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A study on semiflows generated by cooperative full-range CNNs

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ABSTRACT

The paper considers the full-range (FR) model of cellular neural networks (CNNs) characterized by ideal hard-limiter nonlinearities with two vertical segments in the current–voltage characteristic. It is shown that when the FRCNNs are cooperative, i.e., there are excitatory interconnections between distinct neurons, the generated solution semiflow is monotone and that monotonicity implies some fundamental restrictions on the geometry of omega-limit sets. The result on monotonicity is a generalization to the class of differential inclusions describing the dynamics of FRCNNs of a classic result due to Kamke for cooperative ordinary differential equations. The paper also points out difficulties to use the standard theory of eventually strongly monotone (ESM) semiflows for addressing convergence of FRCNNs. By means of counterexamples, it is shown that, even assuming the irreducibility of the interconnections, the semiflow generated by a cooperative FRCNN is not ESM; furthermore, also the LIMIT SET DICHOTOMY can be violated.

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KEY WORDS: cellular neural networks; differential inclusions; monotone semiflows; cooperative systems

1. INTRODUCTION

During the last two decades, considerable efforts were devoted to develop effective technologies for the electronic implementation of cellular neural network (CNN) processors with an increasingly large number of neurons [1–3]. An important step has been the introduction of the so called full-range (FR) model of CNNs [4], where the original pwl neuron activation of the standard (S) CNN model [5] was replaced by a hard-limiter nonlinearity that reduces the range where the state variables can evolve. This improved range has enabled to achieve lower power consumption and higher cell densities and has led to the development of the family of ACE chips in CMOS technology [4,6–9]. The ACE chips are the core of the CNN universal machine [10], where they are integrated with stored programmable algorithms and software on the system level. More recently, nanoscale electronic technologies have emerged as a powerful mean for implementing massively parallel large-scale CNN processors [11–13]. We refer the reader to [14,15], for the first successful attempts to bridge the gap between nanoscale devices and CMOS integrated devices in the implementation of CNNs, and to [16], for a comparison among several topographic and non-topographic image-processing architectures recently developed within the CNN community.

In this paper, we consider the FR model of CNNs with ideal hard-limiter nonlinearities with two vertical segments in the current–voltage (i-v) characteristic [4]. Actually, the current is zero for $|v| < v_\sigma$, where v_σ is a threshold set for simplicity to +1 Volt, whereas the current can assume any value in the interval $[0, +\infty)$ (resp., $(-\infty, 0]$) for $v = v_\sigma$ (resp., $v = -v_\sigma$). Due to the presence of multi-valued nonlinearities, the dynamics of FRCNNs can be rigorously described by means of a system of differential inclusions [4,17]. This is in contrast with the SCNN model whose dynamics is described by a system of ordinary differential equations.

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Since the FRCNN model is used as an effective implementation of the SCNN model, it is required that the two models display an analogous dynamical behavior for the same set of parameters (interconnections and inputs). Recent works have given results along two contrasting directions. On one hand, it has been shown that there are relevant classes of FRCNNs displaying analogous convergence and stability properties as the corresponding SCNNs. These include: (1) FRCNNs with symmetric neuron interconnections, where convergence can be analyzed by means of a generalized Lyapunov approach based on an extended version of LaSalle's invariance principle [18], or by means of the principle of trajectories with finite length and the Łojasiewicz inequality [19]; (2) a general class of FRCNNs defined by gradient-type systems [18]; (3) FRCNNs with nonsymmetric Lyapunov diagonally stable, or M-matrices [20], for which it is possible to prove general results on global asymptotic stability. On the other hand, a counterexample by Corinto and Gilli has shown that there are classes of convergent nonsymmetric SCNNs for which the corresponding FRCNNs display non-vanishing oscillations [21]. The example actually implies that, in general, it is not possible to extend to FRCNNs results on convergence already known for SCNNs. Such an extension, when possible, needs to be investigated on a case-by-case basis.

In this paper, we further investigate on the equivalence between dynamical properties of the SCNN model and the FRCNN model in the important case where there are nonsymmetric excitatory interconnections between neurons, i.e., the CNNs are *cooperative* [22,23]. In the fundamental paper [24], Chua and Roska have proved that the semiflow generated by a cooperative SCNN is monotone and that the semiflow is eventually strongly monotone (ESM), when the interconnection matrix is irreducible (the template is cell linking) and the pwl neuron activation of the SCNN model is approximated by a *sigmoid* (C^1, bounded and strictly increasing) activation. By the standard theory of ESM semiflows, it follows that the semiflow generated by a cell-linking CNN with sigmoid activations satisfies the LIMIT SET DICHOTOMY and is almost quasi-convergent, i.e., the generic solution converges toward the set of equilibrium points [24–26]. The main contribution in this paper is a rigorous proof that, as it happens for cooperative SCNNs, the semiflow generated by cooperative FRCNN is monotone (Theorem 1). Theorem 1 is a generalization to semiflows generated by a class of differential inclusions of a classic result due to Kamke for cooperative ordinary differential equations [27] (see also [23, Sect. 1] or [26, Prop. 1.1, p. 32]). Then, the paper shows that monotonicity of the semiflow implies restrictions on the geometry of omega-limit sets for cooperative FRCNNs, as the property of NON-ORDERING OF LIMIT SETS. Finally, the paper highlights some basic differences. The main difference is that, even assuming a cell-linking template, the semiflow generated by a cooperative FRCNN is not ESM; moreover, the LIMIT SET DICHOTOMY can fail. Then, it is not possible to prove convergence of cell-linking FRCNNs by directly applying the theory of ESM semiflows. Furthermore, differently from SCNNs, it is not possible to define a flow for FRCNNs. Indeed, while the ordinary differential equations describing SCNNs can be solved both forward and backward in time, the FRCNN equations cannot be solved backward in time. This implies that there are difficulties to reduce the analysis of *competitive* FRCNNs to that of cooperative FRCNNs by the typical trick of time reversal [25, Ch. 3].

Notation

Let $\mathbb{R}^n$ be the real n-space. Given matrix $A \in \mathbb{R}^{m \times n}$, by A' we denote the transpose of A. In particular, by E_n, we denote the $n \times n$ identity matrix. Given $x, y \in \mathbb{R}^n$, we denote by $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ the scalar product of x and y. With $\|x\| = \sqrt{\langle x, x \rangle}$ we mean the Euclidean norm of x. By $B(z, r) = \{y \in \mathbb{R}^n : \|y - z\| < r\}$, we denote an n-dimensional open ball with center $z \in \mathbb{R}^n$ and radius r in Euclidean norm. Given a set $D \subset \mathbb{R}^n$, by $\mathrm{bd}(D)$ and $\mathrm{int}(D)$, we denote the boundary of D and the interior of D, respectively. With $\mathrm{dist}(x, D) = \inf_{y \in D} \|x - y\|$, we denote the distance of $x \in \mathbb{R}^n$ from D, while $\mathrm{diam}(D) = \sup_{x, y \in D} \|x - y\|$ is the diameter of set D.

1.1. *Preliminaries*

1.1.1. Tangent and normal cones. This section reports the definition of tangent and normal cones to a closed convex set and some related properties used throughout the paper. The reader is referred to [28–30] for a more thorough treatment.

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Let $Q \subset \mathbb{R}^n$ be a nonempty, closed, convex set. The tangent cone to Q at $x \in Q$ is defined as [29,30]

$$T_Q(x) = \left\{ v \in \mathbb{R}^n : \liminf_{\rho \to 0^+} \frac{\text{dist}(x + \rho v, Q)}{\rho} = 0 \right\}$$

while the normal cone to Q at $x \in Q$ is given by

$$N_Q(x) = \{ p \in \mathbb{R}^n : \langle p, v \rangle \leq 0, \forall v \in T_Q(x) \}.$$

It is known that $T_Q(x)$ and $N_Q(x)$ are nonempty closed convex cones in $\mathbb{R}^n$. The two cones, evaluated at some points of a closed convex set $Q \subset \mathbb{R}^2$, are depicted in Figure 1.

Property 1

([17]) If Q coincides with the hypercube $K_n = [-1, 1]^n$, then $N_{K_n}(x)$ and $T_{K_n}(x)$ have the following expressions. For any $x \in K_n$ we have $T_{K_n}(x) = H(x) = (h(x_1), \ldots, h(x_n))'$, where

$$h(\rho) = \begin{cases} [0, +\infty), & \rho = -1 \\ (-\infty, +\infty), & \rho \in (-1, 1) \\ (-\infty, 0], & \rho = 1. \end{cases}$$

Furthermore, we have $N_{K_n}(x) = \Lambda(x) = (\lambda(x_1), \ldots, \lambda(x_n))'$, where

$$\lambda(\rho) = \begin{cases} (-\infty, 0], & \rho = -1 \\ 0, & \rho \in (-1, 1) \\ [0, +\infty), & \rho = 1. \end{cases}$$

The cones in Property 1, evaluated at some points of the set $K_2 = [-1, 1]^2 \subset \mathbb{R}^2$, are reported in Figure 2. For any $x \in \mathbb{R}^n$, the projection of x on Q is the unique point $\mathcal{P}_Q x \in Q$ satisfying

$$\|x - \mathcal{P}_Q x\| = \text{dist}(x, Q) = \min_{y \in Q} \|y - x\|.$$

If Q is a closed, convex cone, then $\mathcal{P}_Q x$ is said to be the orthogonal projection of x on Q. The next result provides a characterization of the orthogonal projection.

