An Alternative Approach to Study Bifurcation from a Limit Cycle in Periodically Perturbed Autonomous Systems

(Dedicated to Prof. R. Johnson on the occasion of his 60th birthday)

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Abstract. The goal of this paper is to present a new method to prove bifurcation of a branch of asymptotically stable periodic solutions of a T-periodically perturbed autonomous system from a T-periodic limit cycle of the autonomous unperturbed system. The method is based on a linear scaling of the state variables to convert, under suitable conditions, the singular Poincaré map (with 2 singularity conditions) associated to the perturbed autonomous system into an equivalent non-singular equation to which the classical implicit function theorem applies directly. As a result we obtain the existence of a unique branch of T-periodic solutions (usually found for bifurcations of co-dimension 2) as well as a relevant property of the spectrum of their derivatives. Finally, by a suitable representation formula of the classical Malkin bifurcation function, we show that our conditions are equivalent to the existence of a non-degenerate simple zero of the Malkin function. The novelty of the method is that it permits to solve the problem without explicit reduction of the dimension of the state space as it is usually done in the literature by the Lyapunov-Schmidt method.

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1 Introduction

Our motivation for this paper is the problem of bifurcation of T-periodic solutions in the perturbed autonomous system

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon) \tag{1.1}$$

from a T-periodic cycle x_0 of the unperturbed one (that corresponds to $\varepsilon = 0$). Here $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $g \in C^1(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n)$ is T-periodic. Though this problem has been solved long time ago in papers by Malkin [14] and Loud [11], relevant investigations appear in the literature also our days (see e.g. [17], [5]), partly because this perturbation problem serves as a prototypic example of bifurcation of codimension 2. Indeed, since the unperturbed system is autonomous the cycle x_0 forms a curve of fixed points for the respective Poincaré map $\mathcal{P}_\varepsilon$ (over period T) with $\varepsilon = 0$, thus making the matrix $(\mathcal{P}_0)'(x_0(\theta)) - I$ singular for any $\theta \in [0, T]$. Furthermore, the necessary condition for v_ε to be a fixed point of $\mathcal{P}_\varepsilon$ for $\varepsilon > 0$ is that the limit: $\lim_{\varepsilon \to 0} \dfrac{\mathcal{P}_\varepsilon(v_\varepsilon) - \mathcal{P}_0(v_\varepsilon)}{\varepsilon}$ is orthogonal to the singular eigenspace of $(\mathcal{P}_0^*)'(x_0(\theta_0)) - I$ where $v_\varepsilon \to x_0(\theta_0)$ as $\varepsilon \to 0$ and $\mathcal{P}_0^*$ is the Poincaré map associated to the system adjoint to $\dot{y} = f'(x_0(t))y$. It is known since [14], [11] that these two necessary conditions are generically sufficient. This paper provides a method and prove a result of this kind by means of a straightforward application of the implicit function theorem to an auxiliary scaled map bypassing the traditional step of dimension reduction (known as Lyapunov-Schmidt reduction), see ([3], Ch. 2, § 4).

For this, In Section 2 the problem of finding fixed points of the Poincaré map associated to (1.1) is reduced to the problem of the existence of zeros for the equation of the form

$$\Phi(v, \varepsilon) := P(v) + \varepsilon Q(v, \varepsilon) = 0 \tag{1.2}$$

emanating from a zero v_0 of P under the assumption that $P'(v_0)$ is singular and $\Pi Q(v_0, 0) = 0$, where Π is an orthogonal projector on the eigenspace of $P'(v_0)$ corresponding to the zero eigenvalue. Due to the singularity of $P'(v_0)$ it is not possible to employ directly to (1.2) the classical implicit function theorem to show the existence and uniqueness of a branch $\{v_\varepsilon\}$, $\varepsilon > 0$ small, of solutions of the equation $\Phi(v, \varepsilon) = 0$ (typical for codimension 2 bifurcations, see [9]). By means of a linear scaling of the variables $v \in \mathbb{R}^n$ we convert the problem of finding zeros of (1.2) to the problem of finding zeros of a map $\Psi(w, \varepsilon)$ for which there exists a unique $w_0 \in \mathbb{R}^n$ such that $\Psi(w_0, 0) = 0$ and $\Psi'_w(w_0, 0)$ is not singular. Therefore, the new equation $\Psi(w, \varepsilon) = 0$ can be solved by means of the classical implicit function theorem to conclude the existence and uniqueness of a branch of zeros $\{w_\varepsilon\}$, for $\varepsilon > 0$ small.

