NONSMOOTH BIFURCATION PROBLEMS IN FINITE DIMENSIONAL SPACES VIA SCALING OF VARIABLES*

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Abstract

We consider a nonsmooth bifurcation equation depending on a small parameter $\varepsilon > 0$. In Theorem 1 we provide conditions ensuring the existence of branches of solutions, smoothly depending on ε, emanating from a curve of solutions of the bifurcation equation when $\varepsilon = 0$. Several examples will illustrate the different types of bifurcation that occur in the present nonsmooth case.

Key words: nonsmooth bifurcation, branches of solutions, implicit function theorem.

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1 Introduction

In this paper we consider a bifurcation equation of the form

$$P(x) + \varepsilon Q(x, \varepsilon) = 0 \tag{1}$$

where $P : \mathbb{R}^n \to \mathbb{R}^n$ and $Q : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ and $\varepsilon > 0$ is a small parameter. Assuming the existence of a parametrized smooth curve $\theta \mapsto x_0(\theta), \theta \in [0, 1]$, of zeros of the function P, namely $P(x_0(\theta)) = 0$ for any $\theta \in [0, 1]$, we look for conditions ensuring the existence of smooth families of solutions $x(\varepsilon), \varepsilon > 0$ small, of (1) originating from the curve $x_0(\theta)$ at some $\theta_0 \in [0, 1]$. In [19] the solutions of (1) represent the fixed points of the Poincaré map $\mathcal{P}_\varepsilon$ associated to a periodically perturbed autonomous dynamical system of the form

$$x' = f(x) + \varepsilon\, g(t, x, \varepsilon). \tag{2}$$

where $f \in C^2(\mathbb{R}^n, \mathbb{R}^n), g \in C^1(\mathbb{R} \times \mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ is T-periodic and $\varepsilon > 0$ is a small parameter. In fact, defining

$$P(x) = \mathcal{P}_0(x) - x \quad \text{and} \quad Q(x, \varepsilon) = \frac{\mathcal{P}_\varepsilon(x) - \mathcal{P}_0(x)}{\varepsilon}, \tag{3}$$

we can write $\mathcal{P}_\varepsilon(x) - x = P(x) + \varepsilon Q(x, \varepsilon)$.

Precisely, in [19] the existence of a branch of zeros of (1), parametrized by $\varepsilon > 0$, bifurcating from the curve $x_0(\theta)$ is proven. For system (2) this is equivalent to the existence of a family of periodic solutions originating from the limit cycle of the unperturbed autonomous system. The uniqueness and asymptotic stability of these bifurcating periodic solutions are also discussed. Existence, uniqueness, and asymptotic stability of periodic solutions emanating from a limit cycle of an autonomous system when it is periodically perturbed is a very classical problem, see [4, 23, 25]. The regularity assumptions on the functions f and g in (2) guarantee that $P \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $Q \in C^1(\mathbb{R}^n \times [0, 1], \mathbb{R}^n)$.

On the other hand, in many problems involving dynamical systems such a regularity for the associated bifurcation equation is not available. In fact, many dynamical systems arising in applications are nonsmooth. Recently, considerable attention has been paid to the study of the dynamics of nonsmooth systems by means of a bifurcation analysis, see [10] and the extensive references therein. In this review paper the authors have focused on the state of art of the bifurcation analysis in nonsmooth dynamical systems by presenting a large number of results and examples. A precise classification of the different types of the nonsmooth systems under consideration is given. The study of the related bifurcations of equilibria and periodic orbits is also provided.

One relevant example of nonsmooth systems is represented by the piecewise smooth (PWS) systems [11]. In particular, by using a method developed in [15] and tools from the smooth and nonsmooth analysis several authors have performed both qualitative analysis and control of the local properties of the Poincaré map associated to PWS systems. The local analysis has permitted to study and classify the dynamical behaviors of such systems in terms of the different discontinuity-induced bifurcations that can occur in presence of collisions of a limit cycle with a surface at which the dynamics of the PWS system changes, see [8, 18, 21, 24]. Since the occurrence of such collisions can generate a sudden jump from a periodic to a chaotic evolution it is important to control the local properties of the Poincaré map to obtain a desidered dynamical behavior, see [1, 9].

Following [10] other important classes of nonsmooth systems are represented by the discontinuous PWS systems (Filippov systems) and the impacting systems. For the first class it is of particular interest to investigate the existence and the properties of the so-called sliding bifurcation for the Filippov regularization of the discontinuous systems, see e.g. [20, 26]. For the second one, recently the dynamical analysis and the control of impacting systems has received a new attention, in fact their complex dynamics exhibit interesting bifurcation behavior as illustrated, for instance, in [7, 12].

Many other books and papers have been devoted to the study of nonsmooth dynamical systems, more examples and motivations can be found in [2, 3, 5, 16]. Concerning the theoretical setting to approach the dynamical analysis of such systems we refer to the recent books [13, 14], where the author provides rigorous mathematical tools for handling bifurcation and chaos in a broad variety of nonlinear problems like the study of oscillations and vibrations in mechanical systems.

In this paper we consider the equation (1) in the case when the function P is continuous but not Fréchet differentiable at the points $x_0(\theta), \theta \in [0,1]$, at which we want to study the existence of bifurcating branches of solutions of (1). The function Q can be either discontinuous or not Fréchet differentiable at these points. We will only assume the existence of the limits at $x_0(\theta)$, of the derivatives P', Q' of the restrictions of P and Q to suitably defined open subsets of $\mathbb{R}^n$. We suppose that the limits of P' have zero as a simple eigenvalue with corresponding eigenvector $x_0'(\theta)$. As we will see, in the present nonsmooth case, there could exist more than one branch of solutions emanating from $x_0(\theta)$ for some $\theta_0 \in [0,1]$. The advantage of our general bifurcation theorem is that it does not require the usual Lyapunov-Schmidt reduction but only a suitable scaling of the state variables, which reduces significantly the required analysis.

In Section 2, Proposition 1 provides a simple case of the bifurcation equation (1) in $\mathbb{R}^2$, for which we establish sufficient conditions for the existence of the various types of bifurcation that can occur when P is not differentiable at the bifurcation point $x_0(\theta_0)$. Precisely, when the function P fails to be differentiable at $x_0(\theta_0)$ we can have the existence of a single branch as well as that of two branches of zeros of (1) emanating from $x_0(\theta_0)$ or even the nonexistence of solutions near $x_0(\theta_0)$. We end Section 2 by showing three very simple examples for which the different conditions of Proposition 1 are satisfied.