Property 2

([28, Prop. 2, p. 24]) Assume that $Q \subset \mathbb{R}^n$ is a closed, convex cone, and let $x, z \in \mathbb{R}^n$. We have $z = \mathcal{P}_Q x$ if and only if the following three conditions hold:

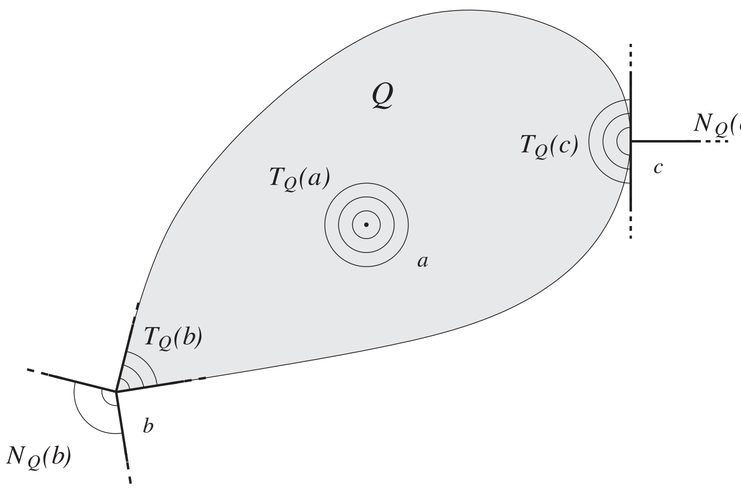

Figure 1. Set $Q \subset \mathbb{R}^2$ and cones T_Q and N_Q at points a, b and c of Q (the cones are shown translated into the corresponding points of Q).

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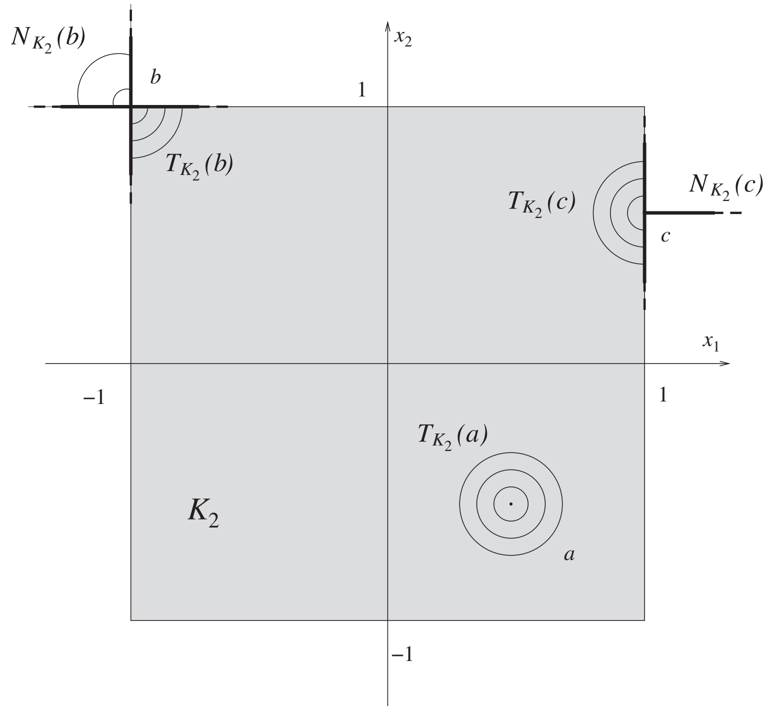

Figure 2. Set $K_2 = [-1, 1]^2$ and cones T_{K_2}, N_{K_2} at points a, b and c of K_2.

a. $z \in Q$;
b. $\langle x - z, y \rangle \leq 0$ for any $y \in Q$;
c. $\langle x - z, z \rangle = 0$.

Property 3
Let $\rho \in K_1 = [-1, 1] \subset \mathbb{R}$ and $u \in \mathbb{R}$. Then, we have

$$\mathcal{P}_{T_{K_1}}(\rho)u = \begin{cases} 0, & \rho u = |u| \\ u, & \rho u \neq |u|. \end{cases} \tag{1}$$

Moreover, let $x \in K_n$ and $v \in \mathbb{R}^n$. Then,

$$\mathcal{P}_{T_{K_n}(x)} v = (\mathcal{P}_{T_{K_1}(x1)} v_1, \ldots, \mathcal{P}_{T_{K_1}(x_n)} v_n)'.$$

Proof
The expression (1) for $\mathcal{P}_{T_{K_1}(\rho)} u$ can be easily derived. Now, let us denote by $z = (\mathcal{P}_{T_{K_1}(x_1)} v_1, \ldots, \mathcal{P}_{T_{K_1}(x_n)} v_n)'$ and choose any $y = (y_1, \ldots, y_n)' \in T_{K_n}(x)$. Since we have $T_{K_n}(x) = (h(x_1), \ldots, h(x_n))'$, it follows that $y_i \in h(x_i) = T_{K_1}(x_i)$ for any $i \in \{1, \ldots, n\}$. Hence, $|v_i - z_i| = |v_i - \mathcal{P}_{T_{K_1}(x_i)} v_i| \leq |v_i - y_i|$ for any $i \in \{1, \ldots, n\}$. It follows that $\|v - y\|^2 = \sum_{i=1}^n |v_i - y_i|^2 \geq \sum_{i=1}^n |v_i - z_i|^2 = \|v - z\|^2$. Thus, we have $z = \mathcal{P}_{T_{K_n}(x)} v$.

1.1.2. Monotone and eventually strongly monotone semiflows. Here, we briefly review some properties of monotone semiflows on $\mathbb{R}^n$ needed in the paper. We refer the reader to the surveys [25,26] for a comprehensive treatment.

We consider the space $\mathbb{R}^n$ with the following vector-order relations [26]. Given $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned} x \leq y &\Leftrightarrow x_i \leq y_i, i = 1, 2, \ldots, n \\ x < y &\Leftrightarrow x \leq y, x \neq y \\ x \ll y &\Leftrightarrow x_i < y_i, i = 1, 2, \ldots, n \end{aligned}$$

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If U, V are subsets of $\mathbb{R}^n$, $U \le V$ (resp., $U < V$) means that we have $x \le y$ (resp., $x < y$) for any $x \in U$ and $y \in V$. We say that a subset U of $\mathbb{R}^n$ is *unordered* if it is not possible to find points $u, v \in U$ related by $u < v$.

Let us denote with Q a nonempty, closed subset of $\mathbb{R}^n$. By a *semiflow* on Q, we mean a continuous map $\Phi : [0, +\infty) \times Q \to Q, (t, x) \to \Phi_t(x)$, satisfying $\Phi_0(x) = x, \Phi_t(\Phi_s(x)) = \Phi_{t+s}(x)$, for all $t, s \ge 0$ and $x \in Q$. A flow on Q is a continuous map $\Phi : \mathbb{R} \times Q \to Q, (t, x) \to \Phi_t(x)$, such that $\Phi_0(x) = x$ and $\Phi_t(\Phi_s(x)) = \Phi_{t+s}(x)$, for all $t, s \in \mathbb{R}$ and $x \in Q$. We suppose henceforth that for any $x \in Q$, the positive orbit through x, $\{\Phi_t(x) : t \in [0, +\infty)\}$, has compact closure. The *omega-limit set*, $\omega(x)$, of x is the set of points $y \in Q$ such that there exists a sequence $t_k \to +\infty$, as $k \to +\infty$, such that $\Phi_{t_k}(x) \to y$ as $k \to +\infty$. It is known that $\omega(x)$ is a nonempty, compact, connected subset of Q. Moreover, $\omega(x)$ is positively invariant, i.e., for any $p \in \omega(x)$, we have $\Phi_t(p) \in \omega(x)$ for any $t \ge 0$. The supremum, $\sup\omega(x)$, of $\omega(x)$ is the unique point $s_x \in \mathbb{R}^n$ such that $s_x \ge \omega(x)$ and $z \ge \omega(x)$ implies $z \ge s_x$. The infimum, $\inf\, \omega(x)$, of $\omega(x)$ is defined dually by substituting $\ge$ with $\le$. An equilibrium point (EP) of Φ is a point $\xi \in Q$ such that $\Phi_t(\xi) = \xi, t \ge 0$. We will denote by $E \subset Q$ the set of EPs of Φ.

Let Φ be a semiflow on Q. We say that Φ is:

- *monotone* if for any $x, y \in Q$ such that $x \le y$ we have $\Phi_t(x) \le \Phi_t(y)$, $t \ge 0$;
- *eventually strongly monotone* (ESM) if Φ is monotone, and for any $x, y \in Q$ such that $x < y$, there exists $\bar{t} > 0$ such that we have $\Phi_t(x) \ll \Phi_t(y)$, $t > \bar{t}$;
- *strongly order preserving* (SOP) if Φ is monotone, and for any $x, y \in Q$ such that $x < y$, there exist open neighborhoods U, V of x and y, respectively, and $\bar{t} \ge 0$ such that we have $\Phi_{\bar{t}}(U \cap Q) \le \Phi_{\bar{t}}(V \cap Q)$.

It is known that if Φ is ESM, then it is SOP, while the converse does not hold in general [26, Prop. 1.2].

2. SEMIFLOWS GENERATED BY FRCNNS

We consider the FR model of CNNs, introduced by Rodríguez-Vázquez et al. in [4], whose dynamics can be described by the system of differential inclusions [4,6,17]

$$\dot{x} \in -x + Ax + I - \Lambda(x) \tag{F}$$

where $x \in K = K_n = [-1, 1]^n$ is the vector of neuron state variables, $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is the neuron interconnection matrix, which is in general nonsymmetric, $I \in \mathbb{R}^n$ is the biasing input and $\Lambda(x) = (\lambda(x_1), \ldots, \lambda(x_n))'$ is a diagonal mapping where

$$\lambda(\rho) = \begin{cases} (-\infty, 0], & \rho = -1 \\ 0, & \rho \in (-1, 1) \\ [0, +\infty), & \rho = 1 \end{cases}$$

is the multi-valued characteristic of an ideal hard-limiter nonlinearity, as in Figure 3, constraining the state variables x_i to evolve within the closed hypercube K for all times.[1]

The FRCNN (F) is said to be cooperative if we have

$$a_{ij} \ge 0 \quad \forall i, j \in \{1, \ldots, n\}, i \ne j.$$

This means that there are excitatory interconnections between distinct neurons.

[1]We remark that, by incorporating the term -x in the nonlinearity, we can write model (F) in the equivalent way

$$\dot{x} \in Ax + I - (x + \Lambda(x)) = Ax + I - \tilde{\Lambda}(x)$$

Where $\tilde{\Lambda}(x) = (\tilde{\lambda}(x_1), \ldots, \tilde{\lambda}(x_n))'$ is a diagonal mapping such that

$$\tilde{\lambda}(\rho) = \begin{cases} (-\infty, -1], & \rho = -1 \\ \rho, & \rho \in (-1, 1) \\ [1, +\infty), & \rho = 1 \end{cases}$$

is a hard-limiter nonlinearity with a (normalized) positive slope +1 in the range (−1,1). Such a positive slope can better model the real characteristic of the hard-limiter nonlinearity implemented in electronic circuits.