Our bifurcation equation $\Psi(w, \varepsilon) = 0$ is, therefore, formally different from that given by Lyapunov-Schmidt reduction (see e.g. [4]). That is why we show in Section 4 that applying our scaling based approach to the perturbed autonomous system (1.1) leads to the same classical Malkin-Loud (or sometimes called Melnikov) bifurcation function. We end up, therefore, with the statement that a well known classical result on the existence, uniqueness and asymptotic stability of a family of T-periodic solution of (1.1) bifurcating from the T-periodic limit cycle x_0 of the autonomous system $\dot{x} = f(x)$ (see Malkin [14], Loud [11], Blekhman [1]) follows from our bifurcation theorem, while avoiding the Lyapunov-Schmidt reduction reduces the analysis significantly.

A first result in this direction has been obtained by the authors in [7] by means of a version of the implicit function theorem for directionally continuous functions, see [2]. The idea of using the linear scaling has been, therefore, already reported at the conference [7]. But the approach in [7] is based on the employ of isochronous surfaces of the Poincaré map transversally intersecting the limit cycle x_0 that requires a non-trivial information about smoothness of these surfaces, while the considerations in this paper rely on very basic facts of analysis only. Long after submitting this paper we got aware of a similar approach employed in [15] for finding certain solutions of reaction-diffusion equations.

The paper is organized as follows. In Section 2 we derive the conditions for the Poincaré map $\mathcal{P}_\varepsilon$ of the perturbed system (1.1) that are necessary for bifurcation of periodic solutions from a cycle of the unperturbed system. In Section 3 we focus our analysis on general maps of form (1.2) that satisfy the two necessary conditions coming from Section 2. We first reduce the abstract equation (1.2) to an equivalent equation for which in Theorem 1 we provide conditions under which it satisfies the assumptions of the classical implicit function theorem. Furthermore, in Theorem 2 we establish a relevant property of the spectrum of the derivative of this equation along the branch which permits to study the asymptotic stability of the bifurcating zeros. In Section 4, under the standard assumption that the Malkin's bifurcation function associated to (1.1) has non-degenerate zeros, the results stated in Section 3 permit to show (Theorem 3) the existence of a parametrized family of T-periodic solutions of (1.1) bifurcating from the T-periodic limit cycle of the unperturbed system as well as their asymptotic stability. The main tools to prove Theorem 3 consist in a representation formula for the Malkin's bifurcation function in terms of the T-periodic perturbation of the autonomous system and of a formula for its derivative. These formulas are stated in Lemma 2 and Lemma 3 respectively.

2 Typical properties of the Poincaré map of a periodically perturbed autonomous system

The system under consideration is the following

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon). \tag{2.1}$$

where $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, $g \in C^1(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n)$ is T-periodic and $\varepsilon > 0$ is the bifurcation parameter. We assume that the unique solution of any Cauchy problem associated to (2.1) is defined on $[0, T]$.

Let z_0 be a T-periodic function of the adjoint system

$$\dot{z} = -(f'(x_0(t)))^* z$$

of the linearized system

$$\dot{y} = f'(x_0(t))y.$$

where x_0 is a T-periodic limit cycle of the autonomous unperturbed system $\dot{x} = f(x)$. By Perron's Lemma [16] we have that

$$\langle \dot{x}_0(t), z_0(t) \rangle = \langle \dot{x}_0(0), z_0(0) \rangle$$

for any $t \in [0, T]$. Without loss of generality we may assume that $\langle \dot{x}_0(0), z_0(0) \rangle = 1$, see ([13], Lemma 1).

Let $\theta_0 \in [0, T]$, we define the projector $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$\Pi\xi = \dot{x}_0(\theta_0) \langle \xi, z_0(\theta_0) \rangle .$$

Finally, we convert the problem of finding T-periodic solutions to (2.1) into the fixed point problem for the associated Poincaré map $\mathcal{P}_\varepsilon$ as illustrated in the following. We consider the function $x : [0, T] \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ given by

$$x(t, v, \varepsilon) = x(t)$$

for all $t \in [0, T]$, where $x(t)$ is the solution of system (2.1) such that $x(0) = v$. The Poincaré map for system (2.1) is defined by

$$\mathcal{P}_\varepsilon(v) = x(T, v, \varepsilon).$$

The functions P and Q of (1.2) are defined as $P(v) = \mathcal{P}_0(v) - v$, $Q(v, \varepsilon) = \dfrac{\mathcal{P}_\varepsilon(v) - \mathcal{P}_0(v)}{\varepsilon}$ that leads to

$$\mathcal{P}_\varepsilon(v) - v = P(v) + \varepsilon Q(v, \varepsilon).$$

Observe that, since $P(x_0(\theta)) = 0$ for any θ, we have that $P'(x_0(\theta))\,\dot{x}_0(\theta) = 0$ and so