Proposition 1 is a very particular case of the main result of the paper proved in Section 3, namely Theorem 1. In fact, Theorem 1 is set in $\mathbb{R}^n$ where we assume that the smooth curve $\theta \mapsto x_0(\theta), \theta \in [0,1]$, of zeros of P is the intersection of a finite number of smooth $(n-1)-$dimensional surfaces intersecting each other transversally. We assume that P is continuous and not Fréchet differentiable at $x_0(\theta)$, $\theta \in [0,1]$; Q is either discontinuous or not Fréchet differentiable at the points of the smooth curve $\theta \mapsto \kappa(\theta), \theta \in [0,1]$, which is the intersection of a finite number of smooth $(n-1)-$dimensional surfaces intersecting each other transversally. We also assume that there exists $\theta_0 \in [0,1]$ such that $x_0(\theta_0) = \kappa(\theta_0)$ and so the above surfaces divide any sufficiently small neighborhood of $x_0(\theta_0)$ in disjoint open sets D_k and Δ_j respectively. We formulate conditions on $\theta_0, D_k, \Delta_j, P$ and Q to ensure the applicability of the classical Implicit Function Theorem in the open set $D_k \cap \Delta_j$ to a suitably defined function $\Psi_{k,j}(w, \varepsilon)$ whose zeros, for $\varepsilon > 0$ sufficiently small, are solutions to (1). Specifically, these conditions are expressed in terms of vectors, denoted by w_k^j, which are defined by means of the functions P, Q, the limits at $x_0(\theta_0)$ of the derivatives of their restrictions to the open sets $D_k \cap \Delta_j$ and the Riesz projector associated to the eigenvector $x'(\theta_0)$ corresponding to the simple zero eigenvalue of

$$P_k'(x_0(\theta_0)) = \lim_{x \to x_0(\theta_0),\, x \in D_k} P_k'(x).$$

Such conditions ensure that $x_0(\theta_0) + \delta w_k^j \in D_k \cap \Delta_j$, for $\delta > 0$ sufficiently small and that w_k^j is a simple zero of $\Psi_{k,j}(w, 0)$, this in turn guarantees the existence of a branch of solutions to (1) lying in the open set $D_k \cap \Delta_j$ of the form $x_{k,j}(\varepsilon) = x_0(\theta_0) + \varepsilon w_k^j + o(\varepsilon)$.

Furthermore, Proposition 2 and Lemma 2 show that the existence of the simple zero θ_0 of the Malkin bifurcation function $M_{k,j}(\theta)$ associated to the problem, see [25], is equivalent to the existence of a simple zero w_k^j of $\Psi_{k,j}(w, 0)$, where $w = \dfrac{1}{\varepsilon}(x - x_0(\theta_0))$. Therefore, as pointed out in Remark 4, the conditions of Theorem 1 can be equivalently formulated in terms of the existence of a simple zero θ_0 for the Malkin bifurcation function $M_{k,j}(\theta)$. We end the paper by presenting in Section 4 an example of a periodically perturbed autonomous dynamical system in $\mathbb{R}^2$ which satisfies all the assumptions of Theorem 1 and having two branches of solutions originating from $x_0(0)$.

2 An Introductory Simple Case in $\mathbb{R}^2$

Consider the equation $P(x, y) + \varepsilon Q(x, y, \varepsilon) = 0$, with $P : \mathbb{R}^2 \to \mathbb{R}^2$ and $Q : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2$. Assume that the equation $P(x, y) = 0$ has one-dimensional set of solutions given by $x_0(\theta) = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$, $\theta \in [0, 1]$, that P is continuous in $\mathbb{R}^2$ and that it is not Fréchet differentiable at $x_0(\theta)$, $\theta \in [0, 1]$. Assume that P is twice continuously differentiable at any point $(x, y) \in \mathbb{R}^2$ with $y \neq 0$ and that, for simplicity, the map Q is smooth in $\mathbb{R}^2$. Denote by $P_\pm'(x, y)$ and $P_\pm''(x, y)$ the derivatives for $y > 0$ and $y < 0$ respectively. Assume that these derivatives have the limits

$$\lim_{(x,y) \to (\hat{x}, 0\pm)} P_\pm'(x, y) := P_\pm'(\hat{x}, 0) \quad \text{and} \quad \lim_{(x,y) \to (\hat{x}, 0\pm)} P_\pm''(x, y) := P_\pm''(\hat{x}, 0)$$

for all $\hat{x} \in [0, 1]$, $x_0'(\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Moreover, zero is an eigenvalue of $P_\pm'(x_0(\theta))$ with eigenvector $x_0'(\theta)$, we assume that it is simple. In what follows by span(v) we denote the linear space spanned by the vector v.

Let $\pi_\pm(\theta) : \mathbb{R}^2 \to \text{span}(x_0'(\theta))$ be the Riesz projector given by $\pi_\pm(\theta)v = \langle v, z_\pm(\theta) \rangle x_0'(\theta)$, where $z_\pm(\theta)$ is the eigenvector of the adjoint operator $(P_\pm'(x_0(\theta)))^*$, corresponding to the

zero eigenvalue, such that $\langle z_\pm(\theta), x_0'(\theta)\rangle = 1$.
Therefore, the one-dimensional subspace $E_\pm(\theta) = Ker(\pi_\pm(\theta))$ is invariant with respect to $P_\pm'(x_0(\theta))$, and $\mathbb{R}^2 = Im(\pi_\pm(\theta)) \oplus Ker(\pi_\pm(\theta))$ and $\pi_\pm(\theta)P_\pm'(x_0(\theta)) = 0$, see e.g. [17]. Introduce now the Malkin bifurcation function, see [23, 25], $M_\pm(\theta)$ as follows:

$$M_\pm(\theta)x_0'(\theta) = \langle Q(x_0(\theta), 0), z_\pm(\theta)\rangle x_0'(\theta).$$

Hence $M_\pm(\theta) = 0$ if and only if $\pi_\pm(\theta)Q(x_0(\theta), 0) = 0$.

Here and in the following $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$. Finally, we denote by $n_\pm(x_0(\theta))$ the normal vector to $x_0(\theta)$ directed towards $y > 0$ and $y < 0$ respectively and by $F'_{(r)}(v, \varepsilon), v \in \mathbb{R}^n$, the derivative of $F(v, \varepsilon)$ with respect to the r-th variable, $r \in \{1, 2\}$.

The following result is a straightforward consequence of Theorem 1 of Section 3.

Proposition 1. *Assume that there exists $\theta_0 \in [0, 1]$ such that*

1. $\pi_\pm(\theta_0)Q(x_0(\theta_0), 0) = 0$;
2. *The operator*

$$\begin{aligned}\Lambda_\pm(\theta_0) :=& \pi_\pm(\theta_0)P_\pm''(x_0(\theta_0))(-P_\pm'(x_0(\theta_0))|_{(I-\pi_\pm(\theta_0))\mathbb{R}^2})^{-1}Q(x_0(\theta_0), 0)\pi_\pm(\theta_0) + \\ &+ \pi_\pm(\theta_0)Q'_{(1)}(x_0(\theta_0), 0)\pi_\pm(\theta_0)\end{aligned}$$

is invertible.