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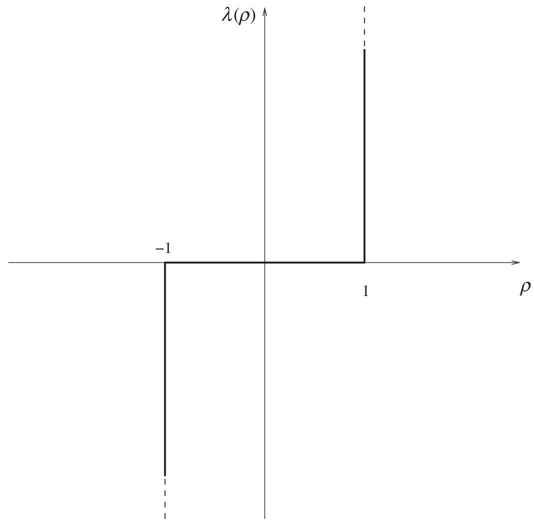

Figure 3. Hard-limiter nonlinearity in the FRCNN model.

System (F) is a special case of the class of differential variational inequalities (DVIs) [28,17,18]

$$\dot{x} \in f(x) - N_K(x) \tag{2}$$

where $x \in K$, $N_K(x)$ is the normal cone to K at point $x \in K$, and $f \in C^1(D)$, where D is a convex open subset of $\mathbb{R}^n$ such that $K \subset D$. It has been observed in [17] that for all $x \in K$, we have $N_K(x) = \Lambda(x)$ (see also Property 1), hence (F) can be obtained from (2) by letting $f(x) = -x + Ax + I$, $x \in D$.

The vector field f is said to be cooperative on D [22] if we have

$$\frac{\partial f_i(x)}{\partial x_j} \geq 0 \quad \forall x \in D \quad \forall i, j \in \{1, \ldots, n\}, i \neq j.$$

In the case $f(x) = -x + Ax + I$, $x \in D$, we obtain $\partial f_i(x)/\partial x_j = a_{ij} \geq 0$ for all $i \neq j$.

By a solution $x(t)$, $t \geq 0$, of the DVI (2) we mean a function $x(\cdot)$ that is absolutely continuous on any compact interval in $[0, +\infty)$ and is such that: (a) $x(t) \in K$ for all $t \geq 0$; (b) $\dot{x}(t) \in f(x(t)) - N_K(x(t))$ for almost all (a.a.) $t \in [0, +\infty)$.

The following hold.

Property 4
For any $x \in K$, there exists a unique solution $x(t)$, $t \geq 0$, of the DVI (2) with initial condition $x(0) = x$. Moreover, there exists at least an EP of (2).

Proof
See [18, Properties 2, 3].

Property 5
Any solution $x(t)$, $t \geq 0$, of the DVI (2) is a solution of the projected differential equation

$$\dot{x}(t) = \mathcal{P}_{T_K(x(t))} f(x(t)), \quad \text{for a.a. } t \geq 0$$

and conversely.

Proof
Follows from [28, Prop. 2, p. 266].

Property 6
The function $\Phi : [0, +\infty) \times K \to K$ defined as $\Phi_t(x) = x(t)$, where $x(t)$, $t \geq 0$, is the solution of the DVI (2) such that $x(0) = x \in K$, is the semiflow generated by (2). Moreover, Φ is continuous in $[0, +\infty) \times K$.

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Proof
Fix any $x \in K$. By definition, we have $\Phi_0(x) = x(0) = x$. Now, fix any $r, s \geq 0$. Let $y = \Phi_r(x)$ and denote by $y(t)$, $t \geq 0$, the solution of (2) with initial condition $y(0) = y = \Phi_r(x)$, hence $y(s) = \Phi_s(y) = \Phi_s(\Phi_r(x))$. To show that $y(s) = \Phi_{r+s}(x)$, it suffices to observe that the function $\varphi(t) = x(r+t)$ is a solution of (2) with $\varphi(0) = x(r) = y$ and so, by Property 4, $\varphi(t) = y(t)$, $t \geq 0$.

Let us now show the continuity of Φ. Let $x(t) = \Phi_t(x)$ and $y(t) = \Phi_t(y)$ be solutions of (2) for $t \geq 0$. From Property 10 in Appendix 1, we have $\|x(t) - y(t)\| \leq \|x(0) - y(0)\| e^{Lt}$ for any $t \geq 0$, where $0 \leq L < +\infty$ is the Lipschitz constant of f restricted to the compact set K. Fix any $(t, x) \in [0, +\infty) \times K$ and $\varepsilon > 0$. Let $T = t + 1$. Since $x(\cdot)$ is continuous, we can find $r_1(\varepsilon, x, t) > 0$ such that $\|x(t) - x(s)\| < \varepsilon/2$ for any $s \in (t - r_1(\varepsilon, x, t), t + r_1(\varepsilon, x, t))$. Moreover, let $\rho(\varepsilon, t) = \varepsilon e^{-LT}/2 = \varepsilon e^{-L(t+1)}/2$. If $y \in B(x, \rho(\varepsilon, t)) \cap K$, then by applying Property 10, we conclude that $\|x(s) - y(s)\| < \varepsilon/2$ for any $s \in [0, T]$. Let $r(\varepsilon, x, t) = \min\{1, r_1(\varepsilon, x, t), \rho(\varepsilon, t)\}$. Let $(s, y) \in [0, +\infty) \times K$ be such that $|s - t| + \|y - x\| < r(\varepsilon, x, t)$. Note that $|s - t| < r(\varepsilon, x, t) \leq r_1(\varepsilon, x, t)$ and $\|y - x\| < r(\varepsilon, x, t) \leq \rho(\varepsilon, t)$. Furthermore, since $|s - t| < 1$, we have $s \in [0, T]$. Thus $\|\Phi_s(y) - \Phi_t(x)\| = \|y(s) - x(t)\| \leq \|y(s) - x(s)\| + \|x(s) - x(t)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

It is of interest to point out the following differences between the semiflow generated by the DVI (2) and semiflows generated by ordinary differential equations. First of all, we remark that, as shown in [17, Example 1], the semiflow Φ generated by (2) is in general *not injective*. Namely, there may exist points $x, y \in K$, $x \neq y$, and $\bar{t} > 0$ such that $x(\bar{t}) = \Phi_{\bar{t}}(x) = y(\bar{t}) = \Phi_{\bar{t}}(y)$, hence $\Phi_t(x) = \Phi_t(y)$ for all $t \geq \bar{t}$. Furthermore, as shown in the next result, in general there are solutions of (2) that *cannot be prolonged backward in time* (see also Example 2 in Sect. 3).

Property 7
Suppose that there exists $x \in \mathrm{bd}(K)$ such that $-f(x) \notin T_K(x)$ and let $x(t)$, $t \geq 0$, be the unique solution of the DVI (2) with initial condition $x(0) = x$. Then, $x(\cdot)$ cannot be prolonged for times $t < 0$, so that it is not possible to define a flow $\Phi : (-\infty, +\infty) \times K \to K$ for (2).

Proof
Since $x \in \mathrm{bd}(K)$, we have $\mathcal{I} = \{i \in \{1, 2, \ldots, n\} : |x_i| = 1\} \neq \varnothing$. We have $T_K(x) = (T_{K_1}(x_1), T_{K_1}(x_2), \ldots, T_{K_1}(x_n))'$ (Property 1), hence there exists $k \in \{1, \ldots, n\}$ such that $-f_k(x) \notin T_{K_1}(x_k)$. Moreover, we have $k \in \mathcal{I}$. In fact, if we had $k \notin \mathcal{I}$, we would have $T_{K_1}(x_k) = \mathbb{R}$ and hence $-f_k(x) \in T_{K_1}(x_k)$. Without loss of generality, we can assume that $x_k = 1$. Since $T_{K_1}(x_k) = T_{K_1}(1) = (-\infty, 0]$, we have $f_k(x) < 0$. Due to the continuity of f, we can find $r \in (0, 1/2)$ such that $f_k(y) \leq f_k(x)/2 < 0$ for any $y \in B(x, r) \cap K$. On the other hand, $T_{K_1}(y_k) \supseteq (-\infty, 0]$ for any $y \in B(x, r) \cap K$. Thus, $\left[\mathcal{P}_{T_K(y)} f(y)\right]_k = \mathcal{P}_{T_{K_1}(y_k)} f_k(y) = f_k(y)$ for any $y \in B(x, r) \cap K$. Now, suppose for contradiction that $\Phi_t(x)$ can be prolonged backward in time, i.e., there exist $T > 0$ and $z \in K$ such that $\Phi_T(z) = x$. Let us denote by $z(t)$, $t \geq 0$, the solution of (2) with initial condition $z(0) = z$. Then, there exists $\delta \in (0, T]$ such that $z(t) \in B(x, r)$ for any $t \in [T - \delta, T]$. We have $\dot{z}_k(t) = [\mathcal{P}_{T_K(z(t))} f(z(t))]_k = \mathcal{P}_{T_{K_1}(z_k(t))} f_k(z(t)) = f_k(z(t))$ for a.a. $t \in [T - \delta, T]$ (Property 5). It follows that $1 = x_k = z_k(T) = z_k(T - \delta) + \int_{T-\delta}^{T} \dot{z}_k(s) ds \leq 1 + \int_{T-\delta}^{T} f_k(z(s)) ds \leq 1 + \frac{f_k(x)}{2} \delta < 1$, since $f_k(x) < 0$, which is a contradiction.

3. MONOTONICITY OF SEMIFLOWS GENERATED BY COOPERATIVE FRCNNS

In this section, we prove that the semiflow generated by a cooperative DVI (2) is monotone. The proof can be considered as a generalization of a classic result of Kamke for cooperative ordinary differential equations to the class of differential inclusions (2).

The following holds.

