, we need only to show that, as before, the operator CK admits an integral representation such as

$$CK(u) = \int_0^T \Phi(s)B(s)u(s)\, ds,$$

where $\Phi(\cdot)$ is a suitable $p \times n$ matrix. To this aim, let

$$L_0: \text{Ker } C_0 \to (L^1)^n$$

be the restriction of L to Ker C_0. By assumption,

$$\text{Ker } C_0 \cap \text{Ker } L = \{0\};$$

thus, L_0 is one-to-one. Moreover, since

$$L: (AC)^n \to (L^1)^n$$

and

$$C_0|\text{Ker } L: \text{Ker } L \to \mathbb{R}^n$$

are both onto, it turns out that also

$$L_0 = L|\text{Ker } C_0: \text{Ker } C_0 \to (L^1)^n$$

is onto, and hence an isomorphism of Banach spaces. We can therefore

consider the bounded linear operator

$$CL_0^{-1}: (L^1)^n \to \mathbb{R}^p,$$

that, by the Riesz representation theorem of bounded linear functionals on L^1, may be written in the integral form

$$CL_0^{-1}(y) = \int_0^T \Phi(s)y(s)\, ds,$$

where $\Phi(\cdot)$ is a $p \times n$ matrix with L^∞-coefficients. So,

$$CK(u) = \int_0^T \Phi(s)B(s)u(s)\, ds,$$

and the argument used to prove Theorems 2.1 and 2.2 works for D-controllable systems in the case where $D|\text{Ker } L$ is one-to-one.

Let us now assume

$$s = \dim(\text{Ker } L \cap \text{Ker } D) > 0.$$

Then, the Hahn-Banach theorem ensures the existence of a bounded linear operator

$$S: (AC)^n \to \mathbb{R}^s,$$

such that the restriction

$$S|(\text{Ker } L \cap \text{Ker } D)$$

is onto. This fact implies that (3) is D-controllable by U iff it is $S \oplus D$-controllable by U. On the other hand $(S \oplus D)|\text{Ker } L$ is clearly one-to-one, so the following two results can be stated.

Theorem 2.3. Let

$$D: (AC)^n \to \mathbb{R}^q$$

be a surjective bounded linear operator, and assume that (3) is D-controllable. Let

$$p = q + s - n,$$

where s is the number of linearly independent solutions of the system

$$\dot{x}(t) = A(t)x(t), \qquad t \in J,$$

$$Dx = 0.$$

Then, (3) is D-controllable by means of a p-dimensional space of the type

$$U = \text{span}\{\chi_{t_1}^{i_1}, \ldots, \chi_{t_p}^{i_p}\}.$$

Theorem 2.4. Assume that the system (3) is D-controllable and that $m = 1$. Let p be as in Theorem 2.3. Then, (3) is D-controllable by means of a space of piecewise constant functions possessing at most $p - 1$ discontinuities at fixed switching times iff any solution x of (3) with $u(t) \equiv 1$ is such that $Dx \neq 0$.

3. Concluding Remarks

The finite-dimensional control space U which replaces the infinite-dimensional control space U_B in the above controllability problems seems to be particularly suitable in applying topological methods, such as degree theory and homotopy invariance results, in order to derive local and global controllability results for nonlinear control processes of the form

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad t \in J, \tag{9}$$

where

$$f: J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

satisfies some Carathéodory-type conditions. In fact, such methods require *a priori* bounds on the solution pairs (x, u) of nonlinear control processes such as (9), and these bounds, even in the linear case, cannot be obtained if the dimension of the control space exceeds that specified on this paper.

Some sufficient conditions for the local and global controllability of system (9), via these methods, are contained in the next paper.

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