J. Math. Anal. Appl. 429 (2015) 833–848

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Exponential stability of positive semigroups in Banach spaces

Ivan Gudoshnikov [a], Mikhail Kamenskii [a], Paolo Nistri [b,*]

[a] Department of Mathematics, Voronezh State University, 394006 Voronezh, Russia
[b] Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, Via Roma 56, 53100 Siena, Italy

ARTICLE INFO

Article history:
Received 5 December 2014
Available online 16 April 2015
Submitted by T. Domínguez Benavides

Keywords:
Ordered Banach space
Positive semigroup
Spectral radius
Exponential stability

ABSTRACT

The paper establishes a link between the stability of the semigroup $e^{(-\Gamma+M)t}$ and the spectral radius of $\Gamma^{-1}M$ in ordered Banach spaces. On the one hand our result allows utilizing simple estimates for the eigenvalues of $-\Gamma + M$ in order to provide general conditions for the convergence of the successive approximation scheme for semilinear operator equations. On the other hand, this paper helps examining the stability of the semigroup $e^{(-\Gamma+M)t}$ for those classes of matrices $-\Gamma$ and M, which lead to observable expressions for $\Gamma^{-1}M$, e.g. when M is a coupling applied to disjoint systems representing Γ. The novelty of the paper is in the development of an infinite-dimensional framework, where an absolute value function induced by a cone is introduced and a way to deal with the lack of global continuity of eigenvalues is presented.

1. Introduction

Due to important applications in analysis (see Huang–Zhang [11], Abbas–Rhoades [1], Ilic–Rakocevic [12]), the fixed point theory for nonlinear operator equations

$$x = \mathcal{M}(x)$$

with a contractive operator $\mathcal{M}$ is being lately rapidly developing for cone metric spaces, where the value of the norm $\|\cdot\|$ is no longer a scalar, but an element $\|\cdot\|_F$ of a Banach space. This paper is stimulated by prospective applications of the contracting mappings theory in cone metric spaces to so-called semilinear operator equations (see Petryshyn [25], Mawhin [22])

$$Gx = \mathcal{M}(x), \tag{1.1}$$

* Corresponding author. Fax: +39 0577 233609.
E-mail addresses: gudoshnikov@yandex.ru (I. Gudoshnikov), mikhailkamenski@mail.ru (M. Kamenskii), pnistri@dii.unisi.it (P. Nistri).

http://dx.doi.org/10.1016/j.jmaa.2015.04.041

where G is a linear invertible unbounded operator and $\mathcal{M}$ is a bounded operator. By introducing the linear operators $\Gamma, M : F \to F$ such that $\|\mathcal{M}(x) - \mathcal{M}(y)\|_F \leqslant M\|x - y\|_F$ and $\|G^{-1}x\|_F \leqslant \Gamma^{-1}\|x\|_F$, the contracting mappings theory in cone metric spaces brings one to the following conclusion: *the convergence of the iterative scheme*

$$x_{n+1} = G^{-1}\mathcal{M}(x_n) \tag{1.2}$$

holds, if

$$\rho(\Gamma^{-1}M) < 1, \tag{1.3}$$

where $\rho(\Gamma^{-1}M)$ stays for the spectral radius of $\Gamma^{-1}M$. In this paper we provide conditions on Γ and M that permit to prove that (1.3) is necessary (Theorem 3.1) and sufficient (Theorem 5.1) for the exponential stability of the semigroup $e^{(-\Gamma+M)t}$, $t \geq 0$. This achievement allows utilizing simple estimates for the location and asymptotics of eigenvalues of infinite unbounded matrices (being the matrix $-\Gamma + M$) (see e.g. Shivakumar–Williams–Rudraiah [27], Malejki [21]) in order to establish the validity of (1.3) and the convergence of the scheme (1.2). Furthermore, Theorem 3.1 unveils new applications of the stability results available for the semigroup $e^{(-\Gamma+M)t}$ (see Hille–Phillips [10], Dickerson–Gibson [6], Megan–Pogan [23], Fridman [9]) by unveiling their intimate connection to the convergence of the respective iterative scheme (1.2) and, thus, to the solvability of the nonlinear equation (1.1).

On the other hand, our result opens new opportunities to examine the exponential stability of the zero equilibrium of linear differential equations

$$y' = (-\Gamma + M)y$$

in Banach spaces through the inequality (1.3). We refer the reader to the texts by Janas–Naboko [13], Shivakumar [26], Cooke [5], Bernkopf [4] for applications of linear differential equations with infinite unbounded matrices in real-life problems. Compared to Shivakumar–Williams–Rudraiah [27], our result (Theorem 5.1) applies even if $-\Gamma+M$ is not 3-diagonal. When $-\Gamma$ is a diagonal matrix with infinitely increasing coefficients and M is an arbitrary unbounded matrix, the approach taken in [27] is through Gershgorin disks, which fails to ensure stability in wide classes of matrices where our approach still applies. An important example of such a situation is given by the problem of the stability (in the space l_1) of an array of 1-dimensional differential equations $\dot{x}_i = m_i x_i$, $i \in \mathbb{N}$, which generate $-\Gamma$ and which are further coupled via an infinite matrix M of general form (discrete heat model). The stability condition through the Gershgorin disks will here be

$$\sum_j m_{ij}/\Gamma_{ii} < 1 \text{ for all } i \in \mathbb{N}, \quad \text{or} \quad \sum_i m_{ij}/\Gamma_{jj} < 1 \text{ for all } j \in \mathbb{N}. \tag{1.4}$$

While the condition (1.3) leads to

$$\sum_i \sup_{j\in\mathbb{N}}\{m_{ij}\}/\Gamma_{ii} < 1. \tag{1.5}$$

It is not hard to come up with classes of matrices M where (1.5) applies and (1.4) fails. In addition, Theorem 5.1 complements the stability conditions by Triggiani [28] (in control theory) and Fan–Li [8] (in elasticity) in the case where the Banach spaces involved respect the order induced by a cone.

Let us briefly discuss the technique used in this paper touching upon the difficulties that come from an infinite-dimensional context and require new advances in comparison to the available finite-dimensional framework, see Kamenskii–Nistri [14] and the references therein. In order to prove by Theorem 5.1 that (1.3) is a sufficient condition for the exponential stability of the semigroup $e^{(-\Gamma+M)t}$, $t \geq 0$ we need to introduce a multivalued map which, in the framework of the Banach spaces with generating cone, has the relevant properties of the absolute value in $\mathbb{R}$. We will refer to this map as the *positiveness map*. Furthermore, an operator in an infinite dimensional Banach space depending continuously on a given parameter has the property that its eigenvalues have only local continuous dependence on the parameter, instead of the global continuity in the case of finite dimensional spaces. We overcame this difficulty by taking the advantage of the one-sided estimates available for the spectrum of analytic semigroup generators. To control the complex behavior coming from continuous and residual spectrum in Theorem 3.1 we assume that the operator $\Gamma^{-1}M$ is compact. Analogously, to prove Theorem 5.1, in the preliminary Lemma 5.1 we give conditions on Γ and M under which the unbounded linear operator $-\alpha\Gamma + M$, $\alpha' \leq \alpha < +\infty$, where $0 < \alpha' < 1$ has a compact resolvent.

