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PERIODIC CONTROL PROBLEMS FOR A CLASS OF NONLINEAR PERIODIC DIFFERENTIAL SYSTEMS*

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INTRODUCTION

IN THE past few years, considerable attention has been paid to periodic optimization problems in dynamical systems described by the nonlinear differential system

$$\dot{x} = f(t, x, u) \qquad \left(\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}x\right) \tag{1}$$

where x is the real state n-vector at time $t \in I \subset \mathbb{R}$ and u is the control parameter m-vector at time t varying in some compact set Ω of $\mathbb{R}^m$ (to u we will refer simply as a control). In some cases f will be independent of t. (See [1] as a survey paper and [2–6].)

The motivation for studying the above system is twofold. First, the performance of many control process, e.g. chemical processes, cf. [7], can be improved with respect to the optimal steady-state by operating the process in a suitable periodic way, i.e. the optimal steady-state operation of a process is not necessarily the optimal periodic operation. Bailey & Horn [8], Guardabassi [9] and other authors (see [1]) have obtained sufficient conditions for optimal periodic operation. A comparision of analytic methods used in this problem is also given in [8]. Secondly, there exist control problems which naturally lead to consider periodic responses to control functions. For instance, control processes involving biomedical devices (see [4]).

If the dynamics is linear, i.e. $f(t, x, u) = A(t)x + B(t)u$, then for different performance costs, sufficient conditions for the existence of the optimal periodic control u_0^* with corresponding optimal periodic response x_0^* have been discussed by two authors: Markus [4] in the case where A and B are independent of t, and Tonkov [10] for the time-dependent case.

For nonlinear control dynamical systems, in [4–6] given a performance cost, one assumes the existence of an optimal control u_0^*. Under the assumptions that the problem in variations admits only the zero solution and of the existence of *a priori*-boundedness of the periodic solutions of (1), necessary conditions (by means of the maximum principle) for optimality and structural properties of u_0^* and x_0^* are derived. The same procedure is employed by Halanay [2] and Bailey [11] to derive necessary conditions for the optimal control of solutions of linear and nonlinear boundary value problems of more general type. In the linear case the maximum principle is shown to be, by direct calculation, a sufficient condition for optimality.

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Clarke [12] reduces a nonlinear optimal control system described by (1) to a differential inclusion problem (without convexity assumption on the multivalued map) for which a 'true' Hamiltonian function is available, i.e. a Hamiltonian function which is a direct descendant of the classical Hamiltonian in the calculus of variation, and gives a necessary condition for optimality in terms of this function. A comparison with the maximum principle showing the advantages of using the true Hamiltonian is also given. A differential inclusion problem is also treated in [3], where sufficient conditions for the existence of an optimal periodic solution among all the admissible ones (which are assumed to exist) as well as sufficient conditions for optimality for an integral cost function are stated.

Other papers [13, 14] deal with sufficient optimality conditions. The existence of a control is assumed with corresponding periodic solution of (1) satisfying the maximum principle. In [14] a sufficient optimality condition for this control is obtained by means of a suitably modified Hamilton–Jacobi equation. While in [13], second variation methods are employed and the problem of preserving optimality under small perturbations is treated.

No direct method to solve the periodic optimization problem of system (1) is given in the papers cited above.

In this paper a direct method is presented to solve this problem. A class of nonlinearities $f: [0, 1] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is considered, satisfying the following conditions:

(i) $f(t, x, u)$ is integrable with respect to t for all $(x, u) \in \mathbb{R}^n \times \Omega$ and continuous in (x, u) for almost all $t \in [0, 1]$;

(ii) for all $u \in \Omega$

$$|f(t, x, u)| \leq \mu(t) + a|x|$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$, $\mu \in L_1([0, 1], \mathbb{R})$ and $a > 0$. The class of controls is defined by $U = \{u \in L_\infty([0, 1], \mathbb{R}^m): u(t) \in \Omega$ for a.a. $t \in [0, 1]\}$.

(Through this paper it is assumed that μ, u and f are functions of t, extended by 1-periodicity from $[0, 1]$ to $\mathbb{R}$.)

In Section 1, a sufficient condition is given, denoted by (H), which assures, for all $u \in U$, the existence of a solution of (1) satisfying the following periodicity boundary condition

$$x(0) = x(1). \tag{2}$$

Moreover, from this condition and the following convexity assumption a.a

(iii) the set $f(t, x, \Omega) = \{f(t, x, u); u \in \Omega\}$ is convex for each $x \in \mathbb{R}^n$ and for ~~each~~ $t \in [0, 1]$. Some properties of the set K of the initial states of the 1-periodic solutions of (1) corresponding to $u \in U$ are given.