Theorem 1
Suppose that the vector field f is cooperative on D. Then, the semiflow Φ generated by the DVI (2) is monotone, i.e., for any $x, y \in K$ such that $x \leq y$ we have $\Phi_t(x) = x(t) \leq y(t) = \Phi_t(y)$ for any $t \geq 0$.

Before giving the proof of Theorem 1, we discuss some issues related to the result in the theorem.

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Remarks
1) By exploiting the monotonicity of the semiflow Φ generated by a cooperative DVI (2), we can prove the following

Property 8
(CONVERGENCE CRITERION) Let Φ be the monotone semiflow generated by the cooperative DVI (2). If we have $\Phi_t(x) \geq x$ for all times t in an open interval in $[0, +\infty)$, then $\Phi_t(x) \rightarrow \xi \in E$ as $t \rightarrow +\infty$.
The proof of Property 8 is a slight modification of that of [26, Th. 2.1] (we omit the details).

Property 9
(NON-ORDERING OF LIMIT SETS) Let Φ be the monotone semiflow generated by the cooperative DVI (2). For any $x \in K$, there cannot exist points $p, q \in \omega(x)$ such that $p \ll q$.

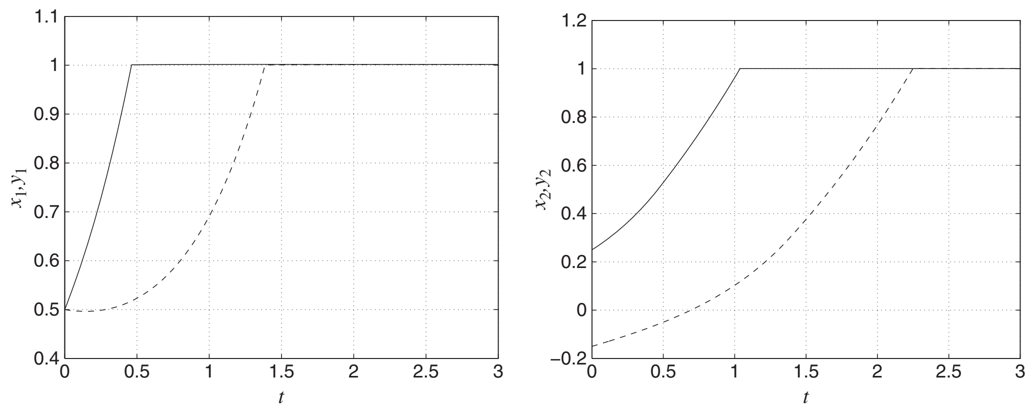
Proof
Suppose for contradiction that there exist $x \in K$ and points $p, q \in \omega(x)$ such that $p \ll q$. Let $r > 0$ be such that $B(p, r) < B(q, r)$. Since $p, q \in \omega(x)$, there exist sequences $\{t_k\}$ and $\{\tau_k\}$, where $t_k, \tau_k \rightarrow +\infty$ as $k \rightarrow +\infty$, such that $x(t_k) \rightarrow p$ and $x(\tau_k) \rightarrow q$ as $k \rightarrow +\infty$. Hence, we can find positive integers i, j such that $t_i < \tau_j$, $x(t_i) \in B(p, r)$ and $x(\tau_j) \in B(q, r)$. Due to the continuity of $x(\cdot)$, there exists $\varepsilon > 0$ such that $x(s) \in B(q, r)$ for any $s \in (\tau_j - \varepsilon, \tau_j + \varepsilon)$ and $t_i \leq \tau_j - \varepsilon$. Choose any $t \in (\tau_j - t_i - \varepsilon, \tau_j - t_i + \varepsilon)$. Since $t_i + t \in (\tau_j - \varepsilon, \tau_j + \varepsilon)$, we have $x(t_i + t) \in B(q, r)$ and so $x(t_i + t) \geq x(t_i)$. Thus, by letting $\tilde{x} = x(t_i)$, we have $\Phi_t(\tilde{x}) = \Phi_t(\Phi_{t_i}(x)) = \Phi_{t_i+t}(x) = x(t_i + t) \geq x(t_i) = \tilde{x}$. By Property 8, we conclude that $x(t) \rightarrow \xi \in E$ as $t \rightarrow +\infty$, i.e., $\omega(x) = \{\xi\}$ is a singleton, which contradicts the existence of $p, q \in \omega(x)$ with $p \ll q$.

2) For cooperative ordinary differential equations, it is known that all the three order relations $\leq$, $<$ and $\ll$ are preserved during the time evolution of the solutions [25, Prop. 1.1, p. 32]. Actually, Theorem 1 states that the solutions of a cooperative DVI (2) preserve the order relation $\leq$. In the next example, we show that the relations $<$ and $\ll$ are instead not preserved by the solutions of the DVI (2). This represents a significant drawback to study the property of eventual strong monotonicity of the semiflow [26] (see also Sect. 4).

Example 1
Let us consider the second-order cooperative FRCNN

$$\dot{x} \in -x + \begin{pmatrix} 3/2 & 2 \\ 1/2 & 3/2 \end{pmatrix} x - N_{K_2}(x)$$

where $x = (x_1, x_2)' \in K_2 = [-1, 1]^2$ and $x = (1/2, -3/20)' < y = (1/2, 1/4)'$. Figure 4 depicts the time evolution of the solutions $x(\cdot)$, $y(\cdot)$ in the time interval [0,3] seconds. It is seen that we have $x(t) \ll y(t)$ for



(a) (b)

Figure 4. (a) Time-domain evolution of $x_1(\cdot), y_1(\cdot)$ and (b) evolution of $x_2(\cdot), y_2(\cdot)$ for a second-order cooperative FRCNN.

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any $t \in (0, 1.39)$. However, the components $y_1(\cdot), y_2(\cdot), x_1(\cdot)$ and $x_2(\cdot)$ reach the saturation level +1 at the instants $t = 0.43$, 1.04, 1.39 and 2.25, respectively. We have $x(t) < y(t)$ for all $1.39 \le t < 2.25$, whereas $x(t) = y(t) = (1, 1)'$, for any $t \ge 2.25$ ($x(\cdot)$ and $y(\cdot)$ converge in finite time to the EP $(1, 1)'$).

3) Ordinary differential equations can be solved both forward in time and backward in time. By time reversal $t \to -t$ a competitive system of differential equations becomes a cooperative system and vice-versa. On this basis, several properties of competitive ordinary differential equations can be directly derived from corresponding properties of cooperative differential equations. For example, while a cooperative system generates a monotone semiflow in the forward time direction, a competitive systems generates a monotone semiflow in the backward time direction (see, e.g., [26, Ch. 3]). We have seen in Property 7 that in general the solutions of the DVI (2) and the FRCNN (F) cannot be defined for negative times (see also the next Example 2 for a detailed discussion). Then, a significant drawback is that there are difficulties to study competitive FRCNNs by reducing them to cooperative FRCNNs via time reversal.

Example 2
Let us consider an inputless cooperative third-order FRCNN defined by the one-dimensional template $[r\ p\ s] = [1\ 2\ 3/2]$ and periodic boundary condition. The FRCNN obeys the DVI (2) where $x = (x_1, x_2, x_3)' \in K_3$, $I = 0$ and

$$f(x) = Ax = \begin{pmatrix} 1 & 3/2 & 1 \\ 1 & 1 & 3/2 \\ 3/2 & 1 & 1 \end{pmatrix} x.$$

First, we want to identify the subsets of $\mathrm{bd}(K_3)$ for which the solutions starting in the same sets cannot be prolonged for negative times. Let $Z_1 = \{x \in K_3 : x_3 = -1, f_3(x) > 0\} = \{x \in K_3 : x_3 = -1, x_2 > 1 - 3x_1/2\} \subset \mathrm{bd}(K_3)$. Fix $x \in Z_1$ and let $x(\cdot)$ be the solution of (2) having x as initial condition at $t = 0$. Since $T_{K_3}(x) = (T_{K_1}(x_1), T_{K_1}(x_2), T_{K_1}(x_3))'$, where $T_{K_1}(x_3) = h(-1) = [0, +\infty)$ (Property 1), we obtain $f_3(x) \notin -T_{K_1}(x_3)$, i.e., $-f(x) \notin T_{K_3}(x)$. Therefore, by Property 7, we conclude that $x(\cdot)$ cannot be prolonged for times $t < 0$. Given the symmetry of vector field f, there are five additional sets $Z_2, \ldots, Z_6 \subset \mathrm{bd}(K_3)$, obtained by an index permutation and sign change, enjoying analogous properties. Therefore, the whole set $Z = Z_1 \cup \ldots \cup Z_6$ is made of initial conditions for which the corresponding solutions cannot be prolonged for times $t < 0$, see Figure 5. Figure 5 also shows the trajectory corresponding to the solution $\hat{x}(\cdot)$ with initial condition $\hat{x} = (0.54, 0.5, 0.46)' \in \mathrm{int}(K_3)$, at $t = 0$. Note that $\hat{x}$ can be prolonged backward in time until it hits the subset $Z_6 \subset \mathrm{bd}(K_3)$ at the instant $t_- = -12.55$. As discussed before, $\hat{x}(\cdot)$ cannot be further prolonged for times $t < t_-$.[2]

Proof of Theorem
Since f is a cooperative vector field in D, then f satisfies a Kamke condition in D, i.e., for each $i \in \{1, \ldots, n\}$ we have $f_i(a) \le f_i(b)$ for any $a, b \in D$ such that $a \le b$ and $a_i = b_i$ [25, Ch. 3]. Consider points $x, y \in K$ such that $x \le y$. If $x = y$, then due to the uniqueness of solutions for (2) (Property 4), we have $x(t) = y(t)$, $t \ge 0$. Then, suppose that $x < y$ and choose any $T > 0$.
We want to prove that

$$x(t) \le y(t),\ t \in [0, T]. \tag{3}$$

To this aim, consider the vector field $f^{(m)} : D \to \mathbb{R}^n$ defined as $f^{(m)}(x) = f(x) + \bar{u}/m$, for any $m \in \mathbb{N}_0 = \{1, 2, \ldots\}$ and $x \in D$, where $\bar{u} = (1, \ldots, 1)'$. Note that $f^{(m)} \in C^1(D)$ and that $f^{(m)}$ satisfies a Kamke

[2]We observe that any solution evolving for positive times in $\mathrm{int}(K_3)$ satisfies the affine system $\dot{x} = Ax$. By time reversal, $t \to -t$, a solution evolving for negative times in $\mathrm{int}(K_3)$ satisfies the affine system $\dot{x} = -Ax$, whose eigenvalues are $\lambda_1 = -3.5$ and $\lambda_{2,3} = 0.25 \pm j0.433$. Let $W^S(0)$ denote the one-dimensional stable manifold of the origin of the linear system $\dot{x} = -Ax, x \in \mathbb{R}^3$. Note that $W^S(0) \cap \mathrm{int}(K_3)$ has measure 0 in $\mathbb{R}^3$. Consider the set $\mathrm{int}(K_3) \setminus W^S(0)$. Due to the presence of eigenvalues $\lambda_{2,3}$ with positive real parts, it can be seen that any solution starting at $\hat{x} \in \mathrm{int}(K_3) \setminus W^S(0)$ at $t = 0$ hits the set $Z \subset \mathrm{bd}(K_3)$ at a finite negative instant t_- depending on the initial condition $\hat{x}$. Any such solution cannot be further prolonged for times $t < t_-$. Then, for a.a. choices of initial conditions in $\mathrm{int}(K_3)$, the corresponding solution cannot be prolonged to $-\infty$.

