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AN INFINITE DIMENSIONAL BIFURCATION PROBLEM WITH APPLICATION TO A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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Abstract. In this paper we consider an infinite dimensional bifurcation equation depending on a parameter $\varepsilon > 0$. By means of the theory of condensing operators, we prove the existence of a branch of solutions, parametrized by ε, bifurcating from a curve of solutions of the bifurcation equation obtained for $\varepsilon = 0$. We apply this result to a specific problem, namely to the existence of periodic solutions bifurcating from the limit cycle of an autonomous functional differential equation of neutral type when it is periodically perturbed by a nonlinear perturbation term of small amplitude.

1. **Introduction.**

In [5] an alternative approach to study bifurcation from a limit cycle in periodically perturbed autonomous systems was proposed. The approach is based on a suitably defined abstract bifurcation equation in finite dimensional spaces of the form $P(x) + \varepsilon Q(x, \varepsilon) = 0$, $\varepsilon > 0$ small. In this paper we extend the applicability of this approach to infinite dimensional spaces with the aim of treating the same bifurcation problem for functional differential equations of neutral type. Specifically, we consider the bifurcation equation

$$P(x, \varepsilon) + \varepsilon Q(x, \varepsilon) = 0, \tag{1}$$

where $P, Q : E \times [0, 1] \longrightarrow E$ are continuous operators, E is a Banach space and $\varepsilon > 0$ is a small parameter. We assume that $P(x, \varepsilon) = x - F(x, \varepsilon)$ and that F and Q are condensing operators in both the variables with respect to the Hausdorff measure of noncompactness. Precisely, F is condensing of constant $q < 1$ and Q is

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condensing of whatever positive constant. This assumption permits to tackle the difficulty of dealing with the space E of infinite dimension. We also assume that the equation $P(x,0)$ has a smooth curve of solutions $x(\theta)$, $\theta \in [0,1]$, that is

$$P(x(\theta),0) = 0,$$

for any $\theta \in [0,1]$. Furthermore, we assume that P is twice differentiable with respect to x continuously in (x,ε) and Q is differentiable with respect to x continuously in (x,ε).

Another relevant difference with [5] is that we consider here the dependence of P on ε. To this regard, observe that if we suppose $P(x,\varepsilon)$ continuously differentiable with respect to both the variables (x,ε) then, by developing the difference $P(x,\varepsilon) - P(x,0)$ in terms of ε, we can reduce 1 to the equation considered in [5]. However, since we intend to apply the sough-after bifurcation results for 1 to a class of functional differential equations of neutral type of the form

$$x'(t) = \varphi(x(t-\varepsilon), x'(t-\varepsilon)) + \varepsilon\psi(t, x(t-\varepsilon), x'(t-\varepsilon)), \tag{2}$$

with φ, ψ continuous functions of their arguments, we cannot assume the continuous differentibility of $P(x,\varepsilon)$ with respect to the pair (x,ε). In fact, the right hand side of 2 contains the term $x'(t-\varepsilon)$ and no reasonable assumptions on its differentiability with respect to the space variables can guarantee the differentiability with respect to ε of the operators P and Q of the bifurcation equation 1 associated to 2. These operators are defined in 32 of Section 3.

On the other hand, if we assume that φ is Lipschitz of constant $K < 1$ with respect to the second variable uniformly in the first and we consider periodic solutions to 2, then the differentiability of the right hand side of 2 with respect to the space variables implies that such periodic solutions are twice differentiable with respect to time t. Therefore, in the case when x is a periodic solution, $x'(t-\varepsilon)$ is differentiable with respect to ε.

Our abstract bifurcation result: Theorem 2.3 will be applied when $x(\theta)$ is a limit cycle of 2 for $\varepsilon = 0$ and x is a periodic solution to 2, hence the previous considerations permit to assume the continuous differentiability of $P(x(\theta),\varepsilon)$, $P'_{(1)}(x(\theta),\varepsilon)$ and $Q(x(\theta),\varepsilon)$ with respect to ε. Here and in what follows $G'_{(j)}$ will indicate the derivative of G with respect to the j-th variable.

The paper is organized as follows. In Section 2, after some needed preliminaries, we formulate and then we prove a general bifurcation result (Theorem 2.3) for the equation 1. Specifically, Theorem 2.3 provides conditions which permit to apply a classical Implicit Function Theorem to a suitably defined function $\Psi(w,\varepsilon)$ whose zeros coincides with the solutions to 1. Precisely, the conditions of Theorem 2.3 ensure the existence of a simple zero w_0 for the function $\Psi(w,0)$, and so the existence of a branch of solutions to 1 parametrized by $\varepsilon > 0$ small. Moreover, Lemma 2.4 shows that the existence of a simple zero of the Malkin function $M(\theta)$, associated to equation 1, see [7] and [8], ensures the conditions of Theorem 2.3.

In Section 3 Theorem 2.3 is then applied to show the existence of periodic solutions of 2. The most part of the work in this Section consists in converting the problem of finding periodic solutions of 2 into the problem of finding fixed points of a suitably introduced map $\mathcal{F}$. In turn, the fixed points of $\mathcal{F}$ coincides with the solutions of 1. Finally, Theorem 3.1 provides conditions for the existence of a branch of solutions of 1 originating from a point $x(\theta_0)$ of the limit cycle $x(\theta)$ of 2 when $\varepsilon = 0$. Specifically, we assume the existence of a simple zero θ_0 of a function $\mathcal{M}(\theta)$ which is the Malkin function $M(\theta)$ of the abstract bifurcation problem 1 associated to 2. Therefore, Lemma 2.4 permits to apply Theorem 2.3. Our arguments in Section 3 are mainly based on results and methods from the theory of condensing operators, see e.g. [1].

2. **The abstract bifurcation result.** This section is devoted to the formulation and the proof of the abstract bifurcation result for 1, namely Theorem 2.3. For this, we need to precise the assumptions and provide some preliminaries. We start by recalling the following definitions. In the sequel E stands for a Banach space.

Definition 2.1. Let $\Omega \subset E$ be a bounded set, the nonnegative number

$$\chi(\Omega) = \inf\{d > 0\,;\, \Omega \text{ has a finite d-net}\}$$

is called the Hausdorff measure of noncompactness of Ω.

Definition 2.2. Let $U \subset E$ be an open set and $T : U \to E$ a continuous operator. We say that F is condensing with respect to the Hausdorff measure of noncompactness of constant $q \geq 0$ if

$$\chi(T(\Omega)) \leq q\chi(\Omega),$$

for any bounded set $\Omega \subset U$.

We now precise the assumptions on the operators $P, Q : E \times [0,1] \to E$, where $\varepsilon \in [0,1]$ is the bifurcation parameter.