The paper is organized as follows. In Section 2 we recall some well known definitions and results concerning the ordered Banach spaces and the semigroup theory which will be used throughout the paper. Section 3 is devoted to the formulation and the proof of Theorem 3.1 which shows that condition (1.3), coupled with additional assumptions on Γ and M, is necessary for the exponential stability of the semigroup $e^{(-\Gamma+M)t}$, $t \geq 0$. The positiveness map is introduced in Section 4, where we provide the relevant properties of such a map useful to our purposes. Finally, in Section 5 we complete our results by proving Theorem 5.1 which states, under suitable assumptions on Γ and M, that the condition (1.3) is sufficient for the exponential stability of the semigroup $e^{(-\Gamma+M)t}$, $t \geq 0$. Despite of many technicalities coming from the infinite-dimensional framework our main result (Theorem 3.1 and Theorem 5.1) is sharp in the sense that, in the case of finite dimension, it transforms exactly into that of Kamenskii–Nistri [14], i.e. the equivalence of (1.3) and the stability of the semigroup $e^{(-\Gamma+M)t}$, $t \geq 0$. An acknowledgment section concludes the paper.

2. Definitions and preliminary results

For the reader's convenience in this section we recall some well known definitions and results concerning both the ordered Banach spaces, see for instance [17,20], and the semigroup theory for which we refer mainly to [7], see also [2,10,18,19,24].

2.1. Ordered Banach spaces

Let F be a real topological linear space. A subset K of F is called a cone, if K is closed, convex, invariant under multiplication by elements of $\mathbb{R}_+ := [0, +\infty)$, and if $K \cap (-K) = \{0\}$. Each cone induces a partial ordering in F, through the rule $u \geq v$ if and only if $u - v \in K$. This ordering is antisymmetric, reflexive, transitive, compatible with the linear structure, i.e. $\alpha \in \mathbb{R}_+$ and $u \geq 0$ imply $\alpha\, u \geq 0$ and, for every $w \in F$, $u \geq v$ implies $u + w \geq v + w$, and the ordering is compatible with the topology, i.e., $u_j \geq 0$, $u_j \to u$, implies $u \geq 0$.

On the other hand, let F be a topological linear space with an ordering $\leq$ which is compatible with the linear structure and with the topology. Then the set $K := \{u \in F : u \geq 0\}$ is a cone in F, *the positive cone*, and this cone induces the given partial ordering on F. Hence, an *ordered normed linear space* (*ordered Banach space*) with positive cone K is a normed linear space (Banach space) together with a partial ordering which is induced by a given cone K. In what follows it will be denoted by (F, K).

Let F be an ordered normed linear space with positive cone K. We shall write $u > 0$ if $u \in K \setminus \{0\} := \dot{K}$.

A cone K is called generating if $F = K - K$ and K is called normal if there exists a $\delta > 0$ such that, for all $u, v \in K$,

$$\|u + v\| \geq \delta \max(\|u\|, \|v\|).$$

The norm $\|\cdot\|$ of the space F is called *semi-monotone* if there exists $N > 0$, such that for any $x, y \in K$, $x \leq y$ implies $\|x\| \leq N\|y\|$. The norm is called *monotone* if $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$. The cone K is normal if and only if the norm is semi-monotone [17, Theorem 1.2, 1.3.3].

2.2. Semigroup theory

Definition 2.1. A family $\{T(t)\}_{t\geqslant 0}$ of bounded linear operators on Banach space X is called a *(one-parameter) semigroup* on X if it satisfies

$$\begin{cases} T(t+s) = T(t)T(s) \quad \forall (t, s \geqslant 0), \\ T(0) = I; \end{cases}$$

Definition 2.2. A semigroup $\{T(t)\}_{t\geqslant 0}$ is called *strongly continuous* or C_0-semigroup if the functions

$$\xi_x : \mathbb{R}_+ \to X$$
$$\xi_x : t \mapsto \xi_x(t) := T(t)x$$

are continuous for all $x \in X$.

Definition 2.3. *The generator* of a strongly continuous semigroup $T(t)$ is the operator, defined on the set

$$D(A) = \{x : \xi_x \text{ is differentiable}\}$$

and acting by the rule

$$A : x \mapsto \lim_{t\downarrow 0} \frac{T(t)x - x}{t}.$$

We will often denote the semigroup $T(t)$ by e^{At}.

Property 2.1. *The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.*

Property 2.2. *If $\{T(t)\}_{t\geqslant 0}$ is a strongly continuous semigroup then there exist $\omega \in \mathbb{R}$ and $N \geqslant 1$, such that for all $t \geqslant 0$*

$$\|T(t)\| \leqslant Ne^{\omega t}.$$

Definition 2.4. Let $\mathcal{T} = \{T(t)\}_{t\geqslant 0}$ be a strongly continuous semigroup. Then its *growth bound* is defined by

$$\omega_0(\mathcal{T}) = \inf \left\{\omega \in \mathbb{R} : \exists N_\omega \text{ such that } \forall (t \geqslant 0) \text{ we have that } \|T(t)\| \leqslant N_\omega e^{\omega t}\right\}.$$

Property 2.3. *Let $\mathcal{T} = \{T(t)\}_{t\geqslant 0}$ be a strongly continuous semigroup then for all $t \geqslant 0$*

$$\rho(T(t)) = e^{\omega_0(\mathcal{T})t}.$$

Definition 2.5. The semigroup $\{T(t)\}_{t\geqslant 0}$ is called *uniformly exponentially stable* if

$$\exists \varepsilon > 0 : \lim_{t\to\infty} e^{\varepsilon t}\|T(t)\| = 0.$$

Property 2.4. *A strongly continuous semigroup* $\{T(t)\}_{t\geqslant 0}$ *is uniformly exponentially stable if and only if its growth bound* $\omega_0(\mathcal{T}) < 0$.

Definition 2.6. Let X be a Banach space and $A : D(A) \subset A \to X$ be a closed operator. Then

$$s(A) := \sup\{\operatorname{Re}\lambda \in \sigma(A)\}$$

is called *spectral bound.*

Definition 2.7. *The essential growth bound* of the semigroup $\mathcal{T} = \{T(t)\}_{t\geqslant 0}$, generated by A, is defined as follows

$$\omega_{ess}(\mathcal{T}) = \omega_{ess}(A) := \inf_{t>0} \frac{1}{t} \ln \|T(t)\|_{ess},$$

where

$$\|S\|_{ess} = \inf\{\|S - C\| : C \text{ is compact}\}.$$

Property 2.5. *Let A be the generator of a strongly continuous semigroup* $\mathcal{T} = \{T(t)\}_{t\geqslant 0}$. *Then*

$$\omega_0(\mathcal{T}) := \max\{\omega_{ess}(\mathcal{T}), s(A)\}.$$

Definition 2.8. Let X be a Banach space and $\Sigma_\delta \overset{\text{def}}{=} \{\lambda \in \mathbb{C} : |\arg\lambda| < \delta\}\backslash\{0\}$.