Lastly, in Section 2, for a general performance cost the existence of an optimal control belonging to U is proved.

SECTION 1. EXISTENCE OF PERIODIC RESPONSES

Let us consider the differential system

$$\dot{x} = f(t, x, u) \tag{1}$$

together with the boundary periodicity condition

$$x(0) = x(1) \tag{2}$$

where $f: [0, 1] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ satisfies conditions (i) and (ii).

Solutions of (1)–(2) are understood in the Carathèodory sense, i.e. absolutely continuous functions $x:[0,1]\to\mathbb{R}^n$ verifying (1) for ~~each~~ a.a. $t\in[0,1]$ with $x(0)=x(1)$.

Under assumptions (i) and (ii), for each $u\in U$ and $x_0\in\mathbb{R}^n$ it is known (cf. [15]) that there exists at least one solution of (1) in $[0,1]$ initiating at x_0.

Let X be the space $AC([0,1],\mathbb{R}^n)$ of the absolutely continuous functions with the norm

$$\|x\|_{AC}=\max_{t\in[0,1]}|x(t)|+\int_0^1|\dot{x}(t)|\,dt.$$

Let $Y=L_1([0,1],\mathbb{R}^n)$ with the norm $\|x\|_{L_1}=\int_0^1|x(t)|\,dt$.

Define $L:D(L)\subset Y\to Y$ by $D(L)=\{x\in X:x(0)=x(1)\}$ and $(Lx)(t)=\dot{x}(t)$ for ~~each~~ a.a. $t\in[0,1]$. Clearly L takes values in Y. For a fixed $u\in U$, let $F_u:Y\to Y$ be defined by $F_u(x)(t)=f(t,x(t),u(t))$ for ~~all~~ $t\in[0,1]$. The assumptions on f imply that F_u maps Y continuously into itself (cf. [16]). a.a.

The problem (1)–(2) takes the form

$$Lx=F_u(x).$$

Notice that

$$N(L)=\{x\in D(L):x(t)=x(0)\text{ for all }t\in[0,1]\},$$

and

$$R(L)=\left\{y\in Y:\int_0^1 y(t)\,dt=0\right\}.$$

Hence $L:D(L)\subset Y\to Y$ is a Fredholm operator of index zero and Y can be decomposed in the topological direct sum $Y=N(L)\oplus R(L)$. Let Q be the projection of Y onto $R(L)$ parallel to $N(L)$ defined by $(Qy)(t)=y(t)-\int_0^1 y(t)\,dt$. Therefore problem (LF_u) is equivalent to the system

$$\begin{aligned} w &= HQF_u(c\mathbb{1}+w) \qquad\qquad (\mathrm{LF}_u)\\ 0 &= (I-Q)F_u(c\mathbb{1}+w)\end{aligned}$$

where $\mathbb{1}(t)=1$ for every $t\in[0,1]$, $c\in\mathbb{R}^n$, $c\mathbb{1}+w\in N(L)\oplus R(L)$ and H is the right inverse of L composed with the compact imbedding of $D(L)$ into Y. System (LF_u) can be regarded as a fixed point problem, namely the solvability of (LF_u) is equivalent to the existence of a fixed point of the map $S:N(L)\times R(L)\to N(L)\times R(L)$ defined by

$$S(c,w)=(c\mathbb{1}+(I-Q)F_u(c\mathbb{1}+w),\,HQF_u(c\mathbb{1}+w)).$$

These methods are widely employed to establish the solvability of some nonlinear differential equations (see [17, 18]), as well as the extensive references therein).

For any $x\in\mathbb{R}^n\backslash\{0\}$ and ~~any~~ a.a. $t\in[0,1]$, let

$$s(t,x)=\sup\left\{\frac{(x,y)}{|x|^2};\,y\in f(t,x,\Omega)\right\},$$

where (x,y) denotes the inner product of x and y in $\mathbb{R}^n$ corresponding to the Euclidean norm $|\cdot|$. Consider

$$s(t)=\limsup_{|x|\to+\infty}s(t,x)$$

we are now in a position of proving the following.

THEOREM 1.1. Suppose that conditions (i) and (ii) are satisfied. Moreover assume that

$$\int_0^1 s(t)\,dt < 0. \tag{H}$$

Then, for each $u \in U$, there exists a solution of (LF_u) of the form

$$x_u = c_u\mathbb{1} + w_u \in N(L) \oplus R(L).$$

Proof. Fixed $u \in U$, the maps $(I-Q)F_u: X \to N(L)$ and $HQF_u: X \to R(L)$ are continuous and compact, i.e. they map bounded sets of X into relatively compact sets. To prove the existence of a solution of (LF_u) the properties of the Leray–Schauder topological degree of the previous maps are utilized. (See [19] for a review on topological degree as well as the extensive referencces therein.)