(H1) There exists a curve $\theta \mapsto x(\theta)$, $\theta \in [0,1]$, $x \in C^1([0,1], E)$ satisfying

$$P(x(\theta), 0) = 0 \quad \text{and} \quad x'(\theta) \neq 0,$$

for any $\theta \in [0,1]$.

(H2) The derivatives $P''_{(1,1)}(x,\varepsilon)$ and $Q'_{(1)}(x,\varepsilon)$ exist and are continuous at any $(x,\varepsilon) \in E \times [0,1]$. The operator F defined as

$$P(x,\varepsilon) = x - F(x,\varepsilon)$$

is condensing in $E \times [0,1]$ with constant $q < 1$; the operator Q is condensing in $E \times [0,1]$ for some positive constant L. Moreover, we assume the existence and the continuity at any $(\theta,\varepsilon) \in [0,1]^2$ of the derivatives $P'_{(2)}(x(\theta),\varepsilon)$, $P''_{(1,2)}(x(\theta),\varepsilon)$ and $Q'_{(2)}(x(\theta),\varepsilon)$.

By deriving $P(x(\theta), 0) = 0$ with respect to the parameter $\theta \in [0,1]$ we obtain that $P'_{(1)}(x(\theta),0)\,x'(\theta) = 0$, hence $0 \in \sigma\left(P'_{(1)}(x(\theta),0)\right)$, i.e. 0 belongs to the spectrum of $P'_{(1)}(x(\theta),0)$ or equivalently $1 \in \sigma(F'(x(\theta),0))$. In virtue of ([1], Thm 1.5.9) the derivative of a condensing operator is condensing, hence by ([1], Thm 2.6.11) 1 is an eigenvalue of $F'(x(\theta),0)$ of finite multiplicity. From now on we assume the following:

(H3) The eigenvalue 1 of $F'(x(\theta),0)$ is simple, for all $\theta \in [0,1]$.

Remark 1. Let $z(\theta)$ be the eigenvector of the adjoint operator $\left(P'_{(1)}(x(\theta),0)\right)^*$, corresponding to the eigenvalue zero, satisfying

$$\|z(\theta)\| = 1 \quad \text{and} \quad \langle x'(\theta), z(\theta)\rangle > 0.$$

This eigenvector is also simple, see [4]. Furthermore, since $P'_{(1)}(x(\theta),0)$ is differentiable with respect to θ so $\left(P'_{(1)}(x(\theta),0)\right)^*$ is. Therefore, $z(\theta)$ is differentiable.

Define the Riesz projector $\pi(\theta)$ associated to the operator $P'_{(1)}(x(\theta),0)$ corresponding to the simple eigenvalue 0 by means of the well-known formula, see [3],

$$\pi(\theta)=\frac{1}{2\pi i}\int_{\gamma}\Big(\lambda I-P'_{(1)}(x(\theta),0)\Big)^{-1}d\lambda, \tag{3}$$

where γ is a circumference centered at the origin containing in the closure of its interior only the zero eigenvalue of $P'_{(1)}(x(\theta),0)$.

Remark 2. We can easily check that

$$\pi(\theta)h=\frac{\langle h,z(\theta)\rangle}{\langle x'(\theta),z(\theta)\rangle}x'(\theta)$$

and $\langle x'(\theta),z(\theta)\rangle\neq 0$ for $\theta\in[0,1]$.

Consider now the function

$$\pi(\theta)\Big[P'_{(2)}(x(\theta),0)+Q(x(\theta),0)\Big]. \tag{4}$$

In this abstract setting it plays the rôle of the following classical Malkin function, see [7], [8].

$$M(\theta)=\Big\langle P'_{(2)}(x(\theta),0)+Q(x(\theta),0),z(\theta)\Big\rangle. \tag{5}$$

In fact, θ_0 is a zero of 4 if and only if $M(\theta_0)=0$. To see this it sufficies to observe that 4 can be rewritten as follows

$$\frac{M(\theta)}{\langle x'(\theta),z(\theta)\rangle}x'(\theta),$$

and that by (H1) $x'(\theta)\neq 0$ for any $\theta\in[0,1]$. Furthermore, as it is showed by 20 of Lemma 2.4, if $M(\theta_0)=0$ then $M'(\theta_0)\neq 0$ if and only if

$$\frac{d}{d\theta}\pi(\theta)\Big[P'_{(2)}(x(\theta),0)+Q(x(\theta),0)\Big]\Big|_{\theta=\theta_0}\neq 0.$$

Assume now that $\theta_0\in[0,1]$ is a zero of 4. Let $x(\theta_0)=v_0,\pi(\theta_0)=\pi,x'(\theta_0)=e_0$ and

$$y_0=-\left(P'_{(1)}(v_0,0)\Big|_{(I-\pi)E}\right)^{-1}\Big(P'_{(2)}(v_0,0)+Q(v_0,0)\Big). \tag{6}$$

We are now in the position to state the abstract bifurcation result.

Theorem 2.3. *Assume (H1)-(H3). Moreover assume that*

$$\pi\Big[P''_{(1,2)}(v_0,0)e_0+Q'_{(1)}(v_0,0)e_0+P''_{(1,1)}(v_0,0)y_0e_0\Big]\neq 0. \tag{7}$$

Then 1 has a solution $x(\varepsilon)$, for $\varepsilon>0$ small, of the form

$$x(\varepsilon)=v_0+\varepsilon w_0+o(\varepsilon), \tag{8}$$

where $w_0=x_0+y_0$, $x_0=\alpha e_0$ with α uniquely determined by the equation

$$\pi\Big[P''_{(1,2)}(v_0,0)x_0+Q'_{(1)}(v_0,0)x_0+\pi P''_{(1,1)}(v_0,0)y_0x_0\Big]$$
$$=-\frac{1}{2}\Big[P''_{(1,1)}(v_0,0)y_0y_0+P''_{(2,2)}(v_0,0)+\pi Q'_{(1)}(v_0,0)y_0\Big].$$

Proof. Since P is twice differentiable with respect to x continuously in the pair (x,ε) we have that

$$\|P'_{(1)}(v_1,\varepsilon)-P'_{(1)}(v_2,\varepsilon)\|\leq L\|v_1-v_2\| \tag{9}$$

for some $L>0$, v_1, v_2 belonging to a neighborhood of $\{x(\theta),\theta\in[0,1]\}$ and $\varepsilon\in[0,1]$. Taking into account that $P(v_0,0)=0$ we have

$$\begin{aligned}&P(v,\varepsilon)+\varepsilon Q(v,\varepsilon)=P(v,\varepsilon)-P(v_0,0)+\varepsilon Q(v,\varepsilon)\\&=P'_{(1)}(v_0,\varepsilon)(v-v_0)+P(v_0,\varepsilon)-P(v_0,0)+\varepsilon Q(v,\varepsilon)+\gamma(v,\varepsilon),\end{aligned}$$

where

$$\begin{aligned}\gamma(v,\varepsilon)&=P(v,\varepsilon)-P(v_0,\varepsilon)-P'_{(1)}(v_0,\varepsilon)(v-v_0)\\&=\int_0^1\left[P'_{(1)}(v_0+\tau(v-v_0),\varepsilon)-P'_{(1)}(v_0,\varepsilon)\right](v-v_0)\,d\tau.\end{aligned}$$