A family of operators $(T(z))_{z\in\Sigma_\delta\cup\{0\}} \subset L(X)$ is called an *analytic semigroup (of angle* $\delta \in (0, \frac{\pi}{2}]$*)* if

(i) $T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\delta$.
(ii) The map $z \mapsto T(z)$ is analytic in Σ_δ.
(iii) $\lim_{\Sigma_{\delta'}\ni z\to 0} T(z)x = x$ for all $x \in X$ and $0 < \delta' < \delta$.

If, in addition,

(iv) $\|T(z)\|$ is bounded in $\Sigma_{\delta'}$ for every $0 < \delta' < \delta$, we call $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ a *bounded analytic semigroup.*

Property 2.6. *Let $A : D(A) \subset F \to F$ be a linear operator defined in a complex Banach space F. If $\vartheta \in \mathbb{R}$, then*

$$e^{-i\vartheta}\sigma(A) = \sigma(e^{i\vartheta}A).$$

Property 2.7. *Let A be a generator of analytic semigroup of angle α. Let $\vartheta \in (-\alpha, \alpha)$. Then $e^{A(e^{i\vartheta}t)}$ is a strongly continuous semigroup and its generator A_ϑ is given by $e^{i\vartheta}A$.*

2.3. Complexification of spaces and operators

We recall now the complexification of the domain of a linear operator and the corresponding extension of the operator. For further details compare [15].

Let $A : F \to F$ be a linear operator, where F is a real Banach space. We define the operator $A_C : F_C \to F_C$, as follows

$$A_C : (x, y) \mapsto (Ax, Ay),$$

where $F_C = F \times F$ is the complex Banach space with norm

$$\|(x,y)\| = \max_{\theta \in [0,2\pi]} \|x \cos\theta + y \sin\theta\|.$$

It is easy to verify the following properties.

Property 2.8. $A_C B_C = (AB)_C$, $(A^{-1})_C = (A_C)^{-1}$.

Property 2.9. *For all real numbers* α, β *we have that* $(\alpha A + \beta B)_C = \alpha A_C + \beta B_C$.

Property 2.10. A_C *is a compact operator, if and only if* A *is a compact operator.*

Property 2.11. $\|A_C\| = \|A\|$ *and by Gelfand's formula:* $\rho(A_C) = \rho(A)$.

Property 2.12. e^{At} *is a strongly continuous semigroup if and only if* $(e^{At})_C$ *is a strongly continuous semigroup and the generator of* $(e^{At})_C$ *is given by* A_C, *i.e.*

$$e^{A_C t} = (e^{At})_C.$$

Moreover, by Property 2.11 we have that

$$\omega_0(e^{A_C t}) = \omega_0(e^{At}).$$

Remark 2.1. If the operator A acts on a real Banach space we will always understand its spectrum, spectral radius and eigenvalue as spectrum, spectral radius and eigenvalue of its complexification A_C.

3. A necessary condition for the stability

We can prove the following result.

Theorem 3.1. *Let* F *be a real Banach space, let* $K \subset F$ *be a generating normal cone. Assume that the norm in* F *is monotone and that*

(1) $\Gamma : D(\Gamma) \subset F \to F$ *is a linear operator such that* $-\Gamma$ *is the generator of a uniformly exponentially stable strongly continuous semigroup* $e^{-\Gamma t}$, $t \geq 0$, *with* $e^{-\Gamma t} \geqslant 0$ *in the ordered Banach space* (F, K) *for any* $t \geq 0$.
(2) $M : F \to F$ *is a bounded linear operator such that* $M \geq 0$ *in* (F, K).
(3) *The composition* $\Gamma^{-1}M$ *is compact.*

Then for all $t > 0$

$$\rho(e^{(-\Gamma+M)t}) < 1 \Longrightarrow \rho(\Gamma^{-1}M) < 1$$

Proof. Since $\Gamma^{-1}M$ is compact its spectrum consists of isolated eigenvalues of finite multiplicity, see [16, III.6.7]. If $\Gamma^{-1}M$ does not have nonzero eigenvalue then $\rho(\Gamma^{-1}M) = 0$ and so theorem is proved. Thus, consider the case when $\Gamma^{-1}M$ has nonzero eigenvalues. Assume, by contradiction, that $\rho(\Gamma^{-1}M) \geqslant 1$, and observe that, since $\omega_0(e^{(-\Gamma)t}) < 0$,

$$\Gamma^{-1}x = R(0,\Gamma)x = -R(0,-\Gamma)x = \int_0^{+\infty} e^{-\Gamma t} x dt \geqslant 0,$$

for any $x \in K$, hence $\Gamma^{-1}M$ is positive as composition of positive operators and it is compact by assumption. Since the cone K is generating, by [20, Theorem 6, §6] there exists an eigenvector $x_0 \in K$ corresponding to the eigenvalue $\lambda = \rho(\Gamma^{-1}M)$. Therefore, we have

$$\Gamma^{-1}Mx_0 = \lambda x_0,$$

and so

$$x_0 = \frac{1}{\lambda}\Gamma^{-1}Mx_0,$$
$$\Gamma x_0 = \Gamma\frac{1}{\lambda}\Gamma^{-1}Mx_0 = \frac{1}{\lambda}\Gamma\Gamma^{-1}Mx_0 = \frac{1}{\lambda}Mx_0 \in K$$

Then

$$\begin{aligned}(-\Gamma + M)x_0 &= Mx_0 - \Gamma x_0 = \Gamma\Gamma^{-1}Mx_0 - \Gamma x_0 = \\ &= \Gamma\lambda x_0 - \Gamma x_0 = \lambda\Gamma x_0 - \Gamma x_0 = (\lambda - 1)\Gamma x_0 \in K.\end{aligned}$$

Consider now the Cauchy problem

$$\begin{cases} y' = (-\Gamma + M)y, \\ y(0) = x_0, \end{cases} \tag{3.1}$$

and its solution $y(t) = e^{(-\Gamma+M)t}x_0$. For $t, \triangle t > 0$ evaluate

$$\begin{aligned}y(t+\triangle t) - y(t) &= e^{(-\Gamma+M)t}e^{(-\Gamma+M)\triangle t}x_0 - e^{(-\Gamma+M)t}x_0 = \\ &= e^{(-\Gamma+M)t}(e^{(-\Gamma+M)\triangle t}x_0 - x_0) = e^{(-\Gamma+M)t}\int_0^{\triangle t} e^{(-\Gamma+M)s}(-\Gamma + M)x_0\,ds = \\ &= e^{(-\Gamma+M)t}\int_0^{\triangle t} e^{(-\Gamma+M)s}(\lambda - 1)\Gamma x_0\,ds.\end{aligned}$$