The procedure is the following

(I) prove that the map $G_u: N(L) \times R(L) \times [0,1] \to N(L) \times R(L)$ defined by

$$G_u(c, w, \lambda) = ((I-Q)F_u(c\mathbb{1} + \lambda w), w - \lambda HQF_u(c\mathbb{1} + w)).$$

is an admissible homotopy in $B_R \times [0,1]$ for $R > 0$, where B_R is the ball in X centered at zero and radius R. Therefore by the homotopy invariance and the product of domains properties of the Leray–Schauder topological degree one has

$$\deg((I-Q)F_u(c\mathbb{1} + w), w - HQF_u(c\mathbb{1} + w), B_R, 0) = \deg((I-Q)F_u(c\mathbb{1}), B_R \cap N(L), 0).$$

(II) Then show that $\deg((I-Q)F_u(c\mathbb{1}), B_R \cap N(L), 0)$ is different from zero and so, by the solution property of the Leray–Schauder topological degree, establish the existence of a fixed point of the map $S(c, w)$ in B_R, or equivalently, the solvability of (LF_u).

Part (I). Consider all the possible 1-periodic solutions of

$$Lx = \nu F_u(x) \tag{3}$$

where $u \in U$ and $\nu \in (0,1]$. It is necessary to show that they are equibounded in $C([0,1], \mathbb{R}^n)$ and so in X by (ii). For this, let x be a solution of (3) for some $u \in U$ and $\nu \in (0,1]$, i.e.

$$\dot{x}(t) = \nu f(t, x(t), u(t)) \qquad \text{for } \cancel{\text{each}}\ \text{a.a. } t \in [0,1]$$

and

$$x(0) = x(1)$$

therefore

$$\frac{d}{dt}\frac{|x(t)|^2}{2} = \nu(x(t), f(t, x(t), u(t))) \le |\mu(t)||x(t)| + a(|x(t)|^2 + 1) - a$$

dividing by $|x(t)|^2 + 1$ and integrating on the interval $[\tau, t]$, where $\tau \in \mathbb{R}$ and $t \in [\tau, \tau + 1]$, produces

$$1/2[\log(|x(t)|^2 + 1) - \log(|x(\tau)|^2 + 1)] < a + \int_0^1 \frac{|\mu(s)||x(s)|}{|x(s)|^2 + 1}\,ds = M(x).$$

By the 1-periodicity of x one has

$$\log \max_{t\in[0,1]} (|x(t)|^2 + 1) < \log \min_{t\in[0,1]} (|x(t)|^2 + 1) + 2M(x). \tag{4}$$

Suppose now that the 1-periodic solutions of (3) are unbounded in $C([0,1],\mathbb{R}^n)$ then there exist sequences $\{\nu_n\} \subset (0,1]$, $\{u_n\} \subset U$ and $\{x_n\} \subset D(L)$ such that

$$\dot{x}_n(t) = \nu_n f(t, x_n(t), u_n(t)) \qquad \text{for } \cancel{\text{each}} \text{ a.a. } t \in [0,1]$$

and

$$\max_{t\in[0,1]} |x_n(t)| \to +\infty \qquad \text{as} \qquad n \to +\infty.$$

From (4) it follows that $\min_{t\in[0,1]} |x_n(t)| \to +\infty$ as $n \to +\infty$ (the sequence $M(x_n)$ being bounded in $\mathbb{R}$). Therefore, for n sufficiently large

$$\nu_n \int_0^1 \frac{(x_n(t), f(t, x_n(t), u_n(t)))}{|x_n(t)|^2}\, dt = \log \left|\frac{x_n(1)}{x_n(0)}\right| = 0$$

but $\nu_n \subset (0,1]$, so

$$\int_0^1 \frac{(x_n(t), f(t, x_n(t), u_n(t)))}{|x_n(t)|^2}\, dt = 0.$$

Hence

$$0 \leq \limsup_{n\to+\infty} \int_0^1 s(t, x_n(t))\, dt. \tag{5}$$

On the other hand, for all $t \in [0,1]$, $|x_n(t)| \to +\infty$ as $n \to +\infty$ hence

$$\limsup_{n\to+\infty} s(t, x_n(t)) \leq \limsup_{|x|\to+\infty} s(t,x) \qquad \text{for all } t \in [0,1]$$

and using Fatou's lemma and (5), (H) is violated.