$$(\mathcal{P}_0)'(x_0(\theta)) - I = P'(x_0(\theta))$$

is a singular $n \times n$ matrix for any $\theta \in [0, T]$. Furthermore, if $\{v_k\}_{k=1}^{\infty}$ is a convergent sequence of fixed points of $\mathcal{P}_{\varepsilon_k}$ such that $\lim_{k\to\infty} v_k = x_0(\theta_0) =: v_0$ and $\lim_{k\to\infty} \varepsilon_k = 0$, then (see [14] or [13])

$$\langle Q(v_0, 0), z_0(\theta_0)\rangle = 0.$$

where $\langle \cdot, \cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^n$. In conclusion, $\Pi P'(v_0) = 0$ and $\Pi Q(v_0, 0) = 0$ are necessary conditions for (2.1) to have those T-periodic solutions whose initial conditions converge to v_0 as $\varepsilon \to 0$.

3 Variables scaling to transform (1.2) into an equivalent non-singular equation

Consider the function $\Phi : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ defined by

$$\Phi(v, \varepsilon) = P(v) + \varepsilon Q(v, \varepsilon) \tag{3.1}$$

where $P \in C^2(\mathbb{R}^n, \mathbb{R}^n), Q \in C^1(\mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ and $\varepsilon > 0$ is a small parameter.
In this Section, assuming the existence of $v_0 \in \mathbb{R}^n$ such that $P(v_0) = 0$ with $P'(v_0)$ singular, we provide a method to show the existence and the uniqueness of the solution v_ε of the equation

$$\Phi(v, \varepsilon) = 0$$

for $\varepsilon > 0$ sufficiently small, without using the usual Lyapunov-Schmidt reduction approach. To this aim we assume the existence of a linear projector $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathrm{Im}\,\Pi \bigoplus \mathrm{Ker}\,\Pi = \mathbb{R}^n$, $\mathrm{Im}\,\Pi$ and $\mathrm{Ker}\,\Pi$ are invariant subspaces under $P'(v_0)$ and $\Pi P'(v_0) = \Pi Q(v_0, 0) = 0$.
Since $P'(v_0)$ is singular we cannot apply the classical implicit function theorem, see e.g. [10], to study the existence of connected components of zeros of Φ emanating from $(v_0, 0)$. Observe that, in general, as it is shown in [11] and [12], there could exist several branches of zeros of Φ emanating from $(v_0, 0)$. In this paper we provide conditions (which are apparently generic when applying the result to differential equations like (1.1), see Section 4) under which the branch is unique. In particular in Section 4, such conditions are expressed in terms of the Malkin bifurcation function associated to (1.1), see [14]. Precisely, in Section 4 we have $v_0 = x_0(\theta_0)$, where x_0 is a one parameter curve of zeros of P and θ_0 is a non-degenerate simple zero of the Malkin bifurcation function. The approach to achieve this result is commonly based on the classical Lyapunov-Schmidt reduction method. In the infinite dimensional case, see [6] and more recently [8].

In this paper we propose a different approach based on an equivalent formulation of the problem. In fact, by means of a scaling of the variables, we rewrite the problem of finding zeros of $\Phi(v,\varepsilon)$, for $\varepsilon>0$ small, as a problem of finding zeros of an equation to which apply the classical implicit function theorem. Precisely, we associate to the map Φ the following function

$$\Psi(w,\varepsilon)=\frac{1}{\varepsilon}\left(\Phi(v_0+\varepsilon w,\varepsilon)-\Pi\Phi(v_0+\varepsilon w,\varepsilon)+\frac{1}{\varepsilon}\Pi\Phi(v_0+\varepsilon w,\varepsilon)\right), \tag{3.2}$$

for any $w\in\mathbb{R}^n$ and any $\varepsilon>0$, and we look for zeros of Ψ branching from some $(w_0,0)$. Indeed, as it is easy to see, $(v,\varepsilon)\in\mathbb{R}^n\times[0,1]$ is a zero of Φ if and only if $\left(\frac{v-v_0}{\varepsilon},\varepsilon\right)$ is a zero of Ψ.

In the sequel the vector space of linear operators $L:\mathbb{R}^n\to\mathbb{R}^n$ will be denoted by $\mathcal{L}(\mathbb{R}^n)$. Next Lemma provides the main properties of the function Ψ.