On the other hand each operator $e^{(-\Gamma+M)t}$ is positive because it can be found using iterative process $v_0 = I$; $v_{i+1} = Qv_i$ for the following positive operator:

$$(Qv)(t) = e^{-\Gamma t} + \int_0^t e^{(-\Gamma)(t-s)}Mv(s)ds.$$

Since we have supposed that $\lambda = \rho(\Gamma^{-1}M) \geqslant 1$ we have that

$$y(t+\triangle t) - y(t) \geqslant 0,$$

i.e.

$$y(t+\triangle t) \geqslant y(t)$$

in (F, K). The positivity of $y(t+\triangle t)$ and $y(t)$ and the monotonicity of the norm of F imply that

$$\|y(t+\triangle t)\| \geqslant \|y(t)\|.$$

This contradicts the assumed stability of the semigroup, thus $\rho(\Gamma^{-1}M) < 1$. □

4. The positiveness map

Let F be a real Banach space and let $K \subset F$ be a generating cone. Define a multivalued map $P : F \to 2^K$ as follows

$$P : x \longmapsto \{u+v \ : \ u, v \in K, u-v = x\}$$

where 2^K is the class of all subsets of the cone K. Since the cone is generating we have that $P(x) \neq \emptyset$ for every $x \in F$.

The map P has the following properties.

Property 4.1. *$\forall x \in F$, $P(x) \subset K$.*

Proof. It is obvious, since any $y \in P(x)$ is a sum of two elements of K. □

Property 4.2. *$\forall x \in F$, $P(x) = P(-x)$.*

Proof. Let $y \in P(x)$, thus there exist $u, v \in K$ such that $u + v = y$ and $u - v = x$. Then $v - u = -x$ implies $y = u+v \in P(-x)$, i.e. $P(x) \subset P(-x)$. Taking $y \in P(-x)$ the same argument shows that $P(x) \subset P(-x)$. □

Property 4.3. *$\forall x \in F$ and $\forall y \in P(x)$ we have $y + x \in K$ and $y - x \in K$.*

Proof. Let $y \in P(x)$, hence by definition there exist $u, v \in K$ such that $u + v = y$ and $u - v = x$. Then $y + x = u + v + u - v = 2u \in K$ and $y - x = u + v - (u - v) = u + v - u + v = 2v \in K$. □

Property 4.4. *$y \geqslant x$ and $y \geqslant -x$ imply $y \in P(x)$.*

Proof. Consider u, v as follows: $u = \frac{1}{2}(y+x) \in K$ and $v = \frac{1}{2}(y-x) \in K$, thus $u+v = y$ and $u-v = x$. □

Property 4.5. *$P(0) = K$.*

Proof. By Property 4.1 we have that $P(0) \subset K$. To prove that $K \subset P(0)$, let $y \in K$ and consider $u = v = \frac{y}{2}$, thus $y \in P(0)$. □

Property 4.6. *$\forall x \in F$ and $\forall b \in P(x)$, $a \geqslant b$ implies $a \in P(x)$.*

Proof. Let $b \in P(x)$, by definition there exist $u, v \in K$ such that $u + v = b$ and $u - v = x$. Consider $a \geqslant b$, namely $a-b \in K$. Let $u_1 = u+\frac{a-b}{2} \in K$ and $v_1 = v+\frac{a-b}{2} \in K$. Then $u_1+v_1 = u+\frac{a-b}{2}+v+\frac{a-b}{2} = b+a-b = a$, $u_1 - v_1 = u + \frac{a-b}{2} - (v + \frac{a-b}{2}) = u - v = x$, hence $a \in P(x)$. □

Property 4.7. *$\forall x \in F$, $P(x)$ is a closed set.*

Proof. Let $\{y_i\} \subset P(x)$ be a sequence such that $y_i \to y$ as $i \to +\infty$, thus $\frac{1}{2}(y_i + x) \to \frac{1}{2}(y + x)$ and $\frac{1}{2}(y_i - x) \to \frac{1}{2}(y - x)$. Since K is closed, by Property 4.3 we get that $u = \frac{1}{2}(y+x) \in K$ and $v = \frac{1}{2}(y-x) \in K$ and so $u + v = y$, $u - v = x$. Hence, $y \in P(x)$ and so $P(x)$ is closed. □

Property 4.8. *$\forall x \in F$ and $\forall \alpha \in \mathbb{R}$ we have that $|\alpha| P(x) \subset P(\alpha x)$.*

Proof. For $\alpha = 0$ we have that $|\alpha|P(x) = 0 \subset K = P(0) = P(\alpha x)$.

For $\alpha > 0$ we have that $|\alpha|P(x) = \alpha P(x)$. Let $y \in P(x)$, by definition there exist $u, v \in K$ such that $u + v = y$ and $u - v = x$. Then $\alpha y = \alpha(u + v) = \alpha u + \alpha v$, with $\alpha u \in K$ and $\alpha v \in K$, moreover $\alpha u - \alpha v = \alpha(u - v) = \alpha x$, hence $\alpha y \in P(\alpha x)$.

For $\alpha < 0$ we have that $|\alpha|P(x) = -\alpha P(x)$. Let $y \in P(x)$, thus there exist $u, v \in K$ such that $u + v = y$ and $u - v = x$. Then $-\alpha y = -\alpha(u + v) = (-\alpha u) + (-\alpha v)$, with $-\alpha u \in K$, and $-\alpha v \in K$, moreover $(-\alpha v) - (-\alpha u) = \alpha(u - v) = \alpha x$, hence $-\alpha y \in P(\alpha x)$. □

Property 4.9. *$\forall x_1, x_2 \in F$ we have that $P(x_1) + P(x_2) \subset P(x_1 + x_2)$.*

Proof. Let $y_1 \in P(x_1)$ and $y_2 \in P(x_2)$, thus there exist $u_1, v_1 \in K$ such that $u_1 + v_1 = y_1$ and $u_1 - v_1 = x_1$ and there exist $u_2, v_2 \in K$ such that $u_2 + v_2 = y_2$ and $u_2 - v_2 = x_2$. Then $y_1 + y_2 = u_1 + v_1 + u_2 + v_2 = (u_1 + u_2) + (v_1 + v_2)$ and $(u_1 + u_2) - (v_1 + v_2) = u_1 - v_1 + u_2 - v_2 = x_1 + x_2$, hence $y_1 + y_2 \in P(x_1 + x_2)$. In conclusion $P(x_1) + P(x_2) \subset P(x_1 + x_2)$. □

Property 4.10. *Let $S : F \to F$ be a linear positive operator. Then $\forall x \in F$ we have that $SPx \subset PSx$.*

Proof.

$$P(x) := \{u + v : u, v \in K, u - v = x\},$$

$$\begin{aligned} SP(x) &= S\{u + v : u, v \in K, u - v = x\} = \\ &= \{Su + Sv : u, v \in K, u - v = x\} \subset P(Su - Sv) = PS(u - v) = PS(x). \end{aligned} \quad □$$

5. Stability theorem

In order to establish the main theorem of the paper, i.e. Theorem 5.1, we need the following preliminary lemmas.