Thus there exists $\rho' > 0$ such that every 1-periodic solution x of (3) satisfies the inequality

$$\max_{t\in[0,1]} |x(t)| < \rho'. \tag{6}$$

Moreover, (H) implies that there exists a positive constant ρ'' such that

$$0 \neq (I - Q)F_u(x) \tag{7}$$

for $x \in D(L)$ and $\max_{t\in[0,1]} |x(t)| \geq \rho''$, and for all $u \in U$.

By means of (6) and (7), notice that the homotopy G_u is admissible in $B_R \times [0,1]$ for each $u \in U$, where B_R is the ball in the norm of X corresponding to the ball $B_{\bar{R}}$ in the norm of $C([0,1],\mathbb{R}^n)$ with $\bar{R} = \max\{\rho', \rho''\}$.

Part (II). For a fixed $u \in U$, let us consider the map $\bar{G}_u : N(L) \times [0,1] \to N(L)$ defined by $\bar{G}_u(c,\lambda) = (\lambda - 1)c\mathbb{1} + \lambda(I - Q)F_u(c\mathbb{1})$. It is necessary to prove the existence of a constant $R_1 \geq R$ such that $\bar{G}_u$ is an admissible homotopy in $B_{R_1} \cap N(L)$. Suppose the contrary, then there exist sequences $\{\lambda_n\} \subset [0,1]$, $\{c_n\} \subset \mathbb{R}^n$ with $|c_n| \to +\infty$ as $n \to +\infty$, and $\{u_n\} \subset U$ such that

$$0 = (\lambda_n - 1)c_n\mathbb{1} + \lambda_n(I - Q)F_{u_n}(c_n\mathbb{1})$$

or equivalently

$$0 = (\lambda_n - 1)c_n + \lambda_n \int_0^1 f(t, c_n, u_n(t))\, dt$$

hence

$$0 = (\lambda_n - 1) + \lambda_n \frac{\left(c_n, \int_0^1 f(t, c_n, u_n(t))\, dt\right)}{|c_n|^2}. \tag{8}$$

By the assumption (H), taking n sufficiently large

$$\frac{\left(c_n, \int_0^1 f(t, c_n, u_n(t))\, dt\right)}{|c_n|^2} < 0$$

which is a contradiction with (8); therefore taking $R = R_1$

$$\deg((I - Q)F_u, B_R \cap N(L), 0) \neq 0.$$

This completes the proof of theorem 1.1. ∎

Remark 1.1. Note that theorem 1.1 remains true and its proof is essentially the same if the assumption (H) is substituted with the following one

$$\int_0^1 m(t)\, dt > 0 \tag{H$'$}$$

where $m(t) = \liminf_{|x| \to +\infty} m(t, x)$, and $m(t, x) = \inf\{(x, y)/|x|^2;\ y \in f(t, x, \Omega)\}$.

Remark 1.2. Define $s_\delta(t, x) = \sup\{(x, y)/|x|^{1+\delta};\ y \in f(t, x, \Omega)\}$, where $\delta \in [0, 1]$. In theorem 1.1 replace (H) by $\int_0^1 s_\delta(t)\, dt < 0$, this allows the case where the nonlinear term f is asymptotically sub-linear with respect to x to be studied. It is the same situation where (ii) takes the form

$$|f(t, x, u)| \leq \mu(t) + a|x|^\delta$$

for all $u \in \Omega$ and $\delta \in [0, 1)$, for which the assumption (H) is not verified. Observe that theorem 1.1 remains true and the proof for $\delta \in [0, 1)$ is the same of the case where $\delta = 1$; in fact

$$\psi(u) = \int_0^u \frac{dv}{v^{(1+\delta)/2}},$$

with $v = |x|^2$, is a positive strictly increasing function such that $\lim_{u \to +\infty} \psi(u) = +\infty$, for all $\delta \in [0, 1)$.

In particular, if f is uniformly bounded and $m = n = 1$ then condition (H) reduces to the following ones

$$\int_0^1 \limsup_{x \to +\infty} \sup_{u \in \Omega} f(t, x, u)\, dt < 0$$

and

$$\int_0^1 \liminf_{x\to-\infty} \inf_{u\in\Omega} f(t,x,u)\,\mathrm{d}t > 0$$

which are Landesman–Lazer type conditions on the asymptotic behaviour of f with respect to x (cf. [20, 21]). Note that if f is time-independent the above conditions are verified if $f(x,u) \leqslant -a < 0$ for $x > k$ and for all $u \in \Omega$, where $k > 0$, and $f(-x,u) = -f(x,u)$. Analogous considerations hold for (H)′.