Lemma 1 *Assume that $P\in C^2(\mathbb{R}^n,\mathbb{R}^n)$ and $Q\in C^1(\mathbb{R}^n\times[0,1],\mathbb{R}^n)$. Let $v_0\in\mathbb{R}^n$ be such that $P(v_0)=0$ and $P'(v_0)$ is singular. Let $\Pi:\mathbb{R}^n\to\mathbb{R}^n$ be a linear projector invariant with respect to $P'(v_0)$ such that $\Pi P'(v_0)=\Pi Q(v_0,0)=0$. Define $\Psi(w,0)$ as follows*

$$\Psi(w,0)=\frac{1}{2}\Pi P''(v_0)ww+\Pi Q'_v(v_0,0)w+\Pi Q'_\varepsilon(v_0,0)+(I-\Pi)P'(v_0)w+(I-\Pi)Q(v_0,0) \tag{3.3}$$

with

$$\Psi'_w(w,0)=\Pi P''(v_0)w+\Pi Q'_v(v_0,0)+(I-\Pi)P'(v_0). \tag{3.4}$$

Then $\Psi\in C^0(\mathbb{R}^n\times\mathbb{R},\mathbb{R}^n)$ and $\Psi'_w\in C^0(\mathbb{R}^n\times\mathbb{R},\mathcal{L}(\mathbb{R}^n))$.

Proof. From (3.2) the Taylor expansion with the rest in the Lagrange's form leads to

$$\begin{aligned}\Pi\Psi(w,\varepsilon) &= \frac{1}{\varepsilon^2}\Pi\Phi(v_0+\varepsilon w,\varepsilon)=\frac{1}{\varepsilon^2}\Pi(P(v_0+\varepsilon w)+\varepsilon Q(v_0+\varepsilon w,\varepsilon))\\ &= \frac{1}{\varepsilon^2}\Pi\Big(P(v_0)+\varepsilon P'(v_0)w+\frac{1}{2}\varepsilon^2P''\left(v_0+\widehat{\varepsilon}(w,\varepsilon)w\right)ww\ +\varepsilon Q(v_0,0)\\ &\quad+\varepsilon^2Q'_v\left(v_0+\widetilde{\varepsilon}(w,\varepsilon)w,\widetilde{\varepsilon}(w,\varepsilon)\right)w+\varepsilon^2Q'_\varepsilon\left(v_0+\widetilde{\varepsilon}(w,\varepsilon)w,\widetilde{\varepsilon}(w,\varepsilon)\right)\Big)\end{aligned}$$

and

$$\begin{aligned}(I-\Pi)\Psi(w,\varepsilon) &= \frac{1}{\varepsilon}(I-\Pi)(P(v_0+\varepsilon w)+\varepsilon Q(v_0+\varepsilon w,\varepsilon))\\ &= \frac{1}{\varepsilon}(I-\Pi)\left(P(v_0)+\varepsilon P'\left(v_0+\overline{\varepsilon}(w,\varepsilon)w\right)w+\varepsilon Q(v_0+\varepsilon w,\varepsilon)\right),\end{aligned}$$

where $\widehat{\varepsilon}(w,\varepsilon),\widetilde{\varepsilon}(w,\varepsilon),\overline{\varepsilon}(w,\varepsilon)\in[0,\varepsilon]$. Using the fact that $P(v_0)=\Pi P'(v_0)=\Pi Q(v_0,0)=0$ we get

$$\begin{aligned}\Psi(w,\varepsilon) &= \frac{1}{2}\Pi P''\left(v_0+\widehat{\varepsilon}(w,\varepsilon)w\right)ww+\Pi Q'_v\left(v_0+\widetilde{\varepsilon}(w,\varepsilon)w,\widetilde{\varepsilon}(w,\varepsilon)\right)w\\ &\quad+\Pi Q'_\varepsilon\left(v_0+\widetilde{\varepsilon}(w,\varepsilon)w,\widetilde{\varepsilon}(w,\varepsilon)\right)+(I-\Pi)P'\left(v_0+\overline{\varepsilon}(w,\varepsilon)w\right)w+(I-\Pi)Q(v_0+\varepsilon w,\varepsilon).\end{aligned}$$

From this formula we conclude that $\Psi\in C^0(\mathbb{R}^n\times\mathbb{R},\mathbb{R}^n)$.

Let us now prove that $\Psi'_w \in C^0(\mathbb{R}^n \times \mathbb{R}, \mathcal{L}(\mathbb{R}^n))$. The Taylor expansion applied to $P'(v_0 + \varepsilon w)$ permits to write

$$\begin{aligned}
\Pi\Psi'_w(w,\varepsilon) &= \frac{1}{\varepsilon^2}\Pi(\varepsilon P'(v_0+\varepsilon w) + \varepsilon^2 Q'_v(v_0+\varepsilon w,\varepsilon)) \\
&= \frac{1}{\varepsilon^2}\Pi\left(\varepsilon P'(v_0) + \varepsilon^2 P''(v_0 + \widetilde{\varepsilon}(w,\varepsilon)w)w + \varepsilon^2 Q'_v(v_0+\varepsilon w,\varepsilon)\right), \\
(I-\Pi)\Psi'_w(w,\varepsilon) &= \frac{1}{\varepsilon}(I-\Pi)\left(\varepsilon P'(v_0+\varepsilon w) + \varepsilon^2 Q'_v(v_0+\varepsilon w,\varepsilon)\right),
\end{aligned}$$