Lemma 5.1. *Let F be a complex Banach space. Assume that*

(1) $\Gamma : D(\Gamma) \subset F \to F$ is a linear operator such that Γ^{-1} is compact.
(2) $M : F \to F$ is a bounded operator.

Moreover, assume that $\rho(\Gamma^{-1}M) < 1$. Then the operator $-\alpha\Gamma + M$ has a compact resolvent for all α such that $\alpha' \leqslant \alpha < +\infty$, where $0 < \alpha' < 1$.

Proof. Let $\varepsilon > 0$ such that $0 < \varepsilon < 1 - \rho(\Gamma^{-1}M)$. We can define a norm on F, see [17, 2.5.2], denoted by $\|\cdot\|_\varepsilon$ which is equivalent to the norm $\|\cdot\|$ on F and satisfying the inequality:

$$\|\Gamma^{-1}M\|_\varepsilon \leqslant \rho(\Gamma^{-1}M) + \varepsilon < \rho(\Gamma^{-1}M) + 1 - \rho(\Gamma^{-1}M) = 1.$$

Now choose α' such that $\|\Gamma^{-1}M\|_\varepsilon < \alpha' < 1$. Clearly, for any α with $\alpha' \leqslant \alpha < +\infty$ the estimate $\|\frac{1}{\alpha}\Gamma^{-1}M\|_\varepsilon < 1$ holds. Therefore, let λ be such that

$$0 < \lambda < \frac{1 - \|\frac{1}{\alpha}\Gamma^{-1}M\|_\varepsilon}{\|\frac{1}{\alpha}\Gamma^{-1}\|_\varepsilon}.$$

Then

$$\|\frac{1}{\alpha}\Gamma^{-1}M\|_\varepsilon + \|\frac{\lambda}{\alpha}\Gamma^{-1}\|_\varepsilon < 1.$$

This means that

$$\rho\left(\frac{1}{\alpha}\Gamma^{-1}M - \frac{\lambda}{\alpha}\Gamma^{-1}\right) \leqslant \|\frac{1}{\alpha}\Gamma^{-1}M - \frac{\lambda}{\alpha}\Gamma^{-1}\|_\varepsilon \leqslant \|\frac{1}{\alpha}\Gamma^{-1}M\|_\varepsilon + \|\frac{\lambda}{\alpha}\Gamma^{-1}\|_\varepsilon < 1.$$

We can expand the resolvent in a Neumann series to obtain

$$R_\lambda(-\alpha\Gamma + M) = (-\alpha\Gamma + M - \lambda I)^{-1} = \left(-\alpha\Gamma\left(I - \frac{1}{\alpha}\Gamma^{-1}M + \frac{\lambda}{\alpha}\Gamma^{-1}\right)\right)^{-1} =$$
$$= \left(I - \frac{1}{\alpha}\Gamma^{-1}M + \frac{\lambda}{\alpha}\Gamma^{-1}\right)^{-1}\left(-\frac{1}{\alpha}\Gamma^{-1}\right) = \sum_{k=0}^{\infty}\left(\frac{1}{\alpha}\Gamma^{-1}M - \frac{\lambda}{\alpha}\Gamma^{-1}\right)^k\left(-\frac{1}{\alpha}\Gamma^{-1}\right). \tag{5.1}$$

From (5.1) it follows that $R_\lambda(-\alpha\Gamma+M)$ is compact. Therefore, the operator $-\alpha\Gamma+M$ has compact resolvent, see [16, III.6.8], for all α such that $\alpha' \leqslant \alpha < +\infty$, where $\alpha' < 1$. □

Lemma 5.2. *Let F be a real Banach space and let $K \subset F$ be a generating cone. Assume that*

(1) *$\Gamma : D(\Gamma) \subset F \to F$ is a linear operator such that $-\Gamma$ is the generator of a uniformly exponentially stable strongly continuous semigroup $e^{-\Gamma t}$, $t \geq 0$, with $e^{-\Gamma t} \geqslant 0$ in the ordered Banach space (F, K) for any $t \geq 0$;*
(2) *$M : F \to F$ is a bounded linear operator such that $M \geq 0$ in (F, K).*

Moreover, assume that $\rho(\Gamma^{-1}M) < 1$. Then the linear operators $-\alpha\Gamma + M$, where $\alpha \in (0, +\infty)$, can possess an eigenvalue with zero real part only if $\alpha < 1$.

Proof. We argue by contradiction, hence we assume that the operator $-\alpha_0\Gamma+M$ has an eigenvalue $i\omega$, $\omega \in \mathbb{R}$, for some $\alpha_0 \geqslant 1$. Denote the corresponding eigenvector by $x_0 + iy_0 \neq 0$. The solution of the equation

$$y' = (-\alpha_0\Gamma_C + M_C)y$$

with initial condition $y(0) = x_0 + iy_0$ is given by

$$y_1(t) = (x_0 + iy_0)(\cos\omega t + i\sin\omega t) =$$
$$= (x_0\cos\omega t - y_0\sin\omega t) + i(x_0\sin\omega t + y_0\cos\omega t).$$

While the solution corresponding to the initial condition $y(0) = x_0 - iy_0$ is given by

$$y_2(t) = (x_0 - iy_0)(\cos\omega t - i\sin\omega t) =$$
$$= (x_0\cos\omega t - y_0\sin\omega t) - i(x_0\sin\omega t + y_0\cos\omega t).$$

Thus their sum is the solution corresponding to the initial condition $y(0) = 2x_0$ and it is given by

$$y^*(t) = 2x_0\cos\omega t - 2y_0\sin\omega t.$$

The solution y^* is periodic of period $T = \frac{2\pi}{\omega}$ and it is non-zero, since $x_0 + iy_0 \neq 0$. Moreover, this solution has no imaginary part, and so, for the rest of the proof we can consider the involved operators as defined in F instead that in F_C. Since y^* can be viewed as the solution of the following non-homogeneous equation

$$\begin{cases} y' = -\alpha_0 \Gamma y + My^*, \\ y(0) = 2x_0; \end{cases}$$

it can be represented as follows

$$y^*(t) = \int_{-\infty}^{t} e^{-\alpha_0 \Gamma(t-s)} My^*(s)ds.$$

Consider now we the family of linear operators $Q_t : F \longmapsto F$ defined by

$$Q_t : x \longmapsto \int_{-\infty}^{t} e^{-\alpha_0 \Gamma(t-s)} Mxds.$$

Notice, that for any $t \in \mathbb{R}$, the operator Q_t coincides with the operator $Q := \frac{1}{\alpha_0}\Gamma^{-1}M$. Indeed, for any $x \in F$, consider

$$Q_t x = \int_{-\infty}^{t} e^{-\alpha_0 \Gamma(t-s)} Mxds = -\int_{+\infty}^{0} e^{-\alpha_0 \Gamma q} Mxdq = \int_{0}^{+\infty} e^{-\alpha_0 \Gamma q} Mxdq =$$
$$= -R(0, -\alpha_0 \Gamma)Mx = \frac{1}{\alpha_0}\Gamma^{-1}Mx$$