From 9 we get

$$\begin{aligned}&\|\gamma(v_1,\varepsilon)-\gamma(v_2,\varepsilon)\|\\=&\|\int_0^1\Big\{\left[P'_{(1)}(v_0+\tau(v_1-v_0),\varepsilon)-P'_{(1)}(v_0,\varepsilon)\right](v_1-v_0)\\&-\left[P'_{(1)}(v_0+\tau(v_2-v_0),\varepsilon)-P'_{(1)}(v_0,\varepsilon)\right](v_2-v_0)\Big\}\,d\tau\|\\&\leq\int_0^1\|P'_{(1)}(v_0+\tau(v_1-v_0),\varepsilon)-P'_{(1)}(v_0+\tau(v_2-v_0),\varepsilon)\|\|v_1-v_0\|d\tau\\&+\int_0^1\|P'_{(1)}(v_0+\tau(v_2-v_0),\varepsilon)-P'_{(1)}(v_0,\varepsilon)\|\|v_2-v_1\|d\tau\\&\leq\int_0^1\tau L\left(\|v_2-v_1\|\|v_1-v_0\|+\|v_2-v_0\|\|v_2-v_1\|\right)d\tau\\&\leq L\max\left(\|v_1-v_0\|,\|v_2-v_0\|\right)\|v_2-v_1\|.\end{aligned}$$

Hence,

$$\|\gamma(v_1,\varepsilon)-\gamma(v_2,\varepsilon)\|\leq L\max\left(\|v_1-v_0\|,\|v_2-v_0\|\right)\|v_2-v_1\|. \tag{10}$$

Since $\gamma(v_0,\varepsilon)=0$ for any $\varepsilon\in[0,1]$, equation 1 is equivalent to

$$\Phi(v,\varepsilon)=0, \tag{11}$$

where $\Phi(v,\varepsilon)=\mathbb{P}(v,\varepsilon)+\varepsilon\mathbb{Q}(v,\varepsilon)+\gamma(v,\varepsilon)$, $\mathbb{P}(v,\varepsilon)=P'_{(1)}(v_0,\varepsilon)(v-v_0)$ and $\mathbb{Q}(v,\varepsilon)=\int_0^1P'_{(2)}(v_0,\tau\varepsilon)\,d\tau+Q(v,\varepsilon)$. Note that

$$\Phi(v_0,0)=\mathbb{P}(v_0,\varepsilon)=0\quad\text{and}\quad\mathbb{P}'_{(1)}(v_0,0)=P'_{(1)}(v_0,0).$$

Let $v=v_0+\varepsilon w$ and observe that for $\varepsilon>0$ equation 11 is equivalent to

$$\Psi(w,\varepsilon)=0, \tag{12}$$

where

$$\Psi(w,\varepsilon)=\frac{1}{\varepsilon}\left(\Phi(v_0+\varepsilon w,\varepsilon)-\pi\Phi(v_0+\varepsilon w,\varepsilon)+\frac{1}{\varepsilon}\pi\Phi(v_0+\varepsilon w,\varepsilon)\right). \tag{13}$$

We can rewrite 13 as follows

$$\Psi(w,\varepsilon)=I_1(w,\varepsilon)+\tilde{I}_1(w,\varepsilon)+I_2(w,\varepsilon)+I_3(w,\varepsilon)+\tilde{I}_3(w,\varepsilon),$$

where

$$\begin{aligned}
I_1(w,\varepsilon) =& \frac{1}{\varepsilon}\left(\mathbb{P}(v_0+\varepsilon w,\varepsilon)+\varepsilon\mathbb{Q}(v_0+\varepsilon w,\varepsilon)\right),\\
\tilde{I}_1(w,\varepsilon) =& \frac{1}{\varepsilon}\gamma(v_0+\varepsilon w,\varepsilon),\\
I_2(w,\varepsilon) =& \frac{1}{\varepsilon}\pi\Phi(v_0+\varepsilon w,\varepsilon),\\
I_3(w,\varepsilon) =& \frac{1}{\varepsilon^2}\pi\left(\mathbb{P}(v_0+\varepsilon w,\varepsilon)+\varepsilon\mathbb{Q}(v_0+\varepsilon w,\varepsilon)\right),\\
\tilde{I}_3(w,\varepsilon) =& \frac{1}{\varepsilon^2}\pi\gamma(v_0+\varepsilon w,\varepsilon).
\end{aligned}$$

By the differentiability of P we have

$$\begin{aligned}
I_1(w,\varepsilon) &= P'_{(1)}(v_0,\varepsilon)w+\int_0^1 P'_{(2)}(v_0,\tau\varepsilon)\,d\tau+Q(v_0+\varepsilon w,\varepsilon)\\
&= P'_{(1)}(v_0,0)w+P'_{(2)}(v_0,0)+Q(v_0,0)+\omega(w,\varepsilon),
\end{aligned}$$

where

$$\omega(w,\varepsilon)\to 0 \quad \text{as} \quad \varepsilon\to 0 \tag{14}$$

uniformly with respect to $w\in B(0,r)$, $r>0$. By 10 we get

$$\|\tilde{I}_1(w,\varepsilon)\|\le L\|\varepsilon w\|\|w\|,$$

hence $\tilde{I}_1(w,\varepsilon)=\omega(w,\varepsilon)$ with ω satisfying 14. Moreover, we have

$$I_2(w,\varepsilon)=-\pi P'_{(1)}(v_0,0)w-\pi P'_{(2)}(v_0,0)-\pi Q(v_0,0)w+\omega(w,\varepsilon).$$

By 4 we obtain

$$I_3(w,\varepsilon)=\frac{1}{\varepsilon}\pi P'_{(1)}(v_0,\varepsilon)w+\pi\int_0^1\tau P_{(2,2)}(v_0,0)\,d\tau+\pi Q'_{(1)}(v_0,0)w+\omega(w,\varepsilon).$$

Since $\pi P'_{(1)}(v_0,0)=0$ we obtain

$$I_3(w,\varepsilon)=\pi P''_{(1,2)}(v_0,0)+\frac{1}{2}\pi P''_{(2,2)}(v_0,0)+\pi Q'_{(1)}(v_0,0)w+\omega(w,\varepsilon).$$