Remark 1.3. Consider the logarithmic norm $\mu(A)$ of a square matrix A defined by means of the formula

$$\mu(A) = \lim_{h\to 0^+} h^{-1}(|I + hA| - 1)$$

(see, for instance [22]). Observe that

$$\mu(A) = \sup_{|x|=1} (x, Ax).$$

If the dynamics is linear, i.e. $f(t,x,u) = A(t)x + B(t)u$, then condition (H) is equivalent to the following one

$$\int_0^1 \sup_{|x|=1} (x, A(s)x)\,\mathrm{d}s < 0.$$

Since for every solution $x(t)$ of the linear system $\dot{x} = A(t)x$ the following inequality holds (see [22], theorem 3, p. 58)

$$|x(1)| \leqslant |x(0)| \exp\left(\int_0^1 \mu(A(s))\,\mathrm{d}s\right)$$

then condition (H), in the linear case, implies that the only periodic solution of $\dot{x} = A(t)x$ is the zero solution. Therefore problem

$$\begin{cases} \dot{x} = A(t)x + B(t)u \\ x(0) = x(1) \end{cases}$$

is solvable for every $u \in U$.

Remark 1.4. Theorem 1.1. can be proved, following the same procedure,, by using the notion of coincidence degree introduced by Mawhin in [23] (cf. also [24] and [27] as well as the extensive references on Landesman-Lazer type problems therein.)

Remark 1.5. It is interesting to observe that the multivalued map $F: Y \multimap Y$ defined by $F(x) = \{z : [0,1] \to \mathbb{R}^n$, z measurable and $z(t) \in f(t, x(t), \Omega)$ for a.a. ~~each~~ $t \in [0,1]\}$, where f satisfies (i)–(iii), has the following properties

(a) for each $x \in Y$, $F(x)$ is a nonempty, closed and convex subset of Y;

(b) $F: Y \multimap Y$ is bounded, i.e. it maps bounded sets into bounded sets;

(c) if $x_n \to x$ in Y and $z_n \in F(x_n)$ then $\lim_{n\to+\infty} \mathrm{dist}\,(z_n, F(x)) = 0$;

(d) for each $x \in Y$, $F(x) = \{F_u(x) : u \in U\}$.

For a proof see [25].

THEOREM 1.2. Let the assumptions (i)–(iii) and (H) be satisfied. Let $I(u)$ the set of the initial states of the 1-periodic solutions of (1) corresponding to $u \in U$. Let $K = \{I(u) : u \in U\}$, then

(a) K is a compact set of $\mathbb{R}^n$;

(b) K is the union of all 1-periodic responses of (1) to controls $u \in U$.

Proof. From theorem 1.1 it follows that every 1-periodic solution of (1) corresponding to the controls $u \in U$ is contained in $B_R \subset X$. Therefore K is bounded in $\mathbb{R}^n$. Moreover, consider a sequence $x_{0,n}$ in K and the corresponding $u_n \in U$ for which

$$x_n(t) = x_{0,n} + \int_0^t f(s, x_n(s), u_n(s))\, ds \qquad \text{with } x_n(1) = x_{0,n}.$$

Hence $\{x_n\}$ is a sequence of equibounded and equicontinuous function in $C([0,1], \mathbb{R}^n)$. By the Ascoli–Arzela theorem, passing to a subsequence, if necessary

$$x_n(t) \to x_0(t) \quad \text{in } C([0,1], \mathbb{R}^n).$$

Put

$$\Phi_n(t) = f(t, x_n(t), u_n(t)),$$

hence

$$\lim_{n \to +\infty} \int_0^t \Phi_n(s)\, ds = \int_0^t \Phi_0(s)\, ds$$

where $\Phi_0(t)$ is an integrable function. Thus

$$\lim_{n \to +\infty} \int_0^1 \chi_E(s)\, \Phi_n(s)\, ds = \int_0^1 \chi_E(s)\, \Phi_0(s)\, ds$$

where χ_E is the characteristic function of a subinterval E. Every measurable set can be approximated by a finite sum of disjoint open intervals, and so the above formula holds for an arbitrary measurable set E in $[0,1]$. Therefore Φ_n converges weakly to Φ_0 in $L_1([0,1], \mathbb{R}^n)$. Let

$$x_0(t) = x_0 + \int_0^t \Phi_0(s)\, ds.$$

Clearly $x_0(0) = x_0(1) = x_0 = \lim_{n \to +\infty} x_{0,n}$, and so to prove that $x_0 \in K$ it remains to show that there exists $u_0 \in U$ such that

$$\Phi_0(t) = f(t, x_0(t), u_0(t)) \quad \text{for } \cancel{\text{each}} \text{ a.a. } t \in [0,1].$$