That is

$$Q_t = \frac{1}{\alpha_0}\Gamma^{-1}M = Q,$$

with $\rho(\Gamma^{-1}M) < 1$, hence we can define an equivalent norm $\|\cdot\|_L$ on F, see [17, 2.5.2], such that

$$\|\Gamma^{-1}M\|_L \overset{\text{def}}{=} \sup_{x \in F, x \neq 0} \frac{\|\Gamma^{-1}Mx\|_L}{\|x\|_L} < 1$$

As a consequence $\forall x, y \in F$ we have

$$\|Q_t x - Q_t y\|_L = \|Q_t(x-y)\|_L = \|\frac{1}{\alpha_0}\Gamma^{-1}M(x-y)\|_L \leqslant \frac{1}{\alpha_0}\|\Gamma^{-1}M\|_L \|x-y\|_L$$

On the other hand, by assumption $\alpha_0 \geqslant 1$, hence Q_t is a contraction and so it has a unique fixed point in F. But Q_t is a linear operator, thus the fixed point is zero. Since the cone K is generating we can define the positiveness map P in F. Now let $\widetilde{P}$ be the set defined as follows

$$\widetilde{P} := \bigcap_{s \in \mathbb{R}} P(y^*(s)).$$

$\widetilde{P}$ is closed since any set $P(y^*(s))$ is closed. We prove that $\widetilde{P}$ is nonempty. For this, let

$$a_1 = 2x_0 + 2y_0,$$
$$a_2 = 2x_0 - 2y_0$$

then

$$2x_0 = \frac{a_1 + a_2}{2},$$
$$2y_0 = \frac{a_1 - a_2}{2}.$$

Since the cone K is generating, there exist $u_1, v_1, u_2, v_2 \in K$, such that

$$a_1 = u_1 - v_1,$$
$$a_2 = u_2 - v_2.$$

Let $\xi = u_1 + v_1 + u_2 + v_2$. Clearly $\xi \in K$ and

$$\begin{aligned}\xi - y^*(t) &= u_1 + v_1 + u_2 + v_2 - 2x_0 \cos\omega t + 2y_0 \sin\omega t = \\ &= u_1 + v_1 + u_2 + v_2 - \frac{a_1 + a_2}{2}\cos\omega t + \frac{a_1 - a_2}{2}\sin\omega t = \\ &= u_1 + v_1 + u_2 + v_2 - \frac{u_1 - v_1 + u_2 - v_2}{2}\cos\omega t + \frac{u_1 - v_1 - u_2 + v_2}{2}\sin\omega t = \\ &= u_1\left(1 - \frac{\cos\omega t}{2} + \frac{\sin\omega t}{2}\right) + v_1\left(1 + \frac{\cos\omega t}{2} - \frac{\sin\omega t}{2}\right) + u_2\left(1 - \frac{\cos\omega t}{2} - \frac{\sin\omega t}{2}\right) + \\ &\quad + v_2\left(1 + \frac{\cos\omega t}{2} + \frac{\sin\omega t}{2}\right).\end{aligned}$$

It is easy to see that all the coefficients above are non-negative, thus $\xi - y^*(t) \in K$, and, by definition, $\xi \geqslant y^*(t)$ for all t. In virtue of the equality $-y^*(t) = y^*(t + \frac{\pi}{\omega})$ and Property 4.4 we get $\xi \in P(y^*(t))$ and so $\xi \in \widetilde{P}$, i.e. $\widetilde{P}$ is nonempty.

Moreover, for all $t \in \mathbb{R}$ and $\widetilde{y} \in \widetilde{P}$ we have:

$$\begin{aligned}Q_t\widetilde{y} &= \int_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)} M\widetilde{y}ds = \int_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)} M\left(\frac{\widetilde{y} + y^*(s)}{2} + \frac{\widetilde{y} - y^*(s)}{2}\right)ds = \\ &= \int_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)} M\frac{\widetilde{y} + y^*(s)}{2}ds + \int_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)} M\frac{\widetilde{y} - y^*(s)}{2}ds \in \\ &\in P\left(\int_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)} M\frac{\widetilde{y} + y^*(s)}{2}ds - \int_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)} M\frac{\widetilde{y} - y^*(s)}{2}ds\right) = \\ &= P\left(\int_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)} M\left(\frac{\widetilde{y} + y^*(s)}{2} - \frac{\widetilde{y} - y^*(s)}{2}\right)ds\right) = \\ &= P\left(\int_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)} My^*(s)ds\right) = P(y^*(t)).\end{aligned}$$

But we have shown that $Q_t = Q$ for all $t \in \mathbb{R}$, hence for any $t \in \mathbb{R}$ we have $Q\widetilde{y} \in P(y^*(t))$, i.e. $Q\widetilde{y} \in \bigcap_{t\in\mathbb{R}} P(y^*(t)) = \widetilde{P}$. This means that the map $\widetilde{Q} : \widetilde{P} \to \widetilde{P}$ defined by

$$\widetilde{Q} : x \longmapsto \frac{1}{\alpha_0}\Gamma^{-1}Mx$$

is well defined. Since the set $\widetilde{P}$ is closed it can be considered as a metric space with the metric generated by the norm of the space F. The map $\widetilde{Q}$ is a contraction as well as Q, then it has fixed point $y_f \in \widetilde{P}$. But y_f is also the fixed point of the map Q, which has zero as the unique fixed point. Hence $y_f = 0 \in \widetilde{P}$. Observe that $\widetilde{P} \subset P(y^*(0))$, thus $0 \in P(y^*(0))$ and so there exist $u_x, v_x \in K : u_x + v_x = 0$ and $u_x - v_x = y^*(0)$. Therefore

$$u_x - v_x = y^*(0) = 2x_0 \cos(\omega 0) - 2y_0 \sin(\omega 0) = 2x_0$$

But K is a cone, thus $u_x, v_x \in K$ and $u_x + v_x = 0$ imply that $u_x = 0$ and $v_x = 0$, i.e. $u_x - v_x = 0 = 2x_0$.

Analogously we obtain that $\widetilde{P} \subset P(y^*(\frac{\pi}{2}))$, thus $0 \in P(y^*(\frac{\pi}{2}))$ and so there exist $u_y, v_y \in K$ such that $u_y + v_y = 0$ and $u_y - v_y = y^*(\frac{\pi}{2}) = -2y_0$. As before $u_y = 0$ and $v_y = 0$ give $u_y - v_y = 0 = -2y_0$. In conclusion, we have shown that $x_0 = 0$ and $y_0 = 0$, contradicting the fact that $x_0 + iy_0 \neq 0$, hence $\alpha_0 < 1$. □

We are now in the position to prove the following result.