Finally,

$$\begin{aligned}
\tilde{I}_3(w,\varepsilon) =& \frac{1}{\varepsilon}\pi\int_0^1 P'_{(1)}(v_0+\tau\varepsilon w,\varepsilon)-P'_{(1)}(v_0,\varepsilon)w\,d\tau\\
=& \pi\int_0^1\tau P''_{(1,1)}(v_0,\varepsilon)ww\,d\tau+\omega(w,\varepsilon)\\
=& \frac{1}{2}\pi P''_{(1,1)}(v_0,0)ww+\omega(w,\varepsilon).
\end{aligned}$$

Letting

$$\begin{aligned}
\Psi(w,0) = {}& (I-\pi)\left[P'_{(1)}(v_0,0)w+P'_{(2)}(v_0,0)+Q(v_0,0)\right]\\
&+\pi P''_{(1,2)}(v_0,0)w+\pi Q'_{(1)}(v_0,0)w\\
&+\frac{1}{2}\pi P''_{(1,1)}(v_0,0)ww+\frac{1}{2}P''_{(2,2)}(v_0,0),
\end{aligned}$$

it results that Ψ is continuous with respect to $(w,\varepsilon)\in B(0,1)\times[0,1]$.

We now prove the existence of $w_0\in E$ such that

$$\Psi(w_0,0)=0.$$

Namely, the existence of $w_0 \in E$ for which

$$\begin{aligned}(I-\pi)\left[P'_{(1)}(v_0,0)\,w_0+P'_{(2)}(v_0,0)+Q(v_0,0)\right]+\pi\Big[P''_{(1,2)}(v_0,0)\,w_0\\ +Q'_{(1)}(v_0,0)\,w_0+\frac{1}{2}P''_{(1,1)}(v_0,0)\,w_0w_0+\frac{1}{2}P''_{(2,2)}(v_0,0)\Big]=0.\end{aligned} \tag{15}$$

Let $x_0=\pi w_0$ and $y_0=(I-\pi)\,w_0$. Applying $(I-\pi)$ to 15 and taking into account that $P'_{(1)}(v_0,0)|_{(I-\pi)E}$ is invertible we get 6 for y_0. If we apply π to 15, since as it is easy to check $\pi P''_{(1,1)}(v_0,0)\,\pi r\,\pi s=0$ for any $r,s\in E$, we obtain the following equation for x_0,

$$\begin{aligned}&\pi P''_{(1,2)}(v_0,0)\,x_0+\pi Q'_{(1)}(v_0,0)\,x_0+\pi P''_{(1,1)}(v_0,0)\,y_0\,x_0\\ &=-\frac{1}{2}\left[P''_{(1,1)}(v_0,0)\,y_0\,y_0+P''_{(2,2)}(v_0,0)+\pi Q'_{(1)}(v_0,0)\,y_0\right].\end{aligned}$$

Since $x_0=\alpha e_0$ for some $\alpha\in\mathbb{R}$, condition 7 allows to uniquely determine α. In conclusion, w_0 is given by

$$w_0=x_0+y_0.$$

To complete the proof of Theorem 2.3 we must show that w_0 is a simple zero of $\Psi(w,0)$. In fact, the application of the Implicit Function Theorem, see [6], to $\Psi(w,\varepsilon)$ at $(w_0,0)$ ensures, by the equivalence of 1 to 12, the existence of a branch of solution to 1 of the form 8. For this, evaluate $\Psi'_{(1)}(w_0,\varepsilon)\,h$, $h\in B(0,1)$. By our assumptions on P and Q we obtain

$$\begin{aligned}\Psi'_{(1)}(w_0,\varepsilon)\,h=&P'_{(1)}(v_0,\varepsilon)\,h+\varepsilon Q'_{(1)}(v_0+\varepsilon w_0,\varepsilon)\,h+\varepsilon P''_{(1,1)}(v_0,0)\,hw_0\\ &-\pi P'_{(1)}(v_0,\varepsilon)\,h-\pi\varepsilon Q'_{(1)}(v_0+\varepsilon w_0,\varepsilon)\,h-\varepsilon\pi P''_{(1,1)}(v_0,0)\,hw_0\\ &+\frac{1}{\varepsilon}\pi P'_{(1)}(v_0,\varepsilon)\,h+\pi Q'_{(1)}(v_0+\varepsilon w_0,\varepsilon)\,h+\pi P''_{(1,1)}(v_0,0)\,hw_0\\ &+\frac{1}{\varepsilon}\omega(\varepsilon w_0)\,h,\end{aligned}$$

where $\frac{\omega(w)}{\|w\|}\to 0$ as $w\to 0$.

Therefore, we have that $\Psi'_{(1)}(w_0,\varepsilon)\,h$ has a limit when $\varepsilon\to 0$ uniformly with respect to $h\in B(0,1)$, that is

$$\begin{aligned}\lim_{\varepsilon\to 0}\Psi'_{(1)}(w_0,\varepsilon)\,h=&(I-\pi)\,P'_{(1)}(v_0,0)\,h\\ &+\pi P''_{(1,2)}(v_0,0)\,h+\pi Q'_{(1)}(v_0,0)\,h+\pi P''_{(1,1)}(v_0,0)\,hw_0.\end{aligned}$$

It remains to show that the operator

$$(I-\pi)\,P'_{(1)}(v_0,0)+\pi P''_{(1,2)}(v_0,0)+\pi Q'_{(1)}(v_0,0)+\pi P''_{(1,1)}(v_0,0)\,w_0 \tag{16}$$

is invertible. $P'_{(1)}(v_0,0)=I-F'_{(1)}(v_0,0)$, where the operator $F'_{(1)}(v_0,0)$ is condensing with constant $q<1$, and the operator

$$-\pi I+\pi F'_{(1)}(v_0,0)+\pi P''_{(1,2)}(v_0,0)+\pi Q'_{(1)}(v_0,0)+\pi P''_{(1,1)}(v_0,0)\,w_0$$

is compact, since it takes value in $\operatorname{span}(e_0)$, thus by ([1], Thm 2.6.11) the operator given in 16 is invertible if we prove that its kernel is trivial. For this, consider

$$\begin{aligned}&(I-\pi)\,P'_{(1)}(v_0,0)\,h+\pi P''_{(1,2)}(v_0,0)\,h\\ &+\pi Q'_{(1)}(v_0,0)\,h+\pi P''_{(1,1)}(v_0,0)\,w_0h=0.\end{aligned} \tag{17}$$

Applying to 17 the projector $I-\pi$ we obtain

$$(I-\pi)\, P'_{(1)}\,(v_0,0)\, h = 0, \tag{18}$$

hence $(I-\pi)\, h = 0$.