Let

$$y_0^\pm = (-P_\pm'(x_0(\theta_0))|_{(I-\pi_\pm(\theta_0))\mathbb{R}^2})^{-1}Q(x_0(\theta_0), 0), \tag{4}$$

$$\begin{aligned} x_0^\pm &= (\Lambda_\pm(\theta_0))^{-1}(-\pi_\pm(\theta_0)Q'_{(2)}(x_0(\theta_0), 0) - \frac{1}{2}\pi_\pm(\theta_0)P_\pm''(x_0(\theta_0))y_0^\pm y_0^\pm - \\ &\quad -\pi_\pm(\theta_0)Q'_{(1)}(x_0(\theta_0), 0)y_0^\pm) \end{aligned} \tag{5}$$

and $w_0^\pm := y_0^\pm + x_0^\pm$.

Then, for $\varepsilon > 0$ sufficiently small, the equation $P(x, y) + \varepsilon Q(x, y, \varepsilon) = 0$

I. *has two branches of solutions emanating from $x_0(\theta_0)$ if both the following conditions*

$$\langle w_0^+, n_+(x_0(\theta_0))\rangle > 0 \quad \text{and} \quad \langle w_0^-, n_-(x_0(\theta_0))\rangle > 0 \tag{6}$$

are satisfied. The branches of solutions have the form

$$\begin{pmatrix} x \\ y \end{pmatrix}(\varepsilon) = x_0(\theta_0) + \varepsilon w_0^\pm + o(\varepsilon); \tag{7}$$

II. *has a branch of solutions if only one of the conditions (6) is verified.*

Remark 1. It is immediate to verify that if both the conditions (6) are violated then (1) does not have any branch of solutions of the form (7).

We now provide three very simple examples to illustrate the different bifurcation cases described in the above Proposition 1. The only aim of the proposed examples is to verify the conditions of Proposition 1 in very simple situations, in fact it is evident that we can get the same conclusions by trivial direct calculations.

Example 1. Let $P(x,y) = \begin{pmatrix} 0 \\ |y| \end{pmatrix}$, $Q(x,y,\varepsilon) = \begin{pmatrix} x+y-1/2 \\ x+y-1 \end{pmatrix}$ and $x_0(\theta) = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$, $\theta \in [0,1]$. Let $\theta_0 = 1/2$ and consider the directional derivatives of P at $x_0(\theta_0)$ along $y > 0$ and $y < 0$ respectively

$$P'_{\pm}(x_0(\theta_0)) = \begin{pmatrix} 0 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Observe that $x_0'(\theta_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is eigenvector of $P'_{\pm}(x_0(\theta_0))$ corresponding to its simple zero eigenvalue, moreover $z_{\pm}(\theta_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence $\pi_{\pm}(\theta_0) : \mathbb{R}^2 \to \operatorname{span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix})$ is given by

$$\pi_{\pm}(\theta_0)v = \pi(\theta_0)v = \langle v, x_0'(\theta_0)\rangle x_0'(\theta_0).$$

Since $Q(x_0(\theta_0), 0) = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$ we have that $\pi(\theta_0)Q(x_0(\theta_0), 0) = 0$, namely $M_{\pm}(\theta_0) = 0$ and so assumption 1 of Proposition 1 is satisfied. The vectors $y_0^{\pm}$ are given by

$$y_0^{\pm} = (-P'_{\pm}(x_0(\theta_0))|_{(I-\pi(\theta_0))\mathbb{R}^2})^{-1} Q(x_0(\theta_0), 0) = \begin{pmatrix} 0 \\ \pm 1/2 \end{pmatrix}.$$

Furthermore, since

$$P''_{\pm}(x_0(\theta_0)) = 0,$$

$$\pi(\theta_0) Q'_{(1)}(x_0(\theta_0), 0) y_0^{\pm} = \begin{pmatrix} \pm 1/2 \\ 0 \end{pmatrix},$$

$$\pi(\theta_0) Q'_{(1)}(x_0(\theta_0), 0) \pi(\theta_0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$Q'_{(2)}(x_0(\theta_0), 0) = 0,$$

we have that assumption 2 of Proposition 1 is verified and $x_0^{\pm} = \begin{pmatrix} \mp 1/2 \\ 0 \end{pmatrix}$, therefore $w_0^{\pm} = \begin{pmatrix} \mp 1/2 \\ \pm 1/2 \end{pmatrix}$. Since

$$\langle w_0^{\pm}, n_{\pm}(x_0(\theta_0))\rangle > 0,$$

where $n_\pm(x_0(\theta_0)) = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$, Proposition 1 ensures the existence of two families of solutions to $P(x,y) + \varepsilon Q(x,y,\varepsilon) = 0$ originating from $x_0(\theta_0) = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix}(\varepsilon) = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} \mp 1/2 \\ \pm 1/2 \end{pmatrix} \varepsilon.$$

Example 2. Let $P(x,y) = \begin{pmatrix} 0 \\ |y| + 2y \end{pmatrix}$, $Q(x,y,\varepsilon) = \begin{pmatrix} x+y-1/2 \\ x+y-1 \end{pmatrix}$ and $x_0(\theta) = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$, $\theta \in [0,1]$. Let $\theta_0 = 1/2$. We have $P'_+(x_0(\theta_0)) = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$ and $P'_-(x_0(\theta_0)) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

As before

$$x_0'(\theta_0) = z_\pm(\theta_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$P''_\pm(x_0(\theta_0)) = 0,$$
$$\pi(\theta_0)Q(x_0(\theta_0),0) = 0,$$
$$\pi(\theta_0)Q'_{(1)}(x_0(\theta_0),0)\pi(\theta_0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$Q'_{(2)}(x_0(\theta_0),0) = 0.$$

In this case we have $y_0^+ = \begin{pmatrix} 0 \\ 1/6 \end{pmatrix}$, $y_0^- = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$, $x_0^+ = \begin{pmatrix} 1/6 \\ 0 \end{pmatrix}$, $x_0^- = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, hence

$$\langle w_0^+, n_+(x_0(\theta_0))\rangle > 0 \qquad \text{and} \qquad \langle w_0^-, n_-(x_0(\theta_0))\rangle < 0.$$

By Proposition 1 we get the existence of a family of solutions of the form

$$\begin{pmatrix} x \\ y \end{pmatrix}(\varepsilon) = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/6 \\ 1/6 \end{pmatrix} \varepsilon.$$

Example 3. Let $P(x,y) = \begin{pmatrix} 0 \\ -|y| \end{pmatrix}$, $Q(x,y,\varepsilon)$, $x_0(\theta)$ and θ_0 as before. We have

$$P'_\pm(x_0(\theta_0)) = \begin{pmatrix} 0 & 0 \\ 0 & \mp 1 \end{pmatrix},$$

proceeding as in Example 1 we obtain $w_0^\pm = \begin{pmatrix} \pm 1/2 \\ \mp 1/2 \end{pmatrix}$ and so $\langle w_0^\pm, n_\pm(x_0(\theta_0))\rangle < 0$. By direct calculation we verify that there is no solutions near $x_0(\theta_0) = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$.