Theorem 5.1. *Let F be a real Banach space, let $K \subset F$ be a generating cone. Assume that*

(1) *$\Gamma : D(\Gamma) \subset F \to F$ is a linear operator such that Γ^{-1} is compact, $-\Gamma_C$ is the generator of an analytic uniformly exponentially stable semigroup, and the operators $e^{-\Gamma t}$, $t \geq 0$, are positive in the ordered Banach space (F, K) for any $t \geq 0$;*
(2) *$M : F \to F$ is a linear bounded operator such that $M \geqslant 0$ in (F, K).*

Then for all $t > 0$ we have that

$$\rho(\Gamma^{-1}M) < 1 \Longrightarrow \rho(e^{(-\Gamma+M)t}) < 1.$$

Proof. Since the semigroup $e^{-\Gamma t}$ is strongly continuous and stable there exist $c \geq 0$ and $\omega_1 < 0$, such that $\|e^{-\Gamma t}\| \leqslant ce^{\omega_1 t}$.

Let $\alpha \geqslant 0$, then $D(-\alpha\Gamma) = D(-\Gamma)$, and $e^{(-\alpha\Gamma)t} = e^{-\Gamma(\alpha t)}$ is a strongly continuous semigroup with $\|e^{(-\alpha\Gamma)t}\| \leqslant ce^{\omega_1 \alpha t}$. From Lemma 5.1 it follows that there exists $\alpha' : 0 < \alpha' < 1$ such that $-\alpha\Gamma_C + M_C$ and $-\alpha\Gamma + M$ are operators with compact resolvent for all $\alpha \geqslant \alpha'$.

Consider now the following family of operators.

$$\{-\alpha\Gamma + M\}_{\alpha \geqslant \alpha'}. \tag{5.2}$$

By [7, Theorem III.1.3, p. 158] the boundedness of M implies the strong continuity of semigroup $e^{(-\alpha\Gamma+M)t}$. Moreover,

$$\|e^{(-\alpha\Gamma+M)t}\| \leqslant ce^{(\alpha\omega_1 + c\|M\|)t}.$$

Since $\omega_1 < 0$, for $\alpha > -\frac{c\|M\|}{\omega_1}$ we have that $\alpha\omega_1 + c\|M\| < 0$, i.e. the whole spectrum $\sigma(-\alpha\Gamma + M)$ belongs to the left complex half-plane.

As $-\Gamma_C$ is the generator of an analytic semigroup, there exists $\vartheta \in (0, \frac{\pi}{2})$, such that the restrictions of $e^{-\Gamma_C z}$ to $\{e^{i\vartheta}t : t \geqslant 0\}$ and $\{e^{-i\vartheta}t : t \geqslant 0\}$ are strongly continuous semigroups. By Property 2.7 their generators are $-e^{i\vartheta}\Gamma_C$ and $-e^{-i\vartheta}\Gamma_C$ respectively. Thus there exist N_i, N_{-i}, ω_i, ω_{-i} such that for any $t \geq 0$ we have

$$\|e^{(-e^{i\vartheta}\Gamma_C)t}\| \leqslant N_i e^{\omega_i t}$$

and

$$\|e^{(-e^{-i\vartheta}\Gamma_C)t}\| \leqslant N_{-i} e^{\omega_{-i} t}.$$

Hence

$$\|e^{(e^{i\vartheta}(-\alpha\Gamma_C + M_C))t}\| \leqslant N_i e^{(\alpha\omega_i + N_i\|M\|)t}$$
$$\|e^{(e^{-i\vartheta}(-\alpha\Gamma_C + M_C))t}\| \leqslant N_{-i} e^{(\alpha\omega_{-i} + N_{-i}\|M\|)t},$$

this implies that

$$\operatorname{Re}\sigma(e^{i\vartheta}(-\alpha\Gamma_C + M_C)) \leqslant \alpha\omega_i + N_i\|M\|$$
$$\operatorname{Re}\sigma(e^{-i\vartheta}(-\alpha\Gamma_C + M_C)) \leqslant \alpha\omega_{-i} + N_{-i}\|M\|$$

and by Property 2.6

$$\operatorname{Re} e^{-i\vartheta}\sigma(-\alpha\Gamma + M) \leqslant \alpha\omega_i + N_i\|M\|$$
$$\operatorname{Re} e^{i\vartheta}\sigma(-\alpha\Gamma + M) \leqslant \alpha\omega_{-i} + N_{-i}\|M\|,$$

i.e. if $\lambda \in \sigma(-\alpha\Gamma + M)$, hence

$$\operatorname{Re}\lambda\cos\vartheta + \operatorname{Im}\lambda\sin\vartheta \leqslant \alpha\omega_i + N_i\|M\|$$
$$\operatorname{Re}\lambda\cos\vartheta - \operatorname{Im}\lambda\sin\vartheta \leqslant \alpha\omega_{-i} + N_{-i}\|M\|.$$

Finally,

$$\operatorname{Im}\lambda \leqslant -\operatorname{ctg}\vartheta\operatorname{Re}\lambda + \frac{\alpha\omega_i + N_i\|M\|}{\sin\vartheta} \tag{5.3}$$
$$\operatorname{Im}\lambda \geqslant \operatorname{ctg}\vartheta\operatorname{Re}\lambda - \frac{\alpha\omega_{-i} + N_{-i}\|M\|}{\sin\vartheta}. \tag{5.4}$$

Therefore, the spectrum of $-\alpha\Gamma + M$ is contained into the sector of complex plane, defined by the previous inequalities (5.3) and (5.4). On the other hand, as noticed before, $-\alpha\Gamma + M$ is an operator with compact resolvent for $\alpha \geqslant \alpha'$, thus its spectrum consists of isolated eigenvalues of finite multiplicity for every α. Clearly, as α changes, the eigenvalues form a continuous, possibly self-intersecting, branches $\mu(\alpha)$. Observe that it may also occur that $|\mu(\alpha)| \to \infty$ as α converges to finite value. For a detailed analysis we refer to [16, Theorem IV.3.16].

Note that the family (5.2) is holomorphic of type (A) and the related branches of eigenvalues $\alpha \to \mu(\alpha)$ are analytic functions.

Consider now the branch of eigenvalues described by the function $\mu : (a, b) \to \mathbb{C}$ (or $\mu : [a, b) \to \mathbb{C}$) with $\alpha' \leqslant a < 1$ and $1 < b \leqslant +\infty$. In what follows we examine the possible situations that may occur. If $b = +\infty$

then for $\alpha > -\frac{c\|M\|}{\omega_1}$, as already observed, $\mathrm{Re}\,\mu(\alpha) < 0$. In this case, if the branch $\mu(\alpha)$ does not cross the imaginary axis then it entirely belongs to the left half plane of $\mathbb{C}$. While if the branch crosses the imaginary axis, then, by Lemma 5.2, the cross can occur only for $\alpha < 1$, thus $\mathrm{Re}\,\mu(1) < 0$. If $b < +\infty$ and for $\alpha \geq b$ the function $\mu(\alpha)$ is not defined, then $|\mu(\alpha)| \to +\infty$ as $\alpha \to b-$.