For all $n \in \mathbb{N}$,

$$\Phi_n(t) \in f(t, x_n(t), \Omega) \quad \text{for } \cancel{\text{each}} \text{ a.a. } t \in [0,1].$$

By the weak convergence of Φ_n to Φ_0

$$\limsup_{n \to +\infty} (y, \Phi_n(t)) \geq (y, \Phi_0(t)) \geq \liminf_{n \to +\infty} (y, \Phi_n(t))$$

for all $y \in \mathbb{R}^n$ and for ~~each~~ a.a. $t \in [0, 1]$. Therefore

$$\limsup_{n \to +\infty} \left[\sup_{u \in \Omega} (y, f(t, x_n(t), u)) \right] \geq (y, \Phi_0(t)) \geq \liminf_{n \to +\infty} \left[\inf_{u \in \Omega} (y, f(t, x_n(t), u)) \right].$$

Thus, by the continuity of f with respect to x

$$\sup_{u \in \Omega} (y, f(t, x_0(t), u)) \geq (y, \Phi_0(t)) \geq \inf_{u \in \Omega} (y, f(t, x_0(t), u))$$

for all $y \in \mathbb{R}^n$ and for a.a. $t \in [0, 1]$.
This implies, via the assumption (iii), that

$$\Phi_0(t) \in f(t, x_0(t), \Omega) \quad \text{for } \cancel{\text{each}}\ \text{a.a. } t \in [0, 1].$$

Roxin proved in [26] the existence of a control $u_0 \in U$ such that

$$\Phi_0(t) = f(t, x_0(t), u_0(t)) \quad \text{for } \cancel{\text{each}}\ \text{a.a. } t \in [0, 1].$$

and so (a) is proved. To prove (b), let $u_0 \in U$ be the control with corresponding 1-periodic response $x_0(t)$ initiating at x_0, i.e.

$$x_0(t) = x_0 + \int_0^t f(s, x_0(s), u_0(s))\, ds \quad \text{with } x_0(1) = x_0.$$

Let $\tau \in \mathbb{R}^+$, then

$$x_0(t + \tau) = x_0 + \int_0^{t+\tau} f(s, x_0(s), u_0(s))\, ds = x_0(\tau) + \int_0^t f(s + \tau, x_0(s + \tau), u_0(s + \tau))\, ds.$$

Notice that $x_0(t + \tau)$ is a periodic function, hence the control $u_0(t + \tau)$ gives a 1-periodic response with initial state $x_0(0 + \tau)$. In conclusion the set $\{x_0(t) : t \in [0, 1]\}$ is contained in K. ◻

SECTION 2. OPTIMIZATION PROBLEMS

In this section, among all solutions $x(t)$ of (1) and (2) for $u \in U$, the optimal solution $x_0^*(t)$ corresponding to a control $u_0^* \in U$ which minimizes a cost functional $C(u)$ of the type considered in [27] is determined.
The following theorem states the existence of such a control

THEOREM 2.1. Consider the nonlinear control process in $\mathbb{R}^n$

$$\dot{x} = f(t, x, u) \tag{1}$$

together with the boundary periodicity condition

$$x(0) = x(1) \tag{2}$$

where the nonlinear term f satisfies the conditions (i)–(iii) and (H) and $u \in U$. Moreover, assume that the following Lipschitz condition holds
(iv) there exists a positive constant Λ such that for every $(\cancel{y}, u) \in [\cancel{0,} 1] \times \Omega$ and a.a. $t \in [0,1]$

$$|f(t, x, u) - f(t, y, u)| < \Lambda |x - y|.$$

Let the cost functional be defined, for each $u \in U$, by

$$C(u) = g(x(1)) + \int_0^1 f^0(t, x(t), u(t))\, dt + \max_{t \in [0,1]} \gamma(x(t))$$

and assume that the following conditions are satisfied.
(v) $f^0\colon [0,1] \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ is integrable with respect to t for all $(x,u) \in \mathbb{R}^n \times \Omega$ and continuous in (x,u) for each $t \in [0,1]$; g, γ are continuous functions in $\mathbb{R}^n$;
(vi) for all $u \in \Omega$

$$|f^0(t,x,u)| \le \mu_1(t) + a_1|x|,$$

where $\mu_1 \in L_1([0,1], \mathbb{R})$ and $a_1 > 0$;
(vii) the set

$$V(t,x) = \{f^0(t,x,\Omega), f(t,x,\Omega)\}$$

is convex in $\mathbb{R}^{n+1}$ for each fixed ~~(t,x)~~ x and a.a. $t \in [0,1]$.
Then there exists an optimal control, i.e. a control $u_0^* \in U$ minimizing $C(u)$.