Finally, if we apply to 17 the projector π, taking into account 18 and the fact that $\pi P''_{(1,1)}\,(v_0,0)\,\pi w_0 \pi h = 0$, we get

$$\pi P''_{(1,2)}\pi h + \pi Q'_{(1)}\,(v_0,0)\,\pi h + \pi P''_{(1,1)}\,(v_0,0)\, y_0 \pi h = 0.$$

From 7 we have $\pi h = 0$. In conclusion, we have $h = 0$. The proof of Theorem 2.3 is now completed. □

The following result concerning the Malkin bifurcation function $M(\theta)$ introduced in 5 holds true.

Lemma 2.4. *Let $\theta_0 \in [0,1]$ be such that $M(\theta_0) = 0$ and $M'(\theta_0) \neq 0$ then 7 holds true.*

Proof. Consider

$$\pi(\theta)\left[P'_{(2)}\,(x(\theta),0) + Q\,(x(\theta),0)\right] = \frac{M(\theta)}{\langle x'(\theta), z(\theta)\rangle} x'(\theta). \tag{19}$$

By deriving 19 with respect to θ we obtain

$$\begin{aligned}
&\pi'(\theta)\left[P'_{(2)}\,(x(\theta),0) + Q\,(x(\theta),0)\right] \\
&+ \pi(\theta)\left[P''_{(1,2)}\,(x(\theta),0)\, x'(\theta) + Q'_{(1)}\,(x(\theta),0)\, x'(\theta)\right] \\
=&\frac{1}{\langle x'(\theta), z(\theta)\rangle}\left[M'(\theta)\, x'(\theta) + M(\theta)\, x''(\theta)\right] \\
&+ M(\theta)\, x'(\theta)\, \frac{d}{d\theta}\left(\frac{1}{\langle x'(\theta), z(\theta)\rangle}\right).
\end{aligned}$$

Let $\theta = \theta_0$, $v_0 = x(\theta_0)$ and $e_0 = x'(\theta_0) \neq 0$. Since $M(\theta_0) = 0$ we have that

$$\begin{aligned}
&\pi'(\theta_0)\left[P'_{(2)}\,(v_0,0) + Q\,(v_0,0)\right] + \pi(\theta_0)\left[P''_{(1,2)}\,(v_0,0)\, e_0 + Q'_{(1)}\,(v_0,0)\, e_0\right] \\
=&\frac{1}{\langle e_0, z(\theta_0)\rangle} M'(\theta_0)\, e_0,
\end{aligned} \tag{20}$$

and $\pi(\theta_0)\left[P'_{(2)}\,(v_0,0) + Q\,(v_0,0)\right] = 0$. From the integral representation of the Riesz projector 3 we obtain

$$\pi'(\theta_0)\, y = \frac{1}{2\pi i}\int_\gamma \left(\lambda I - P'_{(1)}\,(v_0,0)\right)^{-1} P''_{(1,1)}\,(v_0,0)\, e_0 \left(\lambda I - P'_{(1)}\,(v_0,0)\right)^{-1} y\, d\lambda.$$

For notational convenience we let

$$\hat{E} = (I-\pi(\theta_0))\, E,\ z_0 = P'_{(2)}\,(v_0,0) + Q\,(v_0,0) \text{ and } \left(\lambda I - P'_{(1)}\,(v_0,0)\right)^{-1} = R(\lambda).$$

Since $z_0 \in \hat{E}$ we have

$$\begin{aligned}\pi'(\theta)\ z_0 =& \frac{1}{2\pi i}\int_\gamma R(\lambda)\, P''_{(1,1)}(v_0,0)\, e_0 R(\lambda)\Big|_{\hat{E}} z_0\, d\lambda \\ =& \frac{1}{2\pi i}\int_\gamma R(\lambda)\, \pi(\theta_0)\, P''_{(1,1)}(v_0,0)\, e_0 R(\lambda)\Big|_{\hat{E}} z_0\, d\lambda \\ &+ \frac{1}{2\pi i}\int_\gamma R(\lambda)\,(I-\pi(\theta_0))\, P''_{(1,1)}(v_0,0)\, e_0 R(\lambda)\Big|_{\hat{E}} z_0\, d\lambda\end{aligned}$$

The second integral is zero, since the integrand is an analytic function of λ in $\overline{\operatorname{int}(\gamma)}$. For the first integral we consider the Taylor series of the function $\lambda \mapsto R(\lambda)\,|_{\hat{E}}$ in $\operatorname{int}(\gamma)$ and the Laurent series for $\lambda \mapsto R(\lambda)\,|_{\pi(\theta_0)E}$ which has a pole $\lambda = 0$ of first order in $\operatorname{int}(\gamma)$. We have

$$\pi'(\theta)\, z_0 = \frac{1}{2\pi i}\int_\gamma \frac{\pi(\theta_0)\, P''_{(1,1)}(v_0,0)\, e_0 y_0}{\lambda} d\lambda = \pi(\theta_0)\, P''_{(1,1)}(v_0,0)\, e_0 y_0,$$

where y_0 is given in 6. In conclusion from 20 we obtain

$$\pi(\theta_0)\left[P''_{(1,1)}(v_0,0)\, e_0 y_0 + P''_{(1,2)}(v_0,0)\, e_0 + Q'_{(1)}(v_0,0)\, e_0\right] = \frac{1}{\langle e_0, z(\theta_0)\rangle} M'(\theta_0)\, e_0.$$

Hence 7 is equivalent to $M'(\theta_0) \neq 0$. □

3. **Application to a class of neutral functional differential equations.** In this Section we will show how the abstract result of Theorem 2.3 can be applied to state the existence of periodic solutions bifurcating from the limit cycle of an autonomous differential equation when it is periodically perturbed by a nonautonomous nonlinear perturbation. The considered perturbation introduces a delay in time both in the state and in its derivative which disappears as the perturbation vanishes. Therefore, the resulting perturbed equation turns out to be a functional differential equation of neutral type. Precisely, we consider the equation of the form

$$x'(t) = \varphi(x(t-\varepsilon), x'(t-\varepsilon)) + \varepsilon\psi(t, x(t-\varepsilon), x'(t-\varepsilon), \varepsilon), \tag{21}$$

where $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ are continuous functions, ψ is $T-$periodic in time and $\varepsilon \in [0,1]$ is the perturbation parameter. We also assume

$$\|\varphi(x, y_1) - \varphi(x, y_2)\| \leq K\|y_1 - y_2\|, \tag{22}$$

for some $0 < K < 1$, whenever $x \in E$. Moreover,

$$\|\psi(t, x, y_1, \varepsilon) - \psi(t, x, y_2, \varepsilon)\| \leq L\|y_1 - y_2\|, \tag{23}$$