3 The Main Result: General Formulation in $\mathbb{R}^n$

In this Section we consider the bifurcation equation

$$P(x) + \varepsilon Q(x, \varepsilon) = 0$$

where $P : \mathbb{R}^n \to \mathbb{R}^n$ and $Q : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$.

Assume that the equation $P(x) = 0$ has a parametrized one-dimensional set of solutions $\{x_0(\theta), \theta \in [0, 1]\}$. Denote this set by Γ_P. Assume that $\Gamma_P \subseteq \cap_{i=1}^m M_i^P$, where M_i^P are $(n-1)-$dimensional smooth surfaces in $\mathbb{R}^n$. Then the oriented normal $n_i(x), x \in M_i^P, i = 1, .., m$, to each surface is well-defined. Assume that the given surfaces divide any sufficiently small neighborhood $U(\Gamma_P)$ of the curve Γ_P into $2m$ open domains D_k, $k = 1, .., 2m$. We suppose also that the map P has continuous Fréchet derivatives $P_k', P_k'', k = 1, .., 2m$, in every domain D_k and all these derivatives have limits

$$P_k'(x_p) := \lim_{x \to x_p,\, x \in D_k} P_k'(x) \quad \text{and} \quad P_k''(x_p) := \lim_{x \to x_p,\, x \in D_k} P_k''(x)$$

for all $x_p \in \Gamma_P, k = 1, .., 2m$.

Moreover, assume that Q is either discontinuous or not Fréchet differentiable with respect to the first variable x on some curve $\Gamma_Q = \{\kappa(\theta) : \theta \in [0, 1]\}$, such that $\Gamma_Q \subseteq \cap_{i=1}^l M_i^Q$, where M_i^Q are $(n-1)-$dimensional smooth surfaces in $\mathbb{R}^n$, and that there exists $\theta_0 \in [0, 1]$ such that $x_0(\theta_0) = \kappa(\theta_0)$. All the considerations below concern the existence of solutions in a neighborhood of $x_0(\theta_0)$. Any sufficiently small neighborhood $V(\Gamma_Q)$ of the curve Γ_Q is divided into $2l$ open domains $\Delta_j, j = 1, .., 2l$, by the surfaces $M_i^Q, i = 1, .., l$. Assume that Q is continuously differentiable with respect to x in every Δ_j and with respect to the second variable ε when the first belongs to Δ_j. Moreover, the derivative with respect to ε is assumed to be continuous with respect to (x, ε) with $x \in \Delta_j$. We assume that all these derivatives have the limits

$$Q'_{j(1)}(x_q, \varepsilon) := \lim_{x \to x_q,\, x \in \Delta_j} Q'_{j(1)}(x, \varepsilon)$$

for all $x_q \in \Gamma_Q, j = 1, .., 2l$, and

$$Q'_{(2)}(x, 0) := \lim_{\varepsilon \to 0,\, x \in \Delta_j} Q'_{(2)}(x, \varepsilon)$$

for all $j = 1, .., 2l$. Define

$$Q_j(x_q, \varepsilon) := \lim_{x \to x_q,\, x \in \Delta_j} Q(x, \varepsilon)$$

for all $x_q \in \Gamma_Q, j = 1, .., 2l$. Clearly, if Q is continuous at the point $x_q \in \Gamma_Q$ then $Q_j(x_q, \varepsilon)$ does not depend on j.

In the rest of the paper we assume that

A1) the curves Γ_P and Γ_Q are smooth;

A2) $x_0'(\theta) \neq 0$ for all $\theta \in [0,1]$ and there exists $x_0''(\theta)$ continuous at any $\theta \in [0,1]$;

A3) any pair of the surfaces M_i^P either intersects transversally or coincides. The same for M_i^Q.

As it is easy to see the last condition allows us to introduce a bijection between the $2m$ open sets D_k and the $2m$ vectors of m components

$$\text{sign}\langle \xi, n_1(x_0(\theta_0))\rangle, .., \text{sign}\langle \xi, n_m(x_0(\theta_0))\rangle$$

where ξ is any vector belonging to $D_k, k = 1, .., 2m$. In other words, if we choose a vector $(e_1^k, e_2^k, .., e_m^k)$, with $e_i^k \in \{-1, 1\}, i = 1, .., m$, among the $2m$ vectors above then the condition

$$\langle v, n_i(x_0(\theta_0))\rangle e_i^k > 0,$$

where $i = 1, .., m$, is equivalent to the property $x_0(\theta_0) + \delta v \in D_k$ for $\delta > 0$ sufficiently small. Analogously, we can define $2l$ vectors $(\zeta_1^j, \zeta_2^j, .., \zeta_l^j), \zeta_i^j \in \{-1, 1\}, i = 1, .., l$ and $j = 1, .., 2l$, such that the condition

$$\langle u, \eta_i(x_0(\theta_0))\rangle \zeta_i^j > 0,$$

is equivalent to the property $x_0(\theta_0) + \delta u \in \Delta_j$ for $\delta > 0$ sufficiently small. Here $\eta_i(x_0(\theta_0))$, are the oriented normals to the surfaces M_i^Q, $i = 1, .., l$,

In conclusion, we have defined the two following families of vectors which identify the domains D_k and Δ_j respectively:

$$\{(e_1^k, e_2^k, .., e_m^k) : e_i^k \in \{-1, 1\}, i = 1, 2, .., m; k = 1, 2, .., 2m\},$$

$$\{(\zeta_1^j, \zeta_2^j, .., \zeta_l^j) : \zeta_i^j \in \{-1, 1\}, i = 1, 2, .., l; j = 1, 2, .., 2l\}.$$

Any nonempty subset of the previous families is determined by assigning a subset of the sets of indexes $\{k_{i_p}\}_{i_p=1,..,r_p}$ and $\{j_{i_q}\}_{i_q=1,..,r_q}$ respectively, where $1 \leq r_p \leq 2m$ and $1 \leq r_q \leq 2l$. Lemma 1 and Remark 3 will establish that $x_0'(\theta)$ is an eigenvector corresponding to the zero eigenvalue for all $P_k'(x_0(\theta)), k = 1, .., 2m$, and for θ sufficiently close to θ_0. In the sequel we also assume that

A4) the zero eigenvalue of $P_k'(x_0(\theta)), k = 1, .., 2m$, is simple.