where $\widetilde{\varepsilon}(w,\varepsilon) \in [0,\varepsilon]$. Taking into account that $\Pi P'(v_0) = 0$ we have

$$\Psi'_w(w,\varepsilon) = \Pi P''(v_0+\widetilde{\varepsilon}(w,\varepsilon)w)w + \Pi Q'_v(v_0+\varepsilon w,\varepsilon) + (I-\Pi)P'(v_0+\varepsilon w) + \varepsilon(I-\Pi)Q'_v(v_0+\varepsilon w,\varepsilon)$$

and so $\Psi'_w(w,\varepsilon) \to \Psi'_w(w_0,0)$ as $w \to w_0$ and $\varepsilon \to 0$. This concludes the proof. □

Remark 1 *An example of linear projector which is invariant with respect to $P'(v_0)$ is the Riesz projector $\Pi_R : \mathbb{R}^n \to \mathbb{R}^n$ given by*

$$\Pi_R := \frac{1}{2\pi i}\int_\Gamma (\lambda I - P'(v_0))^{-1}\, d\lambda,$$

where Γ is a circumference centered at 0 and containing in its interior the only zero eigenvalue of $P'(v_0)$. In fact, by the Riesz decomposition theorem the subspaces $Im\,\Pi_R$ and $Ker\,\Pi_R$ are invariant with respect to $P'(v_0)$, $Im\,\Pi_R \bigoplus Ker\,\Pi_R = \mathbb{R}^n$ and $\Pi_R P'(v_0) = 0$.

We can now prove the following.

Theorem 1 *Assume that $P \in C^2(\mathbb{R}^n,\mathbb{R}^n)$ and $Q \in C^1(\mathbb{R}^n \times [0,1],\mathbb{R}^n)$. Let $v_0 \in \mathbb{R}^n$ be such that $P(v_0) = 0$ and $P'(v_0)$ is singular. Let $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear projector (not necessarily one-dimensional) invariant with respect to $P'(v_0)$ with $P'(v_0)$ invertible on $(I-\Pi)\mathbb{R}^n$. Finally, assume that $\Pi P'(v_0) = \Pi Q(v_0,0) = 0$, $\Pi P''(v_0)\Pi\, r\; \Pi\, s = 0$ for any $r,s \in \mathbb{R}^n$, and that*

$$-\Pi P''(v_0)(I-\Pi)\left(P'(v_0)|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(v_0,0) + \Pi Q'_v(v_0,0) \tag{3.5}$$

is invertible on $\Pi\mathbb{R}^n$. Then there exists a unique $w_0 \in \mathbb{R}^n$ such that $\Psi(w_0,0) = 0$ and $\Psi'_w(w_0,0)$ is non-singular.

Proof. We start by showing the existence of a $w_0 \in \mathbb{R}^n$ such that $\Psi(w_0,0) = 0$. First, observe that applying $(I-\Pi)$ to (3.3) we obtain the map $w \to (I-\Pi)P'(v_0)w + (I-\Pi)Q(v_0,0)$ and the equation

$$(I-\Pi)P'(v_0)w + (I-\Pi)Q(v_0,0) = (I-\Pi)P'(v_0)(I-\Pi)w + (I-\Pi)Q(v_0,0) = 0 \tag{3.6}$$

is solvable with respect to $(I-\Pi)w$; in fact by our assumptions

$$w_1 = -\left(P'(v_0)\big|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(v_0,0).$$

is the solution of (3.6) with $w_1 \in (I-\Pi)\mathbb{R}^n$. Now, we solve the equation

$$\frac{1}{2}\Pi P''(v_0)(\Pi w + w_1)(\Pi w + w_1) + \Pi Q'_v(v_0,0)(\Pi w + w_1) + \Pi Q'_\varepsilon(v_0,0) = 0 \tag{3.7}$$

with respect to Πw. By assumption $\Pi P''(v_0)\Pi\, r\, \Pi\, s = 0$ for any $r, s \in \mathbb{R}^n$, moreover $P''(v_0)ab = P''(v_0)ba$, hence we can rewrite equation (3.7) as follows

$$\Pi P''(v_0)w_1\Pi w + \Pi Q'_v(v_0,0)\Pi w = -\frac{1}{2}\Pi P''(v_0)w_1\, w_1 - \Pi Q'_v(v_0,0))w_1 - \Pi Q'_\varepsilon(v_0,0).$$