for some $L > 0$ uniformly with respect to the other variables. Let $\varepsilon_0 \in [0,1]$ be such that

$$K + \varepsilon_0 L = q < 1.$$

Consider the equation 21 for $\varepsilon = 0$, namely

$$x'(t) = \varphi(x(t), x'(t)). \tag{24}$$

By using 22 it follows that 24 is equivalent to the autonomous ordinary differential equation

$$x'(t) = g(x(t)). \tag{25}$$

We assume that 25 has a $T-$periodic limit cycle x_0 and that ψ is $T-$periodic with respect to the first variable. We also assume that $\varphi \in C^2(U)$, where U is a neighborhood of the set

$$\{(x_0(t), x_0'(t)); t \in [0,T]\},$$

and $\psi \in C^1(V)$, where V is a neighborhood of the set

$$\{(t, x_0(t), x_0'(t), \varepsilon); t \in [0,T], \varepsilon \in [0, \varepsilon_0]\}.$$

Under these assumptions the classical Implicit Function Theorem ensures that the function g in 25 is of class C^2 in a neighborhood of the set

$$\{x_0(t); t \in [0,T]\}.$$

Since 25 is an autonomous equation the function $x_\theta(t) = x_0(t+\theta)$ is also a solution of 25 for any $\theta \in [0,T]$. We suppose that the linearized equation

$$y'(t) = g'(x_\theta(t))\, y(t) \tag{26}$$

has the unique linearly independent $T-$periodic solution $x_\theta'(t)$ and that equation 26 does not have Floquet adjoint solution to x_θ' for any $\theta \in [0,T]$. This means that there is no solution to 26 of the form

$$y(t) = v(t) + \frac{t}{T} x_\theta'(t),$$

where v is a $T-$periodic function. In other words, by the Floquet's Theorem, see e.g. [1], [2], 1 is a simple eigenvalue of the translation operator from 0 to T along the trajectories of the equation 26.

Observe that $x_\theta \in C^2$ with respect to θ, since $g \in C^2$. Let $a_\theta(t) = \varphi_{(1)}'(x_\theta(t), x_\theta'(t))$ and $b_\theta(t) = \varphi_{(2)}'(x_\theta(t), x_\theta'(t))$ then 26 can be rewritten in the form

$$y'(t) = (I - b_\theta(t))^{-1} a_\theta(t)\, y(t)$$

in fact by 22 we have $\|b_\theta(t)\| \le K < 1$, for any $t \in [0,T]$ and $\theta \in [0,1]$. Thus, $I - b_\theta(t)$ is invertible. Consider the adjoint equation

$$z'(t) = -a_\theta^*(t)\,(I - b_\theta^*(t))^{-1} z(t), \tag{27}$$

with $\theta \in [0,T]$. From ([1], §4.7.2) it follows that 27 has a unique linearly independent $T-$periodic solution z_θ and does not have Floquet adjoint solution to z_θ for any $\theta \in [0,T]$.

Finally, consider the function

$$\begin{aligned}\mathcal{M}(\theta) = \int_0^T &\langle -x_0''(t+\theta) + \psi(t, x_0(t+\theta), x_0'(t+\theta), 0), \\ &(I - b_\theta^*(t))^{-1} z_0(t+\theta)\rangle dt.\end{aligned} \tag{28}$$

Under all the previous conditions on the equations 21, 25 and 26. We can state the following bifurcation result.

Theorem 3.1. *Assume that $\mathcal{M}(\theta_0) = 0$ and $\mathcal{M}'(\theta_0) \neq 0$ for some $\theta_0 \in [0,T]$. Then, for sufficiently small $\varepsilon > 0$, equation 21 has a unique $T-$periodic solution $x(\varepsilon)$ satisfying*

$$x(\varepsilon) = x_0(\theta_0) + \varepsilon w_0 + o(\varepsilon). \tag{29}$$

Remark 3. The proof of Theorem 3.1 will consist in showing that the function $\mathcal{M}(\theta)$ given in 28 coincides with the Malkin bifurcation function $M(\theta)$ defined in 5. This will be done by introducing for 21 suitable operators P and Q. Once this is done the vector w_0 in 29 of Theorem 3.1 is determined by Theorem 2.3.

In order to prove Theorem 3.1 we need to recall some basic results from the theory of condensing operators, for which we refer to [1].

Result 1. Let $A : E \to E$ be a bounded linear condensing operator of constant q. Then the points of its spectrum outside the disc centered at the origin of radius q are isolated eigenvalues of finite multiplicity.

Result 2. Let $T : U \to E$ be a continuous condensing operator of constant q, where $U \subset E$ is an open set. If T is differentiable at $x_0 \in U$ then $T'(x_0)$ is a linear condensing operator of constant q.

We now introduce an operator $\mathcal{F}$ whose fixed points are $T-$periodic solution of 21. For the details we refer to ([1], p. 185). Let $(\lambda, u) \in \mathbb{R}^n \times C_T$, where $C_T = C([0,T], E)$ is the Banach space of $T-$periodic continuous functions $x : [0,T] \to E$ equipped with the sup-norm. Define the map $f : \mathbb{R}^n \times C_T \to C_T$ as follows

$$f(\lambda, u)(t) = \lambda + \int_0^t u(s)\, ds - t\, m(u),$$

where $m(u) = \frac{1}{T}\int_0^T u(s)\, ds$.

Let $\mathcal{F} : \mathbb{R}^n \times C_T \times [0,1] \to \mathbb{R}^n \times C_T$ be defined as follows

$$\begin{aligned} \mathcal{F}(\lambda, u, \varepsilon)(t) = {} & (\lambda - m(u), \varphi(f(\lambda, u)(t-\varepsilon), u(t-\varepsilon)) \\ & + \varepsilon\psi(t, f(\lambda, u)(t-\varepsilon), u(t-\varepsilon), \varepsilon) + r_\theta(t)\, m(u)\,, \end{aligned} \tag{30}$$

where $r_\theta : [0,T] \to \mathcal{L}(\mathbb{R}^n)$ will be defined in the sequel by means of 31. Here $\mathcal{L}(\mathbb{R}^n)$ denotes the vector space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$. As shown in [1], for given $\varepsilon > 0$, the fixed points (λ, u) of $\mathcal{F}$ are the $T-$periodic solutions x of 21 by setting

$$x(t) = \lambda + \int_0^t u(s)\, ds, \quad (\lambda, u) = \mathcal{F}(\lambda, u, \varepsilon).$$

We have the following result, see ([1], p. 187).