To every $P_k'(x_0(\theta)), k = 1, .., 2m$, we can associate the Riesz projector

$$\pi_k(\theta) : \mathbb{R}^n \to \text{span}(x_0'(\theta)) \qquad \text{given by} \qquad \pi_k(\theta)v = \langle v, z_k(\theta)\rangle x_0'(\theta),$$

where $z_k(\theta)$ is the eigenvector corresponding to the simple zero eigenvalue of the adjoint operator $(P_k'(x_0(\theta)))^*$ for which $\langle x_0'(\theta), z_k(\theta)\rangle = 1$. Moreover, $Ker(\pi_k(\theta)) = E_k(\theta)$

is an $(n-1)-$dimensional subspace of $\mathbb{R}^n$ such that $\mathbb{R}^n = \text{span}(x_0'(\theta)) \oplus E_k(\theta)$ and $P_k'(x_0(\theta))E_k(\theta) \subseteq E_k(\theta)$, see [17].

Finally, define the Malkin functions $M_{k,j}(\theta)$ as follows

$$M_{k,j}(\theta)x_0'(\theta) = \langle Q_j(x_0(\theta),0), z_k(\theta)\rangle x_0'(\theta),$$

whenever $k = 1,..,2m$ and $j = 1,..,2l$. In Proposition 2 it will be shown that $M_{k,j}'(\theta)$ exists in the domains of interest $D_k \cap \Delta_j$.

Under the conditions **A1)-A4)** stated above, we are now in the position to prove the main result of the paper.

Theorem 1. *Assume that there exist $\theta_0 \in [0,1]$, with $x_0(\theta_0) \in \Gamma_P \cap \Gamma_Q$, $1 \le r_p \le 2m$, $1 \le r_q \le 2l$ and indexes $\{k_{i_p}\}_{i_p=1,..,r_p}, \{j_{i_q}\}_{i_q=1,..,r_q}$ such that for any $k \in \{k_{i_p}\}_{i_p=1,..,r_p}$ and $j \in \{j_{i_q}\}_{i_q=1,..,r_q}$ the following conditions hold:*

1. $\pi_k(\theta_0)Q_j(x_0(\theta_0),0) = 0$;

2. $\pi_k(\theta_0)P_k''(x_0(\theta_0))y_k^j\pi_k(\theta_0) + \pi_k(\theta_0)Q_{j(1)}'(x_0(\theta_0),0)\pi_k(\theta_0)$ *is invertible;*

3. $\langle w_k^j, n_i(x_0(\theta_0))\rangle e_i^k > 0, i = 1,..,m,$ *and* $\langle w_k^j, \eta_i(x_0(\theta_0))\rangle \zeta_i^j > 0, i = 1,..,l,$ *where*

$$w_k^j = y_k^j + x_k^j,$$

$$y_k^j = -(P_k'(x_0(\theta_0))|_{(I-\pi_k(\theta_0))\mathbb{R}^n})^{-1}Q_j(x_0(\theta_0),0),$$

and

$$x_k^j = \big(\pi_k(\theta_0)P_k''(x_0(\theta_0))y_k^j\pi_k(\theta_0) + \pi_k(\theta_0)Q_{j(1)}'(x_0(\theta_0),0)\pi_k(\theta_0)\big)^{-1} \cdot \\ \cdot(-\pi_k(\theta_0)Q_{(2)}'(x_0(\theta_0),0) - \frac{1}{2}\pi_k(\theta_0)P_k''(x_0(\theta_0))y_k^jy_k^j - \pi_k(\theta_0)Q_{j(1)}'(x_0(\theta_0),0)y_k^j).$$

Then there exists a local solution to (1) given by

$$x_{k,j}(\varepsilon) = x_0(\theta_0) + \varepsilon w_k^j + o(\varepsilon). \tag{8}$$

Remark 2. Note that if for a pair $(k,j) \in \{1,..,2m\} \times \{1,..,2l\}$ either there exists $i \in \{1,..,m\}$ such that $\langle w_k^j, n_i(x_0(\theta_0))\rangle e_i^k < 0$ or there exists $i \in \{1,..,l\}$ such that $\langle w_k^j, \eta_i(x_0(\theta_0))\rangle \zeta_i^j < 0$ then (1) does not have branches of solutions of the form (8) in $D_k \cap \Delta_j$. Furthermore, we would like to point out that the vectors w_k^j which give rise to the solutions $x_{k,j}(\varepsilon)$ can be explicitly computed by the previous formulas. Moreover, for $\varepsilon > 0$ small, if the solution to (1) is not unique then the Clarke's generalized Jacobian of P_k is not of full rank, see [6].

The proof of Theorem 1 relies on the following Lemmas 1 and 2.

Lemma 1. *$x_0'(\theta_0)$ is an eigenvector corresponding to the simple zero eigenvalue of $P_k'(x_0(\theta_0))$, whenever $k = 1, .., 2m$.*

Proof. Let $\xi \in \mathbb{R}^n$ such that $x_0(\theta) + \xi \in D_k$, for θ close to θ_0, we have

$$\begin{aligned} P\left(x_0(\theta) + \xi(\theta - \theta_0)^2\right) - P\left(x_0(\theta_0)\right) =& P_k'\left(x_0(\theta_0)\right)\left(x_0(\theta) + \xi(\theta - \theta_0)^2 - x_0(\theta_0)\right) + \\ &+ o\left(x_0(\theta) + \xi(\theta - \theta_0)^2 - x_0(\theta_0)\right), \end{aligned}$$

where $\dfrac{o(z)}{z} \to 0$ as $z \to 0$. Then, since $x_0(\theta)$ is differentiable at any $\theta \in [0, 1]$ and $P(x_0(\theta)) = P(x_0(\theta_0)) = 0$, we get

$$\begin{aligned} &P_k'\left(x_0(\theta_0)\right)\left(x_0'(\theta_0)(\theta - \theta_0) + o(\theta - \theta_0) + \xi(\theta - \theta_0)^2\right) + o(\theta - \theta_0) = \\ =&P\left(x_0(\theta) + \xi(\theta - \theta_0)^2\right) - P\left(x_0(\theta)\right) = P_k'\left(x_0(\theta)\right)\xi(\theta - \theta_0)^2 + o((\theta - \theta_0)^2). \end{aligned}$$

Dividing by $\theta - \theta_0$ and passing to the limit as $\theta \to \theta_0$, since the function $\theta \to P_k'(x_0(\theta))$ is bounded, one has $P_k'(x_0(\theta_0))x_0'(\theta_0) = 0$. Thus the projector $\pi_k(\theta_0)$ is defined for any $k = 1, .., 2m$. □

Remark 3. The proof of Lemma 1 holds true for any θ sufficiently close to θ_0, hence for such values of θ the projector $\pi_k(\theta)$ is also well-defined for $k = 1, .., 2m$.