Since by assumption the operator $\Pi P''(v_0)w_1 + \Pi Q'(v_0,0)$ is invertible, the last equation has a unique solution w_2 with $w_2 \in \Pi\,\mathbb{R}^n$. Hence $w_0 = w_2 + w_1$ is a zero of $\Psi(w,0)$.
From Lemma 1 we have that Ψ is continuous at $(w_0,0)$, Ψ'_w exists and is continuous at $(w_0,0)$. To apply the classical implicit function theorem it remains to show that $\Psi'_w(w_0,0)$ is non-singular. We argue by contradiction assuming that there exists $h \neq 0$ such that

$$\Psi'_w(w_0,0)h = \Pi P''(v_0)w_0 h + \Pi Q'_v(v_0,0)h + (I-\Pi)P'(v_0)h = 0. \tag{3.8}$$

Applying $(I-\Pi)$ to (3.8) we obtain $(I-\Pi)P'(v_0)h = 0$ that is $(I-\Pi)h = 0$ and so $h = \Pi h$. Therefore,

$$\begin{aligned}\Pi P''(v_0)w_0 h &= \Pi P''(v_0)\Pi w_0 \Pi h + \Pi P''(v_0)(I-\Pi)w_0\Pi h = \\ &= -\Pi P''(v_0)\left(P'(v_0)|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(v_0,0)\Pi h\end{aligned}$$

and applying Π to (3.8) we obtain

$$-\Pi P''(v_0)\left(P'(v_0)\big|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(v_0,0)\Pi h + \Pi Q'_v(v_0,0)\Pi h = 0.$$

This contradicts our assumption and the proof is completed. □

Remark 2 *The conclusions of Theorem 1 permit to apply the classical implicit function theorem to obtain the existence of a $\delta > 0$ such that the equation $\Psi(w,\varepsilon) = 0$ has, for any $\varepsilon \in [0,\delta]$, a unique solution w_ε such that $\|w_0 - w_\varepsilon\| \leq \delta$. Therefore, for $\varepsilon > 0$ small, there exists a family $\{w_\varepsilon\}$ of zeros of the map Ψ such that $w_\varepsilon \to w_0$ as $\varepsilon \to 0$.*
Moreover, under our regularity assumptions $\varepsilon \to \Phi'_v(v_0 + \varepsilon w_\varepsilon, \varepsilon)$ is a continuous map; thus, for any $\varepsilon > 0$ sufficiently small, there exists an eigenvalue λ_ε of $\Phi'_v(v_0 + \varepsilon w_\varepsilon, \varepsilon)$ with the property that $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$.

We are now in the position to formulate the following result.

Theorem 2 *Assume all the conditions of Theorem 1 and that zero is a simple eigenvalue of $P'(v_0)$. Let $v_0 = x_0(\theta_0)$, where $\theta \to x_0(\theta)$ is a C^2-parametrized curve of zeros of the map P. Let $\{w_\varepsilon\}$ and $\{\lambda_\varepsilon\}$ as in Remark 2. Let $\lambda_* \in \mathbb{R}$ be the eigenvalue of the operator $\Pi P''(v_0)w_0|_{\Pi\mathbb{R}^n} + \Pi Q'_v(v_0,0)|_{\Pi\mathbb{R}^n}$. Then*

$$\lambda_\varepsilon = \varepsilon\lambda_* + o(\varepsilon).$$

Proof. Let l_ε be the unitary eigenvector of $\Phi'_v(v_0 + \varepsilon w_0, \varepsilon)$ associated to the eigenvalue λ_ε, namely

$$\Phi'_v(v_0 + \varepsilon w_\varepsilon, \varepsilon)l_\varepsilon = \lambda_\varepsilon l_\varepsilon. \tag{3.9}$$

Clearly,

$$l_\varepsilon \to \frac{\dot{x}_0(\theta_0)}{\|\dot{x}_0(\theta_0)\|} \quad \text{as} \quad \varepsilon \to 0. \tag{3.10}$$