Result 3. Let φ, ψ be continuous functions satisfying 22 and 23. Then $\mathcal{F}$ is condensing with constant $q = K + \varepsilon L$.

Let ξ_i be a coordinate of $x'_\theta(0)$ different from zero, we define $r_\theta(t)$ as the $n \times n$ matrix whose i-th column is given by

$$\xi_i^{-1}[I - b_\theta(t)]\left[x'_\theta(0) - \frac{1}{T}x'_\theta(t)\right] - \xi_i^{-1}x''_\theta(t) \tag{31}$$

and the others $(n-1)$ columns are zero.

Due to the additive term $r_\theta(t)\, m(u)$ in the definition of $\mathcal{F}$, and by using the condition of defect of the Floquet adjoint solution to x'_θ, the following result holds true ([1], p. 188).

Result 4. The eigenvalue 1 of the operator $\mathcal{F}'_{(1)}(x_0(0), x'_0, 0)$ is simple.

We are now in the position to prove Theorem 3.1.

Proof. Define the operators $P, Q : \mathbb{R}^n \times C_T \times [0,1] \to \mathbb{R}^n \times C_T$ as follows

$$\begin{aligned} P(\lambda, u, \varepsilon) &= \begin{pmatrix} m(u) \\ \varphi\left(f(\lambda, u)(\cdot - \varepsilon), u(\cdot - \varepsilon)\right) - r_\theta(\cdot)\, m(u) - u(\cdot) \end{pmatrix}, \\ Q(\lambda, u, \varepsilon) &= \begin{pmatrix} 0 \\ \psi\left(f(\lambda, u)(\cdot - \varepsilon), u(\cdot - \varepsilon), \varepsilon\right) \end{pmatrix}. \end{aligned} \tag{32}$$

It is clear that the fixed points of the map $\mathcal{F}$ given in 30 coincide with the zeros of the equation

$$P(\lambda, u, \varepsilon) + \varepsilon Q(\lambda, u, \varepsilon).$$

In virtue of Lemma 2.4, the proof will consist in showing that the function $\mathcal{M}(\theta)$ given in 28 coincides with the function $M(\theta)$ given in 5.

In what follows, $(\lambda, u) \in \mathbb{R}^n \times C_T$ is considered as the first variable of the operators P, Q and $\varepsilon > 0$ as the second one. Evaluate

$$\begin{aligned} & P'_{(1)}\left(x_\theta(0), x'_\theta, 0\right)(\mu, v) \\ &= \begin{pmatrix} m(v) \\ a_\theta(\cdot)\, f(\mu, v)(\cdot) + b_\theta(\cdot)\, v(\cdot) - r_\theta(\cdot)\, m(v) - v(\cdot) \end{pmatrix}, \end{aligned} \tag{33}$$

and

$$\begin{aligned} & P'_{(2)}\left(x_\theta(0), x'_\theta, 0\right) \\ &= \begin{pmatrix} 0 \\ -a_\theta(\cdot)\, x'_0(\cdot + \theta) - b_\theta(\cdot)\, x''_0(\cdot + \theta) \end{pmatrix} = \begin{pmatrix} 0 \\ -x''_0(\cdot + \theta) \end{pmatrix}, \end{aligned}$$

recall that $x_\theta(t) = x_0(t + \theta)$, $x'_\theta(t) = x'_0(t + \theta)$, and a_θ, b_θ are given by

$$a_\theta(t) = \varphi'_{(1)}\left(x_0(t + \theta), x'_0(t + \theta)\right), \quad b_\theta(t) = \varphi'_{(2)}\left(x_0(t + \theta), x'_0(t + \theta)\right).$$

We now calculate the eigenvector of the adjoint operator

$$\left(P'_{(1)}\left(x_\theta(0), x'_\theta, 0\right)\right)^*$$

corresponding to the zero eigenvalue. For this, consider the Hilbert space L^2_T of the $T-$periodic functions $v : [0, T] \to \mathbb{R}^n$ equipped with the norm

$$\|v\|^2_{L^2_T} = \frac{1}{T}\int_0^T \|v(t)\|^2 dt,$$

with the scalar product in $\mathbb{R}^n \times L^2_T$ given by

$$\langle (\lambda, u), (\mu, v) \rangle_{\mathbb{R}^n \times L^2_T} = \langle \lambda, \mu \rangle + \frac{1}{T}\int_0^T \langle u(t), v(t) \rangle\, dt.$$

Thus, by 33 we have

$$\begin{aligned}
&\left\langle P'_{(1)}\left(x_\theta\left(0\right), x'_\theta, 0\right)\left(\mu, v\right), \left(\lambda, u\right)\right\rangle \\
&= \left\langle m\left(v\right), \lambda\right\rangle + \frac{1}{T}\int_0^T \left\langle a_\theta\left(t\right) f\left(\mu, v\right)\left(t\right) + b_\theta(t)v(t) - r_\theta\left(t\right) m\left(v\right) - v\left(t\right), u\left(t\right)\right\rangle dt \\
&= \frac{1}{T}\int_0^T \left\langle m\left(v\right), \lambda\right\rangle dt + \frac{1}{T}\int_0^T \left\langle a_\theta(t)\mu, u\left(t\right)\right\rangle dt \\
&\quad + \frac{1}{T}\int_0^T \left\langle a_\theta\left(t\right)\left(\int_0^t v\left(s\right) ds - tm\left(u\right)\right), u\left(t\right)\right\rangle dt \\
&\quad + \frac{1}{T}\int_0^T \left\langle b_\theta(t)v\left(t\right), u\left(t\right)\right\rangle dt - \frac{1}{T}\int_0^T \left\langle r_\theta\left(t\right) m\left(v\right), u\left(t\right)\right\rangle dt - \frac{1}{T}\int_0^T \left\langle v\left(t\right), u\left(t\right)\right\rangle dt \\
&= \frac{1}{T}\left\langle \int_0^T v\left(t\right) dt, \lambda\right\rangle + \frac{1}{T}\int_0^T \left\langle \mu, a_\theta^*\left(t\right) u\left(t\right)\right\rangle dt \\
&\quad + \frac{1}{T}\int_0^T \left\langle \int_0^t v\left(s\right) ds - tm\left(v\right), a_\theta^*\left(t\right) u\left(t\right)\right\rangle dt + \frac{1}{T}\int_0^T \left\langle v\left(t\right), b_\theta^*\left(t\right) u\left(t\right)\right\rangle dt \\
&\quad - \frac{1}{T}\int_0^T \left\langle m\left(v\right), r_\theta^*\left(t\right) u\left(t\right)\right\rangle dt - \frac{1}{T}\int_0^T \left\langle v\left(t\right), u\left(t\right)\right\rangle dt.
\end{aligned}$$