On the other hand, from the estimates (5.3) and (5.4) the fact that $|\mu(\alpha)| \to +\infty$ as $\alpha \to b-$ implies that $\mathrm{Re}\,\mu(\alpha) \to -\infty$ as $\alpha \to b-$. Hence, there exists $\alpha \in (1, b)$ such that $\mathrm{Re}\,\mu(\alpha) < 0$, and so, as before by Lemma 5.2 $\mu(1)$ belongs to the left half plane. In conclusion, the entire spectrum of $-\Gamma + M$ belongs to the left half plane, i.e.

$$s(-\Gamma + M) \leq 0.$$

But $-\Gamma + M$ is an operator with compact resolvent, then the limit of its spectrum is infinity. Since the whole spectrum belongs to the sector defined by (5.3) and (5.4) for $\alpha = 1$. This implies that does not exist a sequence $\{\lambda_i\} \subset \sigma(-\Gamma + M)$ such that $\mathrm{Re}\,\lambda_\mathrm{i} \to 0$, hence

$$s(-\Gamma + M) < 0.$$

Moreover, $e^{(-\Gamma_C + M_C)t}$ is an analytic semigroup, hence it is norm continuous for $t > 0$ (*immediately norm continuous*), thus it is compact for $t > 0$ (*immediately compact*), since its generator has compact resolvent, see [7, Theorem 2.4.29] and the related diagram. This implies that there is no essential spectrum for $t > 0$, and

$$\omega_0(e^{(-\Gamma+M)t}) = s(-\Gamma + M) < 0,$$

i.e. $e^{(-\Gamma+M)t}$ is uniformly exponentially stable. □

Remark 5.1. The results proved in this paper can be applied to differential operators, for instance with Γ as the second derivative. Furthermore, operators with positive Green's function may be positive in appropriate spaces. Conditions for a Green's function to be positive can be found in [3].

Acknowledgments

The first two authors acknowledge the support by RFBR Grant 12-01-00392, the third one acknowledges the support by the research project GNAMPA 2013 "Equazioni di evoluzione degeneri e singolari: controllo ed applicazioni" of the Istituto Nazionale di Alta Matematica. The authors acknowledge Prof. Oleg Makarenkov for the helpful discussions which helped us to improve the paper. Finally, the authors acknowledge the anonymous referee for his/her valuable and accurate suggestions.

References

[1] M. Abbas, B.E. Rhoades, Fixed and periodic point results in cone metric spaces, Appl. Math. Lett. 22 (4) (2009) 511–515.
[2] W. Arendt, et al., One-Parameter Semigroups of Positive Operators, Lecture Notes in Mathematics, vol. 1184, Springer-Verlag, Berlin, 1986.
[3] Y.M. Berezanskii, Expansions in Eigenfunctions of Selfadjoint Operators, American Mathematical Society, Providence, RI, 1968 (translated from Russian).
[4] M. Bernkopf, A history of infinite matrices, Arch. Hist. Exact Sci. 4 (1968) 308–358.
[5] R.G. Cooke, Infinite Matrices and Sequence Spaces, Dover, New York, 1955.
[6] J. Dickerson, J. Gibson, Stability of linear functional equations on Banach spaces, J. Math. Anal. Appl. 55 (1) (1976) 150–155.
[7] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer-Verlag, New York, 2000.
[8] H. Fan, Y. Li, Analyticity and exponential stability of semigroups for the elastic systems with structural damping in Banach spaces, J. Math. Anal. Appl. 410 (2014) 316–322.

[9] A. Friedman, Partial Differential Equations, corrected reprint of the original edition, Robert E. Krieger Publishing Co., Huntington, NY, 1976, i+262 pp.
[10] E. Hille, R.S. Philips, Functional Analysis and Semigroups, American Mathematical Society, Providence, RI, 1957.
[11] L.-G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1468–1476.
[12] D. Ilic, V. Rakocevic, Common fixed points for maps on cone metric space, J. Math. Anal. Appl. 341 (2) (2008) 876–882.
[13] J. Janas, S. Naboko, Infinite Jacobi matrices with unbounded entries: asymptotics of eigenvalues and the transformation operator approach, SIAM J. Math. Anal. 36 (2) (2004) 643–658.
[14] M. Kamenskii, P. Nistri, An averaging method for singularly perturbed systems of semilinear differential inclusions with C_0-semigroups, Set-Valued Anal. 11 (2003) 345–357.
[15] L.V. Kantarovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, New York, 1982 (translated from Russian).
[16] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, Heidelberg, New York, 1966.
[17] M.A. Krasnoselskii, Positive Solutions of Operator Equations, P. Noordhoff Ltd., Groningen, 1964 (translated from Russian).
[18] M.A. Krasnoselskii, P.P. Zabreyko, E.I. Pustilnik, P.E. Sobolevski, Integral Operators in Spaces of Summable Functions, Nauka, Moscow, 1966 (in Russian).
[19] S.G. Krein, Linear Differential Equations in Banach Spaces, American Mathematical Society, Providence, RI, 1971 (translated from Russian).
[20] M.G. Krein, M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, Amer. Math. Soc. Transl. 26 (1950) (translated from Russian).
[21] M. Malejki, Asymptotics of large eigenvalues for some discrete unbounded Jacobi matrices, Linear Algebra Appl. 431 (10) (2009) 1952–1970.
[22] J. Mawhin, Topological Degree Methods in Nonlinear Boundary Value Problems, Expository lectures from the CBMS Regional Conference held at Harvey Mudd College, Claremont, Calif., June 9–15, 1977, CBMS Regional Conference Series in Mathematics, vol. 40, American Mathematical Society, Providence, RI, 1979, 122 pp.
[23] M. Megan, A. Pogan, On exponential h-expansiveness of semigroups of operators in Banach spaces, Nonlinear Anal. 52 (2) (2003) 545–556.
[24] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag New-York Inc., 1983.
[25] W.V. Petryshyn, V. Wolodymyr, Generalized Topological Degree and Semilinear Equations, Cambridge Tracts in Mathematics, vol. 117, Cambridge University Press, Cambridge, 1995, 240 pp.
[26] P.N. Shivakumar, K.C. Sivakumar, A review of infinite matrices and their applications, Linear Algebra Appl. 430 (4) (2009) 976–998.
[27] P.N. Shivakumar, J.J. Williams, N. Rudraiah, Eigenvalues for infinite matrices, Linear Algebra Appl. 96 (1987) 35–63.
[28] R. Triggiani, On the stabilizability problem in Banach space, J. Math. Anal. Appl. 55 (1975) 383–403.