Let $x_0(\theta_0) = v_0$, $x = v_0 + \varepsilon w \in D_k \cap \Delta_j$, $\Phi_{k,j}(v_0 + \varepsilon w, \varepsilon) := P(x) + \varepsilon Q(x, \varepsilon)$, $\pi_k(\theta_0) = \pi_k$ and for any $k = 1, .., 2m$ and $j = 1, .., 2l$ introduce the functions

$$\Psi_{k,j}(w, \varepsilon) = \frac{1}{\varepsilon}\left(\Phi_{k,j}(v_0 + \varepsilon w, \varepsilon) - \pi_k \Phi_{k,j}(v_0 + \varepsilon w, \varepsilon) + \frac{1}{\varepsilon}\pi_k \Phi_{k,j}(v_0 + \varepsilon w, \varepsilon)\right).$$

It is easy to see that, for $\varepsilon > 0$, $\Phi_{k,j}(v_0 + \varepsilon w, \varepsilon) = 0$ if and only if $\Psi_{k,j}(w, \varepsilon) = 0$.

We can now formulate the following.

Lemma 2. *Assume all the conditions of Theorem 1. Then there exists w_k^j such that $v_0 + \delta w_k^j \in D_k \cap \Delta_j$ for sufficiently small $\delta > 0$, $\Psi_{k,j}(w_k^j, 0) = 0$ and $\Psi_{k,j(1)}'(w_k^j, 0)$ is invertible.*

Proof. Under our assumptions, since $\pi_k P_k'(v_0) = 0$ and $\pi_k P_k''(v_0)\pi_k r\, \pi_k s = 0$ for any $r, s \in \mathbb{R}^n$, it can be shown that

$$\lim_{\varepsilon \to 0} \pi_k \Psi_{k,j}(w, \varepsilon) = \frac{1}{2}\pi_k P_k''(v_0)ww + \pi_k Q_{j(1)}'(v_0, 0)w + \pi_k Q_{(2)}'(v_0, 0)$$

and

$$\lim_{\varepsilon \to 0}(I - \pi_k)\Psi_{k,j}(w, \varepsilon) = (I - \pi_k)P_k'(v_0)w + (I - \pi_k)Q_j(v_0, 0).$$

Therefore, defining

$$\Psi_{k,j}(w,0) = \lim_{\varepsilon\to 0} \Psi_{k,j}(w,\varepsilon),$$

the equation $\Psi_{k,j}(w,0) = 0$ is equivalent to the system

$$\begin{cases} (I-\pi_k)P_k'(v_0)w + (I-\pi_k)Q_j(v_0,0) = 0 \\ \dfrac{1}{2}\pi_k P_k''(v_0)ww + \pi_k Q_{j(1)}'(v_0,0)w + \pi_k Q_{(2)}'(v_0,0) = 0 \end{cases}$$

which has the solution

$$w_k^j = x_k^j + y_k^j,$$

where

$$y_k^j = (I-\pi_k)w_k^j = -(P_k'(v_0)|_{(I-\pi_k)\mathbb{R}^n})^{-1}Q_j(v_0,0)$$

and x_k^j is the solution to the equation

$$\pi_k P_k''(v_0)y_k^j x_k^j + \pi_k Q_{j(1)}'(v_0,0)x_k^j = -\pi_k Q_{(2)}'(v_0,0) - \frac{1}{2}\pi_k P_k''(v_0)y_k^j y_k^j - \pi_k Q_{j(1)}'(v_0,0)y_k^j,$$

which is solvable by condition 2.

Furthermore, by condition 3 we have that $v_0 + \delta w_k^j \in D_k \cap \Delta_j$ for sufficiently small $\delta > 0$. Evaluate now the derivative

$$\begin{aligned} \Psi_{k,j(1)}'(w_k^j,\varepsilon) =& (I-\pi_k)P_k'(v_0+\varepsilon w_k^j) + \varepsilon(I-\pi_k)Q_{j(1)}'(v_0+\varepsilon w_k^j,\varepsilon) + \frac{1}{\varepsilon}\pi_k P_k'(v_0) + \\ &+ \pi_k P_k''(v_0+\tilde\varepsilon(w_k^j,\varepsilon)w_k^j)\, w_k^j + \pi_k Q_{j(1)}'(v_0+\varepsilon w_k^j,\varepsilon), \end{aligned}$$

where $\tilde\varepsilon(w_k^j,\varepsilon) \in [0,\varepsilon]$. Therefore,

$$\lim_{\varepsilon\to 0}\Psi_{k,j(1)}'(w_k^j,\varepsilon) = (I-\pi_k)P_k'(v_0) + \pi_k P_k''(v_0)w_k^j + \pi_k Q_{j(1)}'(v_0,0).$$

Defining

$$\Psi_{k,j(1)}'(w_k^j,0) = (I-\pi_k)P_k'(v_0) + \pi_k P_k''(v_0)w_k^j + \pi_k Q_{j(1)}'(v_0,0),$$

we have that $\Psi_{k,j(1)}'(w_k^j,0)h = 0$ if and only if

$$\begin{cases} (I-\pi_k)P_k'(v_0)h = 0 \\ \pi_k P_k''(v_0)y_k^j \pi_k h + \pi_k Q_{j(1)}'(v_0,0)\pi_k h = 0. \end{cases}$$

Hence $(I-\pi_k)h = 0$ and by condition 2 the second equation gives $\pi_k h = 0$. Therefore, $\Psi_{k,j(1)}'(w_k^j,0)$ is invertible. □

Proof of Theorem 1. By Lemma 2 there exists $w_k^j \in \mathbb{R}^n$ such that $x_0(\theta_0)+\delta w_k^j \in D_k \cap \Delta_j$ for $\delta > 0$ sufficiently small, $\Psi_{k,j}(w_k^j,0) = 0$ and $\Psi'_{k,j(1)}(w_k^j,0)$ is invertible.. Defining $\Psi_{k,j}(w,\varepsilon)$ on $\mathbb{R}^n \times [-1,0]$ by parity and continuity we can apply the classical Implicit Function Theorem, see e.g. [22], to conclude the existence of a local solution to (1) of the form

$$x(\varepsilon) = x_0(\theta_0) + \varepsilon w_k^j + o(\varepsilon)$$

for sufficiently small $\varepsilon > 0$. □

Finally, we have the following result.