Using integration by parts and recalling that $m\left(v\right) = \dfrac{1}{T}\int_0^T v\left(t\right) dt$ we have that

$$\begin{aligned}
&\frac{1}{T}\int_0^T \left\langle \int_0^t v\left(s\right) ds - tm\left(v\right), a_\theta^*\left(t\right) u\left(t\right)\right\rangle dt \\
&= \frac{1}{T}\int_0^T \left\langle v\left(t\right), \int_t^T a_\theta^*\left(s\right) u\left(s\right) ds\right\rangle dt - \frac{1}{T}\int_0^T \left\langle v\left(t\right), \frac{1}{T}\int_0^T s\, a_\theta^*\left(s\right) u\left(s\right) ds\right\rangle dt.
\end{aligned}$$

Moreover,

$$\left\langle m\left(v\right), \frac{1}{T}\int_0^T r_\theta^*\left(t\right) u\left(t\right) dt\right\rangle = \frac{1}{T}\int_0^T \left\langle v\left(t\right), \frac{1}{T}\int_0^T r_\theta^*\left(s\right) u\left(s\right) ds\right\rangle dt.$$

Therefore, we obtain

$$\begin{aligned}
&\left\langle P'_{(1)}\left(x_\theta\left(0\right), x'_\theta, 0\right)\left(\mu, v\right), \left(\lambda, u\right)\right\rangle \\
&= \left\langle \mu, \frac{1}{T}\int_0^T a_\theta^*\left(t\right) u\left(t\right) dt\right\rangle + \frac{1}{T}\int_0^T \Big\langle v\left(t\right), \lambda + \int_t^T a_\theta^*\left(s\right) u\left(s\right) ds \\
&\quad + b_\theta^*\left(t\right) u\left(t\right) - \frac{1}{T}\int_0^T r_\theta^*\left(t\right) u\left(t\right) dt - \frac{1}{T}\int_0^T s\, a_\theta^*\left(s\right) u\left(s\right) ds - u\left(t\right)\Big\rangle dt.
\end{aligned}$$

Hence,

$$\left(P'_{(1)}\left(x_\theta\left(0\right), x'_\theta, 0\right)\right)^*\left(\lambda, u\right) = \begin{pmatrix} \frac{1}{T}\int_0^T a_\theta^*\left(t\right) u\left(t\right) dt \\ p_\theta\left(\lambda, u\right) \end{pmatrix},$$

where,

$$
\begin{aligned}
p_\theta\left(\lambda,u\right)(t) = &\lambda+\int_t^T a_\theta^*\left(s\right)u\left(s\right)ds+b_\theta^*\left(t\right)u\left(t\right)-\frac{1}{T}\int_0^T r_\theta^*\left(t\right)u\left(t\right)dt\\
&-\frac{1}{T}\int_0^T s\,a_\theta^*\left(s\right)u\left(s\right)ds-u\left(t\right).
\end{aligned}
$$

Now we look for $(\lambda,u)\neq(0,0)$ such that

$$
\left(P'_{(1)}\left(x_\theta\left(0\right),x'_\theta,0\right)\right)^*\left(\lambda,u\right)=0.
$$

That is

$$
\frac{1}{T}\int_0^T a_\theta^*\left(t\right)u\left(t\right)dt=0,
$$

and

$$
p_\theta\left(\lambda,u\right)(t)=0 \text{ for any } t\in[0,T].
$$

To this end, let

$$
u(t)=\left(I-b_\theta^*\left(t\right)\right)^{-1}z_\theta(t),
$$

where $z_\theta(t)$ is the $T-$periodic solution of 27. Therefore

$$
z'_\theta(t)=-a_\theta^*(t)u(t)
$$

and

$$
\frac{1}{T}\int_0^T a_\theta^*\left(t\right)u\left(t\right)dt=0.
$$

On the other hand

$$
z_\theta\left(t\right)=z_0(0)+\int_t^T a_\theta^*\left(s\right)\left(I-b_\theta^*\left(s\right)\right)^{-1}z_\theta\left(s\right)ds,
$$

and so if we let

$$
\begin{aligned}
\lambda = &z_0(0)-\frac{1}{T}\int_0^T\int_0^t a_\theta^*\left(s\right)\left(I-b_\theta^*\left(s\right)\right)^{-1}z_\theta\left(s\right)dsdt\\
&+\frac{1}{T}\int_0^T r_\theta^*\left(s\right)\left(I-b_\theta^*\left(s\right)\right)^{-1}z_\theta\left(s\right)ds,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
z_\theta\left(t\right) = &\lambda+\int_t^T a_\theta^*\left(s\right)\left(I-b_\theta^*\left(s\right)\right)^{-1}z_\theta\left(s\right)ds\\
&+\frac{1}{T}\int_0^T\int_0^t a_\theta^*\left(s\right)\left(I-b_\theta^*\left(s\right)\right)^{-1}z_\theta\left(s\right)dsdt\\
&-\frac{1}{T}\int_0^T r_\theta^*\left(s\right)\left(I-b_\theta^*\left(s\right)\right)^{-1}z_\theta\left(s\right)ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u\left(t\right) = &\lambda+\int_t^T a_\theta^*\left(s\right)u\left(s\right)ds+\frac{1}{T}\int_0^T\int_0^t a_\theta^*\left(s\right)u\left(s\right)dsdt\\
&+b_\theta^*\left(t\right)u\left(t\right)-\frac{1}{T}\int_0^T r_\theta^*\left(t\right)u\left(t\right)dt.
\end{aligned}
$$

Since

$$\frac{1}{T}\int_0^T s\, a_\theta^*(s)\, u(s)\, ds = \frac{1}{T}\left[t\int_0^t a_\theta^*(s)\, u(s)\, ds\right]_0^T - \frac{1}{T}\int_0^T\int_0^t a_\theta^*(s)\, u(s)\, dsdt$$
$$= -\frac{1}{T}\int_0^T\int_0^t a_\theta^*(s)\, u(s)\, dsdt,$$

we obtain

$$p_\theta(\lambda, u)(t) = 0 \text{ for any } t \in [0, T].$$

In conclusion

$$(\lambda, u_\theta) \quad \text{where} \quad u_\theta(t) = (I - b_\theta^*(t))^{-1}\, z_\theta(t)$$

is the eigenvector of the adjoint operator

$$\left(P'_{(1)}(x_\theta(0), x'_\theta, 0)\right)^*$$

and so, taking into account 33, $\mathcal{M}(\theta)$ given by 28 coincides with the Malkin bifurcation function $M(\theta)$ given by 5. Therefore, in virtue of Lemma 2.4 and the fact that P, Q satisfy (H1)-(H3), Theorem 2.3 applies to conclude the proof. □

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