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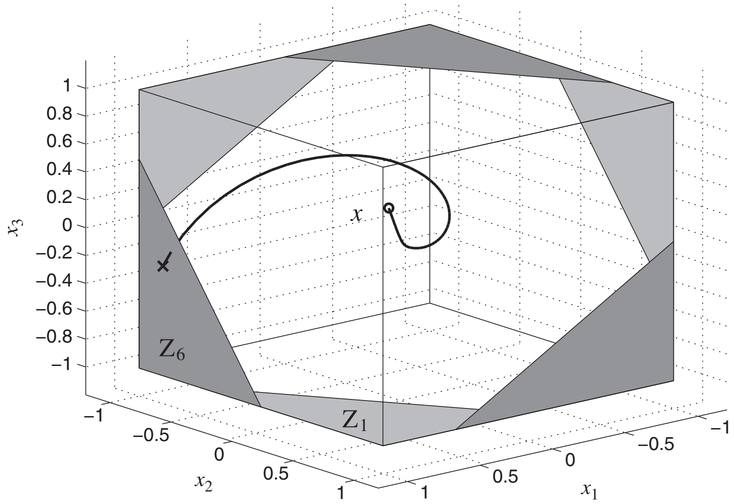

Figure 5. Set $Z=Z_1\cup\ldots\cup Z_6\subset \mathrm{bd}(K_3)$ of initial conditions for which the solutions of a third-order cooperative FRCNN cannot be prolonged for times $t<0$ (gray). The figure also displays the trajectory in the time interval $[-12.55, 0]$ seconds for a solution starting at $\hat{x}=\left(0.54, 0.5, 0.46\right)'\in \mathrm{int}(K_3)$ (point ∘) at t = 0. The solution is seen to hit the set Z_6 (point ×) at the instant $t\_=-12.55$ s.

condition for any $m\in\mathbb{N}_0$. Fix any $m\in\mathbb{N}_0$ and let $z^{(m)}(t)$, $t\geq 0$, be the unique solution of the DVI $\dot{z}\in f^{(m)}(z)-N_K(z)$ such that $z^{(m)}(0)=y$. Let

$$V_m=\left\{\theta\in[0,T]: x(t)\leq z^{(m)}(t), \forall t\in[0,\theta]\right\}.$$

Since $0\in V_m$, we have $V_m\neq\varnothing$. Let $\tau_m=\sup V_m$, where $0\leq\tau_m\leq T$ for any $m\in N_0$. Note that $x(t)\leq z^{(m)}(t)$ for any $t\leq\tau_m$.

As a first step let us show that (3) holds provided that $\tau_m=T$ for any $m\in\mathbb{N}_0$. Suppose for contradiction that there exist $i\in\mathbb{N}_0$ and $\bar{t}\in[0,T]$ such that $x_i(\bar{t})>y_i(\bar{t})$. Then, there exists $\varepsilon>0$ such that $w_i<x_i(\bar{t})$ for any $w\in B(y(\bar{t}),\varepsilon)$. Due to Property 11 in Appendix 1, we can find $\bar{m}\in\mathbb{N}_0$ such that $\left\|y(t)-z^{(\bar{m})}(t)\right\|<\varepsilon$ for any $t\in[0,T]$. It follows that $z^{(\bar{m})}(\bar{t})\in B(y(\bar{t}),\varepsilon)$, i.e., $x_i(\bar{t})>z_i^{(\bar{m})}(\bar{t})$, which is a contradiction.

As a second step, let us show that $\tau_m=T$ for any $m\in\mathbb{N}_0$. We argue again by contradiction assuming that we can find $q\in\mathbb{N}_0$ such that $\tau_q<T$. Let

$$W_q=\left\{i\in\{1,\ldots,n\}:\exists\{t_k^{(i)}\}\subset\left(\tau_q,T\right]: t_k^{(i)}\to\tau_q \text{ and } z_i^{(q)}(t_k^{(i)})<x_i(t_k^{(i)}), \forall k\in\mathbb{N}_0\right\}.$$

Due to our assumption, $W_q\neq\varnothing$. Fix any $i\in W_q$. Since $z_i^{(q)}\left(\tau_q\right)=x_i\left(\tau_q\right)$, the Kamke condition yields

$$f_i\left(x\left(\tau_q\right)\right)\leq f_i\left(z^{(q)}\left(\tau_q\right)\right)<f_i\left(z^{(q)}\left(\tau_q\right)\right)+\frac{1}{q}=f_i^{(q)}\left(z^{(q)}\left(\tau_q\right)\right). \tag{4}$$

As a consequence, due to the continuity of f, there exists $\rho_1>0$ such that $f_i(x(t))<f_i^{(q)}\left(z^{(q)}(t)\right)$ for any $t\in(\tau_q,\tau_q+\rho_1)$. Assume that there exists $\rho_2>0$ such that $\left|z_i^{(q)}(t)\right|<1$ and $|x_i(t)|<1$ for any $t\in(\tau_q,\tau_q+\rho_2)$. Moreover, let $\rho=\min\{\rho_1,\rho_2\}$. By Property 1, we have

$$\dot{x}_i(t)=\left[\mathcal{P}_{T_K(x(t))}f(x(t))\right]_i=\mathcal{P}_{T_{K_1}(x_i(t))}f_i(x(t))=f_i(x(t))<f_i^{(q)}\left(z^{(q)}(t)\right)=\left[\mathcal{P}_{T_K(z^{(q)}(t))}f^{(q)}\left(z^{(q)}(t)\right)\right]_i=\dot{z}_i^{(q)}(t)$$

for any $t\in(\tau_q,\tau_q+\rho)$. Thus, $x_i(t)<z_i^{(q)}(t)$ for any $t\in(\tau_q,\tau_q+\rho)$, while we assumed $i\in W_q$. Hence, there exists a sequence $\{\theta_k\}\subset(\tau_q,T]$, with $\theta_k\to\tau_q$ as $k\to+\infty$, such that $|x_i(\theta_k)|=1$ or $\left|z_i^{(q)}(\theta_k)\right|=1$ for any $k\in\mathbb{N}_0$. In particular, due to the continuity of $x_i(\cdot)$ and $z_i^{(q)}(\cdot)$, we have $z_i^{(q)}\left(\tau_q\right)=x_i\left(\tau_q\right)\in\{-1,1\}$.

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Without loss of generality, we can suppose that $z_i^{(q)}(\tau_q) = x_i(\tau_q) = 1$.
Since $z_i^{(q)}(t_k^{(i)}) < x_i(t_k^{(i)}) \leq 1$ for any $k \in \mathbb{N}_0$, we have $f_i^{(q)}\left(z^{(q)}(\tau_q)\right) \leq 0$. In fact, arguing by contradiction, if we had $f_i^{(q)}(z^{(q)}(\tau_q)) > 0$, then it would result $z_i^{(q)}(t) = 1$ in a whole right neighborhood of τ_q. Thus, $0 \geq f_i^{(q)}(z^{(q)}(\tau_q)) = f_i(z^{(q)}(\tau_q)) + 1/q$, i.e., recalling (4),

$$f_i(x(\tau_q)) \leq f_i\left(z^{(q)}(\tau_q)\right) \leq -\frac{1}{q} < 0.$$

Thus, there exists $\rho^+ > 0$ such that $x_i(t) < 1$ for any $t \in (\tau_q, \tau_q + \rho^+)$. If we had $f_i^{(q)}\left(z^{(q)}(\tau_q)\right) < 0$, then $z_i^{(q)}(t) < 1$ in a whole right interval of τ_q, while we assumed $z_i^{(q)}(\theta_k) = 1$ for any sufficiently large k. Hence, $f_i^{(q)}\left(z^{(q)}(\tau_q)\right) = 0$. Due to the continuity of f_i, there exists $r_1 > 0$ such that $f_i^{(q)}(u) > -1/(3q)$ for any $u \in B(z^{(q)}(\tau_q), r_1)$. By Property 1, we obtain

$$\left[\mathcal{P}_{T_K(u)} f^{(q)}(u)\right]_i = \mathcal{P}_{T_{K_1}(u_i)} f_i^{(q)}(u) \geq -\frac{1}{3q}$$

for any $u \in B(z^{(q)}(\tau_q), r_1) \cap K$. Moreover, since $f_i(x(\tau_q)) \leq -1/q$, we can find $r_2 > 0$ such that $f_i(v) < -2/(3q)$ for any $v \in B(x(\tau_q), r_2)$. let $r = \min\{r_1, r_2\}$. There exists $\rho^* \in (0, \rho^+]$ such that $x(t) \in B(x(\tau_q), r)$ and $z^{(q)}(t) \in B(z^{(q)}(\tau_q), r)$ for any $t \in [\tau_q, \tau_q + \rho^*]$. Fix any $t \in (\tau_q, \tau_q + \rho^*]$. Recalling that $x_i(s) < 1$ for any $s \in (\tau_q, t]$, we obtain

$$x_i(t) = x_i(\tau_q) + \int_{\tau_q}^{t} \dot{x}_i(s) ds = 1 + \int_{\tau_q}^{t} f_i(x(s)) ds \leq 1 - \frac{2}{3q}(t - \tau_q) < 1 - \frac{t - \tau_q}{3q}.$$

On the other hand,

$$z_i^{(q)}(t) = z_i^{(q)}(\tau_q) + \int_{\tau_q}^{t} \dot{z}_i^{(q)}(s) ds = 1 + \int_{\tau_q}^{t} \left[\mathcal{P}_{T_K(z^{(q)}(s))} f^{(q)}\left(z^{(q)}(s)\right)\right]_i ds \geq 1 - \frac{t - \tau_q}{3q} > x_i(t).$$

By choosing a sufficiently large k, we have $t_k^{(i)} \in (\tau_q, \tau_q + \rho^*)$. Hence, $z_i^{(q)}\left(t_k^{(i)}\right) > x_i\left(t_k^{(i)}\right)$, which is a contradiction.