Proof. For each $u \in U$ there exists the $\min_{I(u)} C(u)$, denoted by $a(u)$. This is evident if $I(u)$ is a finite set in $\mathbb{R}^n$. If $I(u)$ is an infinite set in $\mathbb{R}^n$ the assertion follows from the compactness of $I(u)$, which is an immediate consequence of theorem 1.2, and the continuity of $C(u)$ on $I(u)$ (cf. [15]). It is easy to prove now the following relation

$$\inf_U C(u) = \inf_U \min_{I(u)} C(u) = \inf_K C(u).$$

Since the 1-periodic responses x to the controls $u \in U$ are uniformly bounded $a = \inf_K C(u)$ is finite. Thus select a sequence $\{x_{0,n}\} \subset K$ of initial states and a sequence $\{u_n\} \subset U$, with $x_{0,n} \in I(u_n)$, such that

$$a = \lim_{n \to +\infty} a(u_n) = \lim_{n \to +\infty} \left\{ g(x_{0,n}) + \int_0^1 f^0(t, x_n(t), u_n(t))\, dt + \max_{t \in [0,1]} \gamma(x_n(t)) \right\}$$

where $x_n(t)$ is the 1-periodic response to the control $u_n(t)$ with initial state $x_{0,n}$, i.e.

$$x_n(t) = x_{0,n} + \int_0^t f(s, x_n(s), u_n(s))\, ds \text{ and } x_n(1) = x_{0,n}.$$

Put

$$x_n^0(t) = \int_0^t f^0(s, x_n(s), u_n(s))\, ds \text{ and } \hat{f} = (f^0, f).$$

The functions $\hat{x}_n(t) = (x_n^0(t), x_n(t))$ are equibounded and equicontinuous in $C([0,1], \mathbb{R}^{n+1})$, thus by the Ascoli–Arzelà Theorem, passing to a subsequence if necessary

$$\hat{x}_n(t) \to \hat{x}_0^*(t) \quad \text{in } C([0,1], \mathbb{R}^{n+1})$$

where

$$\hat{x}_0^*(t) = (x^{0*}(t), x_0^*(t)), \quad \text{and } \hat{x}_0^*(0) = \hat{x}_0^* = (0, x_0^*).$$

Therefore, just as in theorem 1.2, $\hat{f}(t, x_n(t), u_n(t))$ converges weakly in $L_1([0, 1], \mathbb{R}^{n+1})$ to some integrable $(n+1)$-vector $\hat{\Phi}_0(t) = (\Phi^0(t), \Phi(t))$. Define

$$\hat{x}_0^*(t) = \hat{x}_0^* + \int_0^t \hat{\Phi}_0(s)\, ds$$

in virtue of (v)–(vii), and following the procedure used in the proof of theorem 1.2 the existence of a control $u_0^* \in U$ can be stated such that

$$\hat{\Phi}_0(t) = \hat{f}(t, x_0^*(t), u_0^*(t))$$

hence x_0^* is the 1-periodic solution of (1) corresponding to the control u_0^* and the initial state x_0^*. Thus

$$a = g(x_0^*) + \int_0^1 f^0(t, x_0^*(t), u_0^*(t))\, dt + \max_{t \in [0,1]} \gamma(x_0^*(t)) = C(u_0^*)$$

which proves the existence of a control $u_0^* \in U$ minimizing $C(u)$. ∎

If more assumptions on the regularity of f with respect to its arguments and on the functions appearing in $C(u)$ are made and if the problem in variations

$$\begin{cases} \dot{y} = \dfrac{\partial f}{\partial x}(t, x_0^*(t), u_0^*(t))y \\ y(0) = y(1) \end{cases}$$

admits only the zero solution, then further information on the optimal control can be derived from the maximum principle.

In fact (see [2]), by using the approach of Hestenes [28], it can be shown that u_0^* satisfies the maximum principle. Note that the term $\max_{t \in [0,1]} \gamma(x(t))$ in the performance cost $C(u)$ can be written as an integral term and treated as in [5].

In particular the results of this Section and the discussion above can be applied to the case, considered by Markus in [4], where $C(u) = -\max_{t \in [0,1]} x^1(t)$ (x^1 is the first component of the periodic solution $x(t)$ of (1) corresponding to the control $u(t)$) to obtain the existence of an optimal control satisfying a maximum principle.