Now we observe that

$$\Psi'_w(w,\varepsilon) = \frac{1}{\varepsilon}\left(\varepsilon\Phi'_v(v_0 + \varepsilon w, \varepsilon) - \varepsilon\Pi\Phi'_v(v_0 + \varepsilon w, \varepsilon) + \Pi\Phi'_v(v_0 + \varepsilon w, \varepsilon)\right)$$

and using (3.9) we get

$$\Pi\Psi'_w(w_\varepsilon, \varepsilon)l_\varepsilon = \frac{1}{\varepsilon}\Pi\Phi'_v(v_0 + \varepsilon w_\varepsilon, \varepsilon)l_\varepsilon = \frac{1}{\varepsilon}\lambda_\varepsilon \Pi l_\varepsilon \tag{3.11}$$

for any $\varepsilon > 0$ sufficiently small. By Lemma 1 as $\varepsilon \to 0$ we have

$$\Pi\Psi'_w(w_\varepsilon, \varepsilon)l_\varepsilon \quad \to \quad \Pi P''(v_0)w_0\frac{\dot{x}_0(\theta_0)}{\|\dot{x}_0(\theta_0)\|} + \Pi Q'_v(v_0,0)\frac{\dot{x}_0(\theta_0)}{\|\dot{x}_0(\theta_0)\|}.$$

From this, by (3.11) we have that $\dfrac{\lambda_\varepsilon}{\varepsilon} \to a \in \mathbb{R}$ as $\varepsilon \to 0$ and

$$\Pi P''(v_0)w_0\dot{x}_0(\theta_0) + \Pi Q'_v(v_0,0)\dot{x}_0(\theta_0) = a\dot{x}_0(\theta_0).$$

Therefore, $a = \lambda_*$, and the proof is completed. □

4 The Malkin bifurcation function

This Section is devoted to show, by means of Theorem 3, that if θ_0 is a non-degenerate simple zero of the Malkin function [14]

$$M(\theta) = \int_0^T \langle g(t, x_0(t+\theta), 0), z_0(t+\theta)\rangle\, dt,$$

then the conditions of Theorem 1 are satisfied and thus Remark 2 and Theorem 2 ensures the existence of a unique branch of asymptotically stable periodic solutions of (1.1) emanating from the family of periodic solutions represented by the limit cycle x_0 of the unperturbed system corresponding to $\varepsilon = 0$.

To this end, with x_0, z_0, Π, P, Q as introduced in Section 2, we have the following two results. The first one provides a representation formula for the Malkin's bifurcation function, the second one a formula for its derivative.

Lemma 2 *For any $\theta \in [0, T]$ the limit $Q(v, 0) := \lim_{\varepsilon \to 0} Q(v, \varepsilon)$ exists and*

$$M(\theta) = \langle Q(x_0(\theta), 0), z_0(\theta) \rangle .$$

Moreover, $Q \in C^1(\mathbb{R}^n \times [0, 1], \mathbb{R}^n)$.

The proof of Lemma 2 can be found in [13]. For the reader convenience we report it in what follows.
Proof. Differentiating with respect to time one can see that the function $y(t) = \dfrac{\partial}{\partial \varepsilon} x(t, x_0(\theta), \varepsilon)$ evaluated at $\varepsilon = 0$ solves, for any $\theta \in [0, T]$, the Cauchy problem

$$\dot{y} = f'(x_0(t + \theta))y + g(t, x_0(t + \theta), 0), \quad y(0) = 0.$$

A direct computation shows that

$$\frac{d}{dt} \langle y(t), z_0(t + \theta) \rangle = \langle g(t, x_0(t + \theta), 0), z_0(t + \theta) \rangle$$

and, integrating over the period, yields

$$M(\theta) = \langle y(T), z_0(\theta) \rangle = \langle Q(x_0(\theta), 0), z_0(\theta) \rangle ,$$

which is the assertion. □

Lemma 3 *For any $\theta \in [0, T]$ we have*

$$M'(\theta) = \left\langle -P''(x_0(\theta))(I - \Pi) \left(P'(x_0(\theta))\big|_{(I-\Pi)\mathbb{R}^n} \right)^{-1} Q(x_0(\theta), 0)\dot{x}_0(\theta) + Q'_v(x_0(\theta), 0)\dot{x}_0(\theta), z_0(\theta) \right\rangle . \tag{4.1}$$

Proof. From Section 2 we have that $\langle \dot{x}_0(\theta), z_0(\theta) \rangle = 1$ for any $\theta \in [0, T]$. As a consequence, by the definition of the projector Π, we get

$$\langle \xi, z_0(\theta) \rangle = \langle \Pi\, \xi, z_0(\theta) \rangle , \tag{4.2}$$

for any $\theta \in [0, T]$. Therefore

$$\left\langle P'(x_0(\theta))h, z_0(\theta) \right\rangle = \left\langle \Pi\, P'(x_0(\theta))(I - \Pi)h, z_0(\theta) \right\rangle = 0,$$

for any $\theta \in [0, T]$ and any $h \in \mathbb{R}^n$. Then, by deriving with respect to θ, we obtain

$$\left\langle P'(x_0(\theta))h, \dot{z}_0(\theta) \right\rangle = \left\langle -P''(x_0(\theta))\dot{x}_0(\theta)h, z_0(\theta) \right\rangle ,$$