4. COOPERATIVE FRCNNS WITH IRREDUCIBLE INTERCONNECTIONS

The paper is mainly devoted to the study of monotonicity of semiflows generated by cooperative FRCNNs. In doing so, we didn't require the assumption of irreducibility of the neuron interconnection matrix. Here, we show by means of counterexamples that the semiflow generated by a cooperative FRCNN with an irreducible interconnection matrix is in general *not* ESM nor SOP (Sect. 4.1), moreover also the LIMIT SET DICHOTOMY might be *violated* (Sect. 4.2). Then, we briefly compare these results with those valid for other CNN models (Sect. 4.3).

4.1. *Counterexample to ESM and SOP Properties*

Let us consider the third-order FRCNN

$$\dot{z} \in -z + Az - N_{K_3}(z) + I = \bar{f}(z) - N_{K_3}(z) + I \tag{5}$$

where $I = (0, 0, -0.5)'$, $z \in K_3 = [-1, 1]^3$ and

$$A = \begin{pmatrix} -1 & 1.5 & 0.5 \\ 0 & -1 & 0.5 \\ 0.2 & 0.2 & 2 \end{pmatrix}$$

is cooperative and irreducible. Fix any $z \in K_3$ with $z_3 = 1$ and let $z(t), t \geq 0$, be the solution of (5) with initial condition $z(0) = z$. Since $\bar{f}_3(x) = x_3 + 0.2x_1 + 0.2x_2 - 0.5 > 0$, for any $x \in K_3$ such that $x_3 = 1$,

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we have $z_3(t)=1$ for all $t\geq 0$. Then, for $t\geq 0$ the vector $(z_1(\cdot), z_2(\cdot))'$ satisfies
$$\dot{w} \in -w + A_2 w - N_{K_2}(w) + I_2 \tag{6}$$

where

$$A_2 = \begin{pmatrix} -1 & 1.5 \\ 0 & -1 \end{pmatrix}$$

and $I_2=(1/2, 1/2)'$. Note that the interconnection matrix A_2 of (6) is not irreducible. Indeed, the state variable z_2 evolves independently of z_1 and we have $z_2(t)=(z_2-1/4)e^{-2t}+1/4$ for any $t\geq 0$.

Now, let us choose points x, $y\in K_3$ such that

$$x_1 < y_1, \quad x_2 = y_2 = h, \quad x_3 = y_3 = 1$$

for some $h\in(-1, 1)$. We want to show that, for any $\varepsilon>0$, there exist points $a\in B(x,\varepsilon)\cap K_3$ and $b\in B(y,\varepsilon)\cap K_3$ such that $a(t)\nleq b(t)$ for any $t\geq 0$. This implies that the semiflow generated by (6) and (5) is not SOP nor ESM. Fix any $\varepsilon>0$ such that $-1\leq h-\varepsilon<h+\varepsilon\leq 1$ and let $a=(x_1, x_2+\varepsilon/2, x_3)'\in B(x,\varepsilon)\cap K_3$ and $b=(y_1, y_2-\varepsilon/2, y_3)'\in B(y,\varepsilon)\cap K_3$. We have $a_2(t)=(h+\varepsilon/2-1/4)e^{-2t}+1/4>(h-\varepsilon/2-1/4)e^{-2t}+1/4=b_2(t)$, and so $a(t)\nleq b(t)$, for any $t\geq 0$.

4.2. *Counterexample to the* LIMIT SET DICHOTOMY

We provide an example of a seventh-order cooperative FRCNN (F), with an irreducible interconnection matrix, for which the LIMIT SET DICHOTOMY fails. To this end, we first establish some preliminary results needed in the construction of the seventh-order FRCNN. We refer the reader to [31] for a counterexample to the LIMIT SET DICHOTOMY for cooperative differential equations.

4.2.1. Periodic solution in a second-order FRCNN. Let us consider the inputless second-order FRCNN

$$\dot{u} \in -u + Au - N_{K_2}(u) \tag{7}$$

where $u=(u_1, u_2)\in K_2=[-1, 1]^2$, the interconnection matrix is given by

$$A = \begin{pmatrix} p & s-r \\ -(s-r) & p \end{pmatrix}$$

and the parameters are assumed to satisfy

$$p > 1, \quad 0 < r < s, \quad p-1 < s-r. \tag{8}$$

Note that there are both positive (s-r) and negative ($-(s-r)$) interconnections between neurons.

It can be verified that for the parameters (8) the origin is the unique EP of (7). Moreover, 0 is a repelling EP, i.e., no solution of (7), except the trivial solution, can converge to 0 as $t\to+\infty$. Due to the uniqueness of the solutions of (7), we can apply a Poincaré-Bendixson-type result [32] in order to conclude that, for any initial condition $u\neq 0$, the solution $u(\cdot)$ converges to $c_u(t)$ as $t\to+\infty$, where $c_u:[0,+\infty)\to K_2$ is a nonconstant periodic solution of (7).

4.2.2. Embedding a periodic solution in a fourth-order cooperative FRCNN.
Consider now the inputless fourth-order FRCNN

$$\dot{u} \in -u + \tilde{A}u - N_{K_4}(u) \tag{9}$$

where $u=(u_1, u_2, u_3, u_4)'\in K_4=[-1, 1]^4$, the interconnection matrix is given by

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$$\tilde{A} = \begin{pmatrix} p & s & 0 & r \\ r & p & s & 0 \\ 0 & r & p & s \\ s & 0 & r & p \end{pmatrix} \tag{10}$$

and the parameters satisfy (8). Note that there are nonnegative interconnections between neurons, i.e., the FRCNN (9) is cooperative.

Consider the vector subspace $\Gamma = \{x = (x_1, x_2, x_3, x_4)' \in \mathbb{R}^4 : x_3 = -x_1 \text{ and } x_4 = -x_2\}$ and the closed convex set $\Gamma^\star = \Gamma \cap K_4$. It is shown in Appendix 2 that $\Gamma^\star$ is positively invariant for the solutions of (9), namely for any $u \in \Gamma^\star$ we have $u(t) \in \Gamma^\star$, $t \geq 0$.

Now, fix any $u \in \Gamma^\star \setminus \{0\}$. Note that $\dot{u}(t) = -u(t) + \tilde{A}u(t) - \gamma(t)$ for a.a. $t \geq 0$, where $\gamma(t) \in N_{K_4}(u(t))$. Let $\bar{u}(t) = (u_1(t), u_2(t))'$ and $\hat{u}(t) = (u_3(t), u_4(t))'$ for any $t \geq 0$. Moreover, let $\bar{\gamma}(t) = (\gamma_1(t), \gamma_2(t))'$. Note that $\bar{\gamma}(t) \in N_{K_2}(t)$ by Property 1.

Since $\Gamma^\star$ is positively invariant, we have $\hat{u}(t) = -\bar{u}(t)$ for any $t \geq 0$, and thus

$$\begin{aligned} \frac{d\bar{u}(t)}{dt} &= \begin{pmatrix} p-1 & s \\ r & p-1 \end{pmatrix} \bar{u}(t) + \begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix} \hat{u}(t) - \bar{\gamma}(t) \\ &= \begin{pmatrix} p-1 & s \\ r & p-1 \end{pmatrix} \bar{u}(t) - \begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix} \bar{u}(t) - \bar{\gamma}(t) \\ &= \begin{pmatrix} p-1 & s-r \\ -(s-r) & p-1 \end{pmatrix} \bar{u}(t) - \bar{\gamma}(t) \end{aligned}$$

for a.a. $t \geq 0$. Then, $\bar{u}(t)$, $t \geq 0$, coincides with the solution of (7) with initial condition $\bar{u} \neq 0$ at $t = 0$. From the results in Sect. 4.2.2, it follows that $u(t) = (\bar{u}(t), -\bar{u}(t))$ converges to a nonconstant periodic solution $(c_{\bar{u}}(t), -c_{\bar{u}}(t))$ of (9) as $t \to +\infty$.

4.2.3. A seventh-order cooperative FRCNN with an irreducible interconnection matrix for which the LIMIT SET DICHOTOMY fails.

Finally, let us consider the seventh-order FRCNN

$$\dot{x} \in -x + Ax + I - N_{K_7}(x) = \hat{f}(x) - N_{K_7}(x) \tag{11}$$

where $x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)' \in K_7 = [-1, 1]^7$, the input $I \in \mathbb{R}^7$ is given by $I = (0, 0, -1, 0, -3, 0, 3)'$, the interconnection matrix is

$$A = \begin{pmatrix} p & s & 0 & r & 0 & 0 & 0 \\ r & p & s & 0 & 0 & 0 & 0 \\ 0 & r & p & s & 0 & 0 & 1 \\ s & 0 & r & p & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and the parameters satisfy (8). Note that the FRCNN (11) is cooperative. Moreover, the interconnection matrix A is irreducible, see the interconnection graph in Figure 6.