Proposition 2. *Assume condition 1 of Theorem 1. Then condition 2 of Theorem 1 is equivalent to* $M'_{k,j}(\theta_0) \neq 0$.

Proof. By Remark 3 we have that $\pi_k(\theta)$ is well defined for θ sufficiently close to θ_0, hence

$$\pi_k(\theta)Q_j(x_0(\theta),0) = M_{k,j}(\theta)x_0'(\theta),$$

with $\langle x_0'(\theta), z_k(\theta)\rangle = 1$. Since π_k is the Riesz projector and x_0 is twice differentiable, we can differentiate both sides of the previous equality obtaining

$$\begin{aligned}\frac{d}{d\theta}\pi_k(\theta)Q_j(x_0(\theta),0) &= \pi_k'(\theta)Q_j(x_0(\theta),0) + \pi_k(\theta)Q'_{j(1)}(x_0(\theta),0)x_0'(\theta) = \\ &= M'_{k,j}(\theta)x_0'(\theta) + M_{k,j}(\theta)x_0''(\theta).\end{aligned}$$

Put $\theta = \theta_0$ in the previous equality, since $M_{k,j}(\theta_0) = 0$ by condition 1 of Theorem 1 we have

$$\pi_k'(\theta_0)Q_j(x_0(\theta_0),0) + \pi_k(\theta_0)Q'_{j(1)}(x_0(\theta_0),0)x_0'(\theta_0) = M'_{k,j}(\theta_0)x_0'(\theta_0). \tag{9}$$

The Riesz projector $\pi_k(\theta)$ associated to $P_k'(x_0(\theta))$ has the integral representation [17]

$$\pi_k(\theta) = \frac{1}{2\pi i}\int_\gamma (\lambda I - P_k'(x_0(\theta)))^{-1} d\lambda,$$

where γ is a closed Jordan curve containing in its interior, which we denote by $int(\gamma)$, the only simple zero eigenvalue of $P_k'(x_0(\theta))$. Since

$$\pi_k'(\theta) = \frac{1}{2\pi i}\int_\gamma (\lambda I - P_k'(x_0(\theta)))^{-1}P_k''(x_0(\theta))x_0'(\theta)(\lambda I - P_k'(x_0(\theta)))^{-1} d\lambda,$$

and $\pi_k(\theta_0)Q_j(v_0,0) = 0$ where $v_0 := x_0(\theta_0)$, we have

$$\begin{aligned}\pi_k'(\theta_0)Q_j(v_0,0) &= \frac{1}{2\pi i}\int_\gamma (\lambda I - P_k'(v_0))^{-1}P_k''(v_0)x_0'(\theta_0)(\lambda I - P_k'(v_0))^{-1}Q_j(v_0,0)d\lambda = \\ =&\frac{1}{2\pi i}\int_\gamma (\lambda I - P_k'(v_0))^{-1}\pi_k(\theta_0)P_k''(v_0)x_0'(\theta_0)(\lambda I - P_k'(v_0))^{-1}|_{(I-\pi_k(\theta_0))\mathbb{R}^n}Q_j(v_0,0)d\lambda + \\ &+ \frac{1}{2\pi i}\int_\gamma (\lambda I - P_k'(v_0))^{-1}(I-\pi_k(\theta_0))P_k''(v_0)x_0'(\theta_0)(\lambda I - P_k'(v_0))^{-1}|_{(I-\pi_k(\theta_0))\mathbb{R}^n}Q_j(v_0,0)d\lambda.\end{aligned}$$

The second integral is zero, since the integrand is an analytic function of λ in $\overline{int(\gamma)}$. For the first integral we consider for the function $\lambda \mapsto (\lambda I - P_k'(v_0))^{-1}|_{(I-\pi_k(\theta_0))\mathbb{R}^n}$ its Taylor series in $int(\gamma)$, while for the function $\lambda \mapsto (\lambda I - P_k'(v_0))^{-1}|_{\pi_k(\theta_0)\mathbb{R}^n}$ which has the pole $\lambda = 0$ of first order in $int(\gamma)$, we consider its Laurent series. It results that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\gamma (\lambda I - P_k'(v_0))^{-1}\pi_k(\theta_0)P_k''(v_0)x_0'(\theta_0)(\lambda I - P_k'(v_0))^{-1}|_{(I-\pi_k(\theta_0))\mathbb{R}^n}Q_j(v_0,0)d\lambda = \\
&=\frac{1}{2\pi i}\int_\gamma (\lambda I - P_k'(v_0))^{-1}\pi_k(\theta_0)P_k''(v_0)x_0'(\theta_0)[(-P_k'(v_0)|_{(I-\pi_k(\theta_0))\mathbb{R}^n})^{-1}Q_j(v_0,0)+ \\
&\quad + \lambda(-P_k'(v_0)|_{(I-\pi_k(\theta_0))\mathbb{R}^n})^{-2}Q_j(v_0,0)+ ...]d\lambda = \\
&=\frac{1}{2\pi i\,\lambda}\int_\gamma \pi_k(\theta_0)P_k''(v_0)x_0'(\theta_0)y_k^j\,d\lambda = \pi_k(\theta_0)P_k''(v_0)y_k^j x_0'(\theta_0),
\end{aligned}$$

i.e.

$$\pi_k'(\theta_0)Q_j(v_0,0) = \pi_k(\theta_0)P_k''(v_0)y_k^j x_0'(\theta_0). \tag{10}$$

Substituting (10) into (9) we finally obtain

$$\pi_k(\theta_0)P_k''(v_0)y_k^j x_0'(\theta_0) + \pi_k(\theta_0)Q_{j(1)}'(v_0,0)x_0'(\theta_0) = M_{k,j}'(\theta_0)x_0'(\theta_0).$$

Since $x_0'(\theta_0) \neq 0$ we have that $M_{k,j}'(\theta_0) \neq 0$ if and only if the operator

$$\pi_k(\theta_0)P_k''(v_0)y_k^j\pi_k(\theta_0) + \pi_k(\theta_0)Q_{j(1)}'(v_0,0)\pi_k(\theta_0)$$

is invertible and so the proof is completed. □

Remark 4. Condition 1 of Theorem 1 is equivalent to $M_{k,j}(\theta_0) = 0$. Moreover, by Proposition 2, condition 2 is equivalent to $M_{k,j}'(\theta_0) \neq 0$. Hence Theorem 1 can be equivalently formulated in terms of the existence of a simple zero of the Malkin function $M_{k,j}(\theta)$ in order to obtain the same conclusions.