for any $\theta \in [0,T]$ and any $h \in \mathbb{R}^n$. Therefore, we can rewrite the left hand side of (4.1) with $(I-\Pi)\left(P'(x_0(\theta))|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(x_0(\theta),0) = h$ as follows

$$\left\langle P'(x_0(\theta))(I-\Pi)\left(P'(x_0(\theta))|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(x_0(\theta),0), \dot{z}_0(\theta)\right\rangle + \left\langle Q'_v(x_0(\theta),0)\dot{x}_0(\theta), z_0(\theta)\right\rangle$$

or equivalently,

$$\langle Q(x_0(\theta),0), \dot{z}_0(\theta)\rangle + \left\langle Q'_v(x_0(\theta),0)\dot{x}_0(\theta), z_0(\theta)\right\rangle,$$

which is the derivative of $M(\theta)$ at any $\theta \in [0,T]$ according to the formula given by Lemma 3. □

Finally, we can prove the following.

Theorem 3 *Assume that there exists $\theta_0 \in [0,T]$ such that $(\mathcal{P}_0)'(x_0(\theta_0))$ has $n-1$ eigenvalues with negative real parts, $M(\theta_0) = 0$ and $M'(\theta_0) < 0$. Then, for $\varepsilon > 0$ sufficiently small, equation (2.1) has a unique T-periodic solution x_ε such that $x_\varepsilon(t) \to x_0(t+\theta_0)$ as $\varepsilon \to 0$ uniformly in $[0,T]$. Moreover the solutions $\{x_\varepsilon\}$ are asymptotically stable.*

Proof. Let $v_0 = x_0(\theta_0)$, from Lemma 2 we have

$$\Pi Q(x_0(v_0),0) = \dot{x}_0(\theta_0)\langle Q(v_0,0), z_0(\theta_0)\rangle = \dot{x}_0(\theta_0)M(\theta_0) = 0.$$

By (4.2) we obtain

$$M'(\theta_0) = \left\langle -\Pi P''(v_0)(I-\Pi)\left(P'(v_0)|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(v_0,0)\dot{x}_0(\theta_0) + \Pi Q'_v(v_0,0)\dot{x}_0(\theta_0), z_0(\theta_0)\right\rangle \neq 0,$$

and so (3.5) is invertible on $\Pi\mathbb{R}^n$. Moreover, from the fact that $P(x_0(\theta)) = 0$ for any $\theta \in [0,T]$, we obtain that

$$P''(v_0)\dot{x}_0(\theta_0)\dot{x}_0(\theta_0) + P'(v_0)x''_0(\theta_0) = 0$$

Since $\Pi P'(v_0)x''_0(\theta_0) = \Pi P'(v_0)\Pi x''_0(\theta_0) = 0$ we have that $\Pi P''(v_0)\,\Pi r\,\Pi s = 0$ for any $r, s \in \mathbb{R}^n$. Therefore, all the conditions of Theorem 1 are satisfied and so, compare Remark 2, equation (2.1) has a unique T-periodic solution x_ε satisfying

$$\left\| w_0 - \frac{x_\varepsilon(0) - v_0}{\varepsilon} \right\| \leq \delta,$$

with $\Psi(w_0, 0) = 0$. Moreover

$$\Pi P''(v_0)w_0\dot{x}_0(\theta_0) + \Pi Q'_v(v_0,0)\dot{x}_0(\theta_0) = \lambda_*\,\dot{x}_0(\theta_0).$$

But

$$\text{sign}\,\lambda_* = \text{sign}\,\left\langle \Pi P''(v_0)w_0\dot{x}_0(\theta_0) + \Pi Q'_v(v_0,0)\dot{x}_0(\theta_0), z_0(\theta_0)\right\rangle = \text{sign}\,M'(\theta_0) = -1$$

Therefore, from Theorem 2 there exists $\lambda_\varepsilon = \varepsilon\lambda_* + o(\varepsilon)$ eigenvalue of $(\mathcal{P}_\varepsilon)'(x_\varepsilon(0)) - I$. This implies that

$$\det\left((\mathcal{P}_\varepsilon)'(x_\varepsilon(0)) - I - \lambda_\varepsilon I\right) = 0.$$

Hence, $\rho_\varepsilon = 1 + \lambda_\varepsilon = 1 + \lambda_*\varepsilon + o(\varepsilon)$ is an eigenvalue of $(\mathcal{P}_\varepsilon)'(x_\varepsilon(0))$ converging to 1 as $\varepsilon \to 0$. Since $\lambda_* < 0$, then $|\rho_\varepsilon| < 1$ for $\varepsilon > 0$ sufficiently small. This ends the proof. □

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