Fix any $x, y \in K_7$ such that $x_1 = y_1 \neq 0, x_2 = y_2, x_3 = y_3 = -x_1, x_4 = y_4 = -x_2, x_5 = y_5 = -1, x_6 < y_6$ and $x_7 = y_7 = 1$. Let us denote by $x(\cdot)$ and $y(\cdot)$ the solutions of (11) having x and y as initial conditions at $t = 0$, respectively. Note that $\dot{x}(t) = -x(t) + Ax(t) + I - \gamma_x(t)$ and $\dot{y}(t) = -y(t) + Ay(t) + I - \gamma_y(t)$ for a.a. $t \geq 0$, where $\gamma_x(t) \in N_{K_7}(x(t))$ and $\gamma_y(t) \in N_{K_7}(y(t))$. Since $x < y$, due the monotonicity of the semiflow (Theorem 1), we have $x(t) \leq y(t)$ for all $t \geq 0$. Moreover, $x_5(t) = y_5(t) = -1$ for all $t \geq 0$. In fact, $x_5 = y_5 = -1$ and $\hat{f}_5(z) = -z_5 + z_2 + I_5 \leq 2 - 3 < 0$ for any $z \in K_7$. Similarly, it can be seen that $x_7(t) = y_7(t) = 1$ for all $t \geq 0$.

Let $\tilde{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))'$, $\tilde{y}(t) = (y_1(t), y_2(t), y_3(t), y_4(t))'$ and consider matrix $\tilde{A}$ in (10). Since $x_7(t) = y_7(t) = 1$ for all $t \geq 0$, and $I_3 = -1$, we have $d\tilde{x}(t)/dt = -\tilde{x}(t) + \tilde{A}\tilde{x}(t) - \tilde{\gamma}_x(t)$ and

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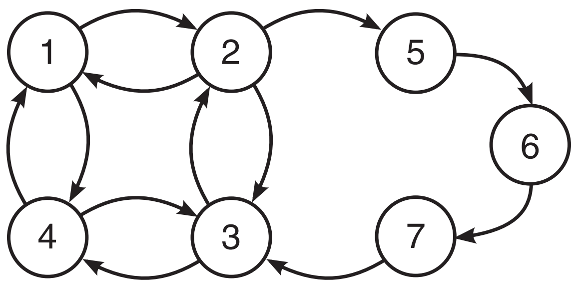

Figure 6. Neuron interconnection graph for a seventh-order FRCNN.

$d\tilde{y}(t)/dt = -\tilde{y}(t) + \tilde{A}\tilde{y}(t) - \tilde{\gamma}_y(t)$ for a.a. $t \geq 0$, where $\tilde{\gamma}_x(t) \in N_{K_4}(\tilde{x}(t)), \tilde{\gamma}_y(t) \in N_{K_4}(\tilde{y}(t))$, i.e., $\tilde{x}(\cdot)$ and $\tilde{y}(\cdot)$ are solutions of the fourth-order FRCNN (9) considered in Sect. 4.2.2. Let $u = (x_1, x_2)' \in K_2$. Note that $\tilde{x}(0) = (u, -u) = \tilde{y}(0)$ with $0 \neq (u, -u) \in \Gamma^{\star}$. Recalling the result in Sect. 2, we conclude that $\tilde{x}(t) = \tilde{y}(t)$ for any $t \geq 0$ and $\tilde{x}(t) = \tilde{y}(t) \rightarrow C(t)$, as $t \rightarrow +\infty$, where $C(t) = (c_u(t), -c_u(t))$ is a nonconstant periodic function. Since $\omega(x)$ and $\omega(y)$ are not included in the set of EPs, by definition, the dichotomy of omega-limit sets would hold if and only if we had $\omega(x) < \omega(y)$, which implies $\sup\omega(x) \leq \inf\omega(y)$. However, since $C(t)$ is a nonconstant periodic solution, there exists $j \in \{1, 2, 3, 4\}$ such that $C_{\min} = \min_{t \in [0,T]}\{C_j(t)\} < \max_{t \in [0,T]}\{C_j(t)\} = C_{\max}$, where $T > 0$ is the period of $C(\cdot)$, and then $[\inf\omega(y)]_j \leq C_{\min} < C_{\max} \leq [\sup\omega(x)]_j$. Therefore, $\omega(x) \not< \omega(\mathrm{y})$, i.e., the dichotomy of omega-limit sets is violated for the seventh-order SCNN (11).

As an illustration, Figure 7 shows the time-domain behavior of $x_1(\cdot)$ and $y_1(\cdot)$ for the solutions $x(t), y(t)$ of (11) with initial conditions $x_1 = y_1 = 0.2$, $x_2 = y_2 = -0.07$, $x_3 = y_3 = -x_1 = -0.2$, $x_4 = y_4 = -x_2 = 0.07$, $x_5 = y_5 = -1$, $x_6 = -.5$, $y_6 = 0.3$, $x_7 = y_7 = 1$. The interconnection parameters are $p = 2$, $r = 1.2$ and $s = 2.8$. As predicted, we have $x_1(t) = y_1(t)$ for all times, and the two components tend to a nonconstant periodic solution sweeping the whole range $[-1, 1]$ allowed by the hard limiter, thus violating the LIMIT SET DICHOTOMY.

4.3. Discussion

In the fundamental paper [24], Chua and Roska have shown that cooperative CNNs with an irreducible interconnection matrix and *sigmoid* (C^1, bounded and strictly increasing) activations generate an ESM semiflow Φ_σ. From the standard theory of ESM semiflows, it follows that Φ_σ satisfies the LIMIT SET DICHOTOMY and is almost quasi-convergent [25,26]. The two counterexamples given in Sect. 4.1 and Sect. 4.2 show that the situation is basically different for cooperative FRCNNs with irreducible interconnection matrices. In fact, since the semiflow is not ESM, and the LIMIT SET DICHOTOMY might fail, it is not possible to directly use the results in [25,26] for addressing convergence of FRCNNs. It is of interest to note that, as discussed in [33], analogous difficulties are encountered for

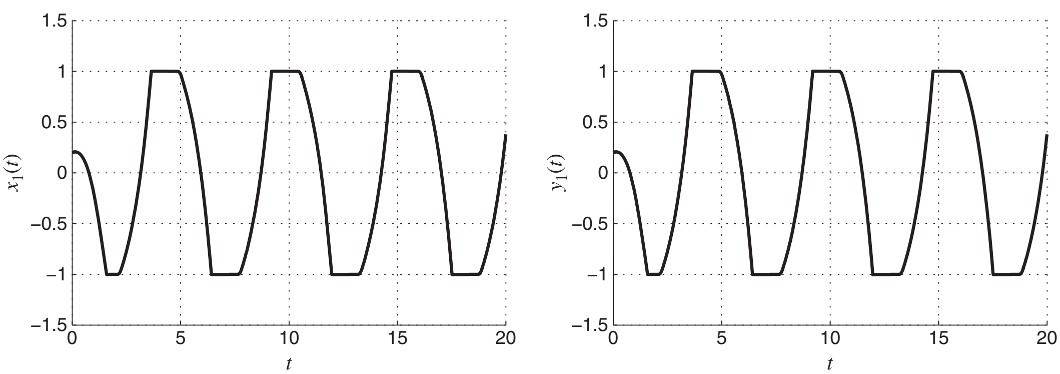

Figure 7. Oscillatory behavior of $x_1(\cdot)$ and $y_1(\cdot)$ for a seventh-order FRCNN.

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cooperative SCNNs with an irreducible interconnection matrix and a typical pwl neuron activation $g(\rho) = (1/2)(|\rho + 1| - |\rho - 1|)$ in place of a sigmoid one as in [24]. Namely, even assuming an irreducible interconnection matrix, the semiflow generated by an SCNN with neuron activations g is not ESM; moreover, there are cases where the LIMIT SET DICHOTOMY can fail.

A significant problem for future research is to discover if there exist subclasses of irreducible templates for which the semiflow of cooperative FRCNNs enjoys the LIMIT SET DICHOTOMY, and is almost quasi-convergent, although the same semiflow is not ESM. It is pointed out that recent works have given a positive answer to an analogous problem for subclasses of irreducible SCNNs with pwl neuron activations $g(\rho) = (1/2)(|\rho + 1| - |\rho - 1|)$ [34] and for the class of fully interconnected Hopfield-type neural networks with the same pwl activations g [35].

5. CONCLUSION

The paper has given a rigorous proof of monotonicity for the solution semiflow generated by cooperative FRCNNs with an ideal multi-valued hard-limiter nonlinearity with vertical segments in the *i-v* characteristic. The paper has also shown that monotonicity of the semiflow implies some fundamental restrictions on the geometry of omega-limit sets. These results parallel those valid for cooperative SCNNs. On the other hand, the paper has pointed out one main difference. Namely, a cooperative CNN with a cell-linking (irreducible) template and sigmoid activations generates an ESM semiflow that satisfies the LIMIT SET DICHOTOMY and is almost quasi-convergent [24], whereas for a FRCNN with an irreducible template, the generated semiflow is not ESM, and the LIMIT SET DICHOTOMY can fail. As a consequence, the convergence properties of cooperative and irreducible FRCNNs cannot be directly derived by means of the standard tools in the literature for ESM semiflows.

APPENDIX I
TWO PROPERTIES OF THE SEMIFLOW

Property 10
Let $x(t)$ and $y(t)$ be two solutions of (2) for $t \geq 0$. Then, we have

$$\|x(t) - y(t)\| \leq \|x(0) - y(0)\| e^{Lt}$$

for any $t \geq 0$, where $0 \leq L < +\infty$ is the Lipschitz constant of f restricted to the compact set K.