REFERENCES

1. GUARDABASSI G., LOCATELLI A. & RINALDI S., Status of periodic optimization of dynamical systems, *J. optim. Theory Appl.* 14, 1–20 (1974).
2. HALANAY A., Optimal control of periodic solutions, *Rev. roum. Math. Pures Appl.* 19, 3–16 (1974).
3. IRISOV A. E. & TONKOV E. L., Sufficient conditions for the existence of optimal periodic solutions of a differential inclusion, *Mat. Fiz. (Kiev)* 27, 13–19 (1980) in Russian.
4. MARKUS L., Optimal control of limit cycles or what control theory can do to cure a heart attack or to cause one, *Symposium on Ordinary Differential Equations, Minneapolis, Minnesota, May 29–30, 1972.* (Edited by HARRIS W. A. & SIBUYA Y.), Berlin (1973).
5. TONKOV E. L., Optimal periodic motions of a guided system, *Mat. Fiz. (Kiev)* 21, 45–59 (1977), in Russian.
6. TONKOV E. L., Optimal control of periodic motions, *Mat. Fiz. (Kiev)* 22, 54–64 (1977) in Russian.
7. HORN F. J. M. & BAILEY J. E., An application of the theorem of relaxed control to the problem of increasing catalyst selectivity, *J. optim. Theory Appl.* 2, 441–449 (1968).
8. BAILEY J. E. & HORN F. J. M., Comparison between two sufficient conditions for improvement of an optimal steady-state process by periodic operation, *J. optim. Theory Appl.* 7, 378–384 (1971).
9. GUARDABASSI G., Optimal steady-state vs periodic control: a circle criterion, *Ric. Automatica* II, 240–252 (1971).

10. TONKOV E. L., A linear optimal control problem with periodic solutions, *Differentsial'nye Uravneniya* **12**, 1007–1011 (1976) in Russian.
11. BAILEY J. E., Necessary conditions for optimality in a general class of nonlinear mixed value control problems, *Int. J. Control* **16**, 311–320 (1972).
12. CLARKE F. H., Optimal control and the true Hamiltonian, *SIAM Rev.* **21**, 157–166 (1979).
13. BITTANTI S., LOCATELLI A. & MAFFEZZONI C., Second variation methods in periodic optimization, *J. optim. Theory Appl.* **14**, 31–49 (1974).
14. MAFFEZZONI C., Hamilton–Jacobi theory for periodic control problems, *J. optim. Theory Appl.* **14**, 21–29 (1974).
15. SANSONE G. & CONTI R., *Equazioni Differenziali Non Lineari*, Edizioni Cremonese, Roma (1956).
16. PRODI G. & AMBROSETTI A., *Analisi Non Lineare*, Editrice Tecnica Scientifica, Pisa (1973).
17. AMANN H., AMBROSETTI A. & MANCINI G., Elliptic equations with non invertible Fredholm linear part and bounded nonlinearities, *Math. Z.* **158**, 179–194 (1978).
18. CONTI G., MASSABÒ I. & NISTRI P., Set-valued perturbations of differential equations at resonance, *Nonlinear Analysis TMA* **4**, 1031–1041 (1980).
19. FUČIK S., NEČAS J., SOUČEK J. & SOUČEK V., Spectral analysis of nonlinear operators, *Lecture Notes in Mathematics* **346**, Springer, Berlin (1973).
20. LANDESMAN E. M. & LAZER A. C., Nonlinear perturbation of linear elliptic boundary value problems at resonance, *J. Math. Mech.* **19**, 609–623 (1970).
21. MAWHIN J., Landesman–Lazer's type problems for nonlinear equations, Seminari dell'Istituto di Matematica Applicata dell'Università degli Studi di Firenze.
22. COPPEL W. A., Stability and asymptotic behaviour of differential equations, *Heat Mathematical Monographs*, D.C. Heath, Boston (1965).
23. MAWHIN J., Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector space, *J. Diff. Eqns* **12**, 610–636 (1972).
24. GAINES R. E. & MAWHIN J. L., Coincidence degree and nonlinear differential equations, *Lecture Notes in Mathematics* **568**, Springer, Berlin (1977).
25. LASRY J. M. & ROBERT R., Analyse nonlinéaire multivoque, Cah. *Math thématiques de la Décision*, Université de Paris, Dauphine.
26. ROXIN E., The existence of optimal controls, *Michigan Math. J.* **9**, 109–119 (1962).
27. LEE E. B. & MARKUS L., *Foundations of Optimal Control Theory*, Wiley, New York (1967).
28. HESTENES M. R., On variational theory and optimal control theory, *SIAM J. Control Optim.* A3, 23–48 (1965).