4 Example

In this section we consider a periodically perturbed autonomous dynamical system of the form

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 - \xi_1(\xi_1^2+\xi_2^2-1)\text{sign}(\xi_1^2+\xi_2^2-1) + \varepsilon\sin t \\
\dot{\xi}_2 &= -\xi_1 - \xi_2(\xi_1^2+\xi_2^2-1)\text{sign}(\xi_1^2+\xi_2^2-1) + \varepsilon\cos t.
\end{aligned} \tag{11}$$

We will show that the corresponding bifurcation equation $P(x) + \varepsilon Q(x,\varepsilon)$, where $\varepsilon > 0$ and P, Q are defined as in (3) by means of the Poincaré map $\mathcal{P}_\varepsilon$ associated to (11), satisfies all the assumptions of Theorem 1 with $n = 2$.

For $\varepsilon = 0$ system (11) has the limit cycle

$$x_0^\theta(t) = \begin{pmatrix} \sin(t+\theta) \\ \cos(t+\theta) \end{pmatrix},$$

$t \in [0, 2\pi]$, whenever $\theta \in [0, 2\pi]$, thus $x_0^\theta(0)$ is the parametrized curve $x_0(\theta)$ of the previous sections.
It is easy to see that the linearized system of (11) has the two linearly independent solutions

$$\begin{pmatrix} \cos(t+\theta) \\ -\sin(t+\theta) \end{pmatrix},$$

and

$$\begin{pmatrix} e^{\pm 2t}\sin(t+\theta) \\ e^{\pm 2t}\cos(t+\theta) \end{pmatrix},$$

$t, \theta \in [0, 2\pi]$. Here and in the following we will denote $\mathrm{sign}(\xi_1^2 + \xi_2^2 - 1)$ simply by $\pm$. Moreover,

$$\begin{pmatrix} \cos(t+\theta) \\ -\sin(t+\theta) \end{pmatrix},$$

is the solution of the adjoint system to the linearized system in both cases $\pm$. The translation operator from 0 to 2π for the linearized system is

$$V_\theta^{\pm}(t) = \begin{pmatrix} \cos(t+\theta) & e^{\pm 2t}\sin(t+\theta) \\ -\sin(t+\theta) & e^{\pm 2t}\cos(t+\theta) \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Calculating

$$Q_\pm(x_0^\theta(0), 0) = \lim_{\varepsilon \to 0} \frac{\mathcal{P}_\varepsilon^{\pm}(x_0^\theta(0)) - \mathcal{P}_0^{\pm}(x_0^\theta(0))}{\varepsilon}.$$

we obtain

$$Q_\pm(x_0^\theta(0), 0) = V_\theta^{\pm}(2\pi) \int_0^{2\pi} \left(V_\theta^{\pm}(s)\right)^{-1} \begin{pmatrix} \sin s \\ \cos s \end{pmatrix} ds.$$

Since the eigenvectors of $V_\theta^{\pm}(t)$ are orthogonal, the Malkin function takes the form

$$M_\pm(\theta) = \langle Q_\pm(x_0^\theta(0), 0), \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \rangle.$$

It can be seen that $M_\pm(0) = 0$ and $M'_\pm(0) = 1/2\,(1 \pm 2e^{\pm 4\pi})(e^{\mp 4\pi} - 1) \neq 0$, hence, by Proposition 2, condition 2 of Theorem 1 is satisfied. We now calculate $y_0^\pm, x_0^\pm$ and then we verify condition (6). For this, let

$$\phi_\pm(\xi_1, \xi_2) = \left(\xi_2 \pm \xi_1(\xi_1^2 + \xi_2^2 - 1), -\xi_1 \pm \xi_2(\xi_1^2 + \xi_2^2 - 1)\right),$$

It can be shown that from (4) we get

$$y_0^{\pm} = \begin{pmatrix} 0 \\ \pm 1/2 \end{pmatrix}.$$

Moreover,

$$P''_{\pm}(x_0^0(0))\, e\, e = V_0^{\pm}(2\pi) \int_0^{2\pi} \left(V_0^{\pm}(s)\right)^{-1} \phi''_{\pm}(\sin s, \cos s)\, v_e^{\pm}(s)\, v_e^{\pm}(s)\, ds$$

where $v_e^{\pm}$ is the solution of the Cauchy problem

$$\begin{cases} \dot{v} = \phi'_{\pm}(\sin t, \cos t)\, v \\ v(0) = e, \end{cases}$$

and $e = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Furthermore,

$$\begin{aligned} Q'_{\pm(1)}(x_0^0(0), 0)\, y_0^{\pm} =& V_0^{\pm}(2\pi) \int_0^{2\pi} \left(V_0^{\pm}(s)\right)^{-1} \phi''_{\pm}(\sin s, \cos s) \cdot \\ & \cdot V_0^{\pm}(s) \int_0^{s} \left(V_0^{\pm}(\tau)\right)^{-1} \begin{pmatrix} \sin\tau \\ \cos\tau \end{pmatrix} d\tau \left(V_0^{\pm}(s)\right)^{-1} y_0^{\pm}\, ds, \end{aligned}$$

and

$$\begin{aligned} Q'_{\pm(2)}(x_0^0(0), 0) =& V_0^{\pm}(2\pi) \int_0^{2\pi} \left(V_0^{\pm}(s)\right)^{-1} \phi''_{\pm}(\sin s, \cos s) \cdot \\ & \cdot V_0^{\pm}(s) \int_0^{s} \left(V_0^{\pm}(\tau)\right)^{-1} \begin{pmatrix} \sin\tau \\ \cos\tau \end{pmatrix} d\tau\, V_0^{\pm}(s) \int_0^{s} \left(V_0^{\pm}(\tau)\right)^{-1} \begin{pmatrix} \sin\tau \\ \cos\tau \end{pmatrix} d\tau ds. \end{aligned}$$

We can show that $x_0^+ > 0$ and $x_0^- < 0$, with $x_0^{\pm}$ given by (5). On the other hand the scalar products in (6) is equal to x_0^+ and x_0^- respectively, thus there are two branches of solutions of the form (7) emanating from $x_0^0(0)$.
Finally, observe that if in (11) we substitute $\text{sign}(\xi_1^2 + \xi_2^2 - 1)$ by $-\text{sign}(\xi_1^2 + \xi_2^2 - 1)$ then we have not branches of solutions to (11) of the form (7) for $\theta_0 = 0$.

5 Conclusion

The paper provides sufficient conditions for the existence of branches of solutions of the form (8) to a nonsmooth $\varepsilon-$parametrized bifurcation equation originating from a curve of solutions of the equation for $\varepsilon = 0$. The considered equation is quite general and the obtained existence results, as the example of Section 4 shows, seem to be useful tools for the bifurcation analysis of nonsmooth dynamical systems. It is worth to notice that, contrary to the smooth case, in this case many different branches of solutions, parametrized by $\varepsilon > 0$, can arise from the same point of the curve of solutions at $\varepsilon = 0$. Such a situation was also described in [20].

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