Proof
Consider the function $\Delta(t) = \|x(t) - y(t)\|^2/2, t \geq 0$. Since Δ is absolutely continuous on each compact interval in $[0, +\infty)$, it is differentiable for a.a. $t \geq 0$, and we have $\dot{\Delta}(t) = < x(t) - y(t), \dot{x}(t) - \dot{y}(t) >$, where $\dot{x}(t) = f(x(t)) - \gamma_x(t)$, $\dot{y}(t) = f(y(t)) - \gamma_y(t)$ and $\gamma_x(t) \in N_K(x(t))$, $\gamma_y(t) \in N_K(y(t))$ for a.a. $t \geq 0$. Then,

$$\begin{aligned}\dot{\Delta}(t) &= \langle x(t) - y(t), \dot{x}(t) - \dot{y}(t) \rangle \\ &= \langle x(t) - y(t), f(x(t)) - \gamma_x(t) - f(y(t)) + \gamma_y(t) \rangle \\ &= \langle x(t) - y(t), f(x(t)) - f(y(t)) \rangle - \langle x(t) - y(t), \gamma_x(t) - \gamma_y(t) \rangle.\end{aligned}$$

Since the normal cone to a compact convex set is a maximal monotone operator [28, Prop. 1, p. 159], we have $\langle x(t) - y(t), \gamma_x(t) - \gamma_y(t) \rangle \geq 0$. It follows that

$$\dot{\Delta}(t) \leq \langle x(t) - y(t), f(x(t)) - f(y(t)) \rangle \leq L\|x(t) - y(t)\|^2 = 2L\Delta(t).$$

By Gronwall's Lemma, we conclude that $0 \leq \Delta(t) \leq \Delta(0)e^{2Lt}$ for any $t \geq 0$. Furthermore, by recalling the definition of Δ, we obtain

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$$\|x(t) - y(t)\| = \sqrt{2\Delta(t)} \leq \sqrt{2\Delta(0)e^{2Lt}} = \|x(0) - y(0)\|e^{Lt}$$

for any $t \geq 0$.

Property 11
Let $f, g \in C^1(D)$, and consider the two DVIs $\dot{x} \in f(x) - N_K(x)$ and $\dot{y} \in g(y) - N_K(y)$. Let $x(t)$, $t \geq 0$, be the solution of the first DVI with initial condition $x \in K$ at $t = 0$ and $y(t)$, $t \geq 0$, the solution of the second DVI with initial condition $y \in K$ at $t = 0$. Then, we have

$$\|x(t) - y(t)\| \leq \sqrt{\|x - y\|^2 + 4M\sqrt{n}t}e^{Lt}$$

for any $x, y \in K$ and $t \geq 0$, where L is as in Property 10 and $M = \max_{z \in K}\{\|f(z) - g(z)\|\}$.

Proof
Consider again the function $\Delta(t) = \|x(t) - y(t)\|^2/2$, $t \geq 0$, and fix any $T \geq 0$. Taking into account that $\dot{x}(t) = f(x(t)) - \gamma_x(t), \dot{y}(t) = g(y(t)) - \gamma_y(t)$, where $\gamma_x(t) \in N_K(x(t))$ and $\gamma_y(t) \in N_K(y(t))$ for a.a. $t \in [0, T]$, we obtain

$$\begin{aligned}\Delta(t) &= \Delta(0) + \int_0^t \dot{\Delta}(s)ds = \Delta(0) + \int_0^t \langle x(s) - y(s), \dot{x}(s) - \dot{y}(s)\rangle ds \\ &= \Delta(0) + \int_0^t \langle x(s) - y(s), f(x(s)) - \gamma_x(s) - g(y(s)) + \gamma_y(s)\rangle ds \\ &= \Delta(0) + \int_0^t \langle x(s) - y(s), f(x(s)) - g(y(s))\rangle ds - \int_0^t \langle x(s) - y(s), \gamma_x(s) - \gamma_y(s)\rangle ds.\end{aligned}$$

We have $\langle x(s) - y(s), \gamma_x(s) - \gamma_y(s)\rangle \geq 0$ for a.a. $s \in [0, t]$, and so

$$\begin{aligned}\Delta(t) &= \Delta(0) + \int_0^t \langle x(s) - y(s), f(x(s)) - g(y(s))\rangle ds - \int_0^t \langle x(s) - y(s), \gamma_x(s) - \gamma_y(s)\rangle ds \\ &\leq \Delta(0) + \int_0^t \langle x(s) - y(s), f(x(s)) - g(y(s))\rangle ds \\ &= \Delta(0) + \int_0^t \langle x(s) - y(s), f(x(s)) - f(y(s))\rangle ds + \int_0^t \langle x(s) - y(s), f(y(s)) - g(y(s))\rangle ds \\ &\leq \Delta(0) + L\int_0^t \|x(s) - y(s)\|^2 ds + M\int_0^t \|x(s) - y(s)\| ds \\ &\leq \Delta(0) + M\mathrm{diam}(K)t + 2L\int_0^t \Delta(s)ds \\ &\leq \Delta(0) + M\mathrm{diam}(K)T + 2L\int_0^t \Delta(s)ds.\end{aligned}$$

Observing that $\mathrm{diam}(K) = 2\sqrt{n}$, and applying the Gronwall's Lemma, we conclude that

$$\frac{1}{2}\|x(t) - y(t)\|^2 = \Delta(t) \leq \left(\Delta(0) + 4M\sqrt{n}T\right)e^{2Lt} = \left(\frac{1}{2}\|x - y\|^2 + 4M\sqrt{n}T\right)e^{2Lt}$$

i.e., $\|x(t) - y(t)\| \leq \sqrt{\|x - y\|^2 + 4M\sqrt{n}T}e^{Lt}$, for all $t \in [0, T]$.
In particular, $\|x(T) - y(T)\| \leq \sqrt{\|x - y\|^2 + 4M\sqrt{n}T}e^{LT}$.

APPENDIX 2
SUBSET $\Gamma^\star$ IS POSITIVELY INVARIANT

To prove that $\Gamma^\star$ is positively invariant for (9) we begin by establishing the following geometrical properties.

A) Let us show that we have $\mathcal{P}_{T_{K_4}(z)}v \in \Gamma \cap T_{K_4}(z)$ for any $z \in \Gamma^\star$ and $v \in \Gamma$. Since by definition $\mathcal{P}_{T_{K_4}(z)}v \in T_{K_4}(z)$, it suffices to show that $\mathcal{P}_{T_{K_4}(z)}v \in \Gamma$, or $\tilde{\mathcal{P}}_{T_{K_4}(z)}v = -\hat{\mathcal{P}}_{T_{K_4}(z)}v$, where $\tilde{\mathcal{P}}$ denotes the first two components and $\hat{\mathcal{P}}$ the last two components. Due to the symmetry of K_2 with respect to the origin, we have $\mathcal{P}_{T_{K_2}(-x)}(-u) = -\mathcal{P}_{T_{K_2}(x)}u$ for any $x \in K_2$ and $u \in \mathbb{R}^2$. Now, let $\tilde{z} = (z_1, z_2)'$, $\hat{z} = (z_3, z_4)'$, $\tilde{v} = (v_1, v_2)'$ and $\hat{v} = (v_3, v_4)'$. Note that $\hat{z} = -\tilde{z}$ and $\hat{v} = -\tilde{v}$. Thus, recalling Property 3, we obtain $\hat{\mathcal{P}}_{T_{K_4}(z)}v = \mathcal{P}_{T_{K_2}(\hat{z})}\hat{v} = \mathcal{P}_{T_{K_2}(-\tilde{z})}(-\tilde{v}) = -\mathcal{P}_{T_{K_2}(\tilde{z})}\tilde{v} = -\tilde{\mathcal{P}}_{T_{K_4}(z)}v$.

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B) Let us prove that $\mathcal{P}_{T_{\Gamma^\star}(z)}v = \mathcal{P}_{T_{K_4}(z)}v$ for any $z \in \Gamma^\star$ and $v \in \Gamma$. Let $h = \mathcal{P}_{T_{K_4}(z)}v$. We want to show that h satisfies the conditions of Property 2 with $Q = T_{\Gamma^\star}(z)$, $x = v$ and $z = h$. Note that $T_\Gamma(z) = \Gamma$, since Γ is a subspace. By [29, Table 4.3, p. 141], we have that

$$T_{\Gamma^\star}(z) = T_{\Gamma \cap K_4}(z) = T_\Gamma(z) \cap T_{K_4}(z) = \Gamma \cap T_{K_4}(z). \tag{12}$$

Then, point (A) implies that $h \in T_{\Gamma^\star}(z)$. Now, fix any $u \in T_{\Gamma^\star}(z)$. By (12) we have $u \in T_{K_4}(z)$. Moreover, by point (b) in Property 2, we have $\langle v - h, u \rangle = \left\langle v - \mathcal{P}_{T_{K_4}(z)}v, u \right\rangle \leq 0$. Finally, due to point (c) in Property 2, we have $\langle v - h, h \rangle = \left\langle v - \mathcal{P}_{T_{K_4}(z)}v, \mathcal{P}_{T_{K_4}(z)}v \right\rangle = 0$.

Now, fix any $x \in \Gamma^\star$ and let $x(t)$, $t \geq 0$, be the solution of (9) with initial condition $x(0) = x$. We want to prove that $x(t) \in \Gamma^\star$ for any $t \geq 0$. To this aim, let us consider the DVI

$$\begin{cases} \dot{y} \in -y + Ay - N_{\Gamma^\star}(y) \\ y(0) = x. \end{cases}$$

Since $\Gamma^\star$ is a nonempty, compact, convex subset of $\mathbb{R}^4$, [18, Properties 2, 3] assure that there exists a solution $y(t) \in \Gamma^\star$ of (9) defined for all $t \geq 0$. In particular, $y(\cdot)$ is an absolutely continuous function on any compact interval in $[0, +\infty)$ and we have $\dot{y}(t) \in -y(t) + Ay(t) - N_{\Gamma^\star}(y(t))$ for a.a. $t \geq 0$. In what follows, we show that $y(t) = x(t)$, hence $x(t) \in \Gamma^\star$, $t \geq 0$. To this end, it suffices to show that $\dot{y}(t) = \mathcal{P}_{T_{K_4}(y(t))}(A - E_4)y(t)$ for a.a. $t \geq 0$, see Properties 4, 5. Fix any $t \geq 0$. Since $y(t) \in \Gamma$, it can be immediately verified that $(A - E_4)y(t) \in \Gamma$. Thus, from point (B), we obtain $\mathcal{P}_{T_{\Gamma^\star}(y(t))}(A - E_4)y(t) = \mathcal{P}_{T_{K_4}(y(t))}(A - E_4)y(t)$. Hence, by applying [28, Proposition 2, p. 266], we obtain $\dot{y}(t) = \mathcal{P}_{T_{\Gamma^\star}(y(t))}(A - E_4)y(t) = \mathcal{P}_{T_{K_4}(y(t))}(A - E_4)y(t)$ for a.a. $t \in [0, +\infty)$.

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