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An Approximation Method for the Existence of Periodic Solutions to Systems with Delay

We describe a method for proving the existence of periodic solutions to n-dimensional systems of the form $z'(t) - Az(t) - Bz(t-\tau) = F[z(t)]$. The proposed method is based on the harmonic balance method and the theory of reproducing kernels.

1. Introduction

In this paper we describe an approach to proving the existence of periodic solutions to autonomous systems of differential equations with delay, of the form

$$z'(t) - Az(t) - Bz(t-\tau) = F[z(t)], \tag{1}$$

with $z : R \to R^m$, A and B $m \times m$ constant matrices, $\tau > 0$ and $F : R^m \to R^m$ continuous. Our approach avoids the heavy machinery associated with the use of the Poincaré map

$$\mathcal{P} : \varphi(\cdot) \longrightarrow z(T + \cdot\,; \varphi)$$

which maps the initial segment $\varphi(s)$, $-\tau \le s \le 0$, into the terminal segment $z(T+s;\varphi)$, $-\tau \le s \le 0$, of the corresponding solution to (1) with initial data $\varphi(\cdot)$. T here is the unknown period.

Our method is based on ideas that go by various names: Cesari's method, Urabe's method (see [6]); the method of harmonic balance, (see [3], [4], [5]), the theory of reproducing kernels (see [1], [2]). Our method extends easily to problems with several delays.

2. The Operator Equation

Our first step is to convert (1) to an integral equation on an appropriate Banach space of functions. Consider (1) as a version of $z'(t) - Az(t) = f(t)$, and add the periodic boundary conditions

$$z(0) = z(T) \qquad (T \text{ unknown}). \tag{2}$$

We define $x(t) = z(Tt/2\pi)$, so $x(\cdot)$ is 2π-periodic if and only if $z(\cdot)$ is T-periodic. Then the differential equation (1) with the boundary conditions (2) becomes

$$\begin{aligned} &\text{(a)} \quad \omega\dot{x}(t) - Ax(t) = Bx(t-\sigma) + F[x(t)], \\ &\text{(b)} \quad x(0) = x(2\pi), \end{aligned} \tag{3}$$

where $\sigma = \omega\tau$, $\omega = 2\pi/T$. Thus we have replaced the unknown period T by the fixed period 2π by introducing the parameter ω into the system. A solution of (3)(a)(b) will not in general yield a periodic solution of (1), because of the delay term; to guarantee a periodic solution we would replace (b) by $x(s) = x(2\pi + s)$, $-\sigma \le s \le 0$.

The Green's matrix for $\omega\dot{x} - Ax = f$, with periodic boundary condition $x(0) = x(2\pi)$, is given by (see [6])

$$G_\omega(t-s) = (1/\omega)[I - e^{2\pi A/\omega}]^{-1} \begin{cases} e^{A(t-s)/\omega}, & 0 \le s \le t, \\ e^{A(2\pi+t-s)/\omega}, & t < s \le 2\pi, \end{cases} \tag{4}$$

assuming $I - e^{2\pi A/\omega}$ is nonsingular. We make this assumption, which permits to write (3) as an operator equation

$$x(t) = \mathcal{T}_\omega[x](t) \equiv \mathcal{G}_\omega\big[BS_{-\sigma}[x] + \mathcal{F}[x]\big](t) \tag{5}$$
$$\equiv \int_0^{2\pi} G_\omega(t-s)\big[Bx(s-\sigma) + F[x(s)]\big]\,ds,$$

where $S_{-\sigma}[x](t) = x(t-\sigma)$ is the shift operator and $\mathcal{F}[x]$ is the Nemytskii operator $\mathcal{F}[x](t) = F[x(t)]$.

The appropriate space for (5) depends on the nonlinearity F and on the desired degree of regularity of solutions. The idea is to take an appropriate space X_0 of functions defined on $[0, 2\pi]$, say $C([0,2\pi], R^m)$ or $L_2((0,2\pi), R^m)$, and create the space of periodic extensions

$$X = \{x \in X_0 \mid x(t+2\pi) = x(t) \text{ for all } t \in R\}$$

using the norm from X_0.

For example, if $m = 1$ and $F[x] = x^3$, then $X_0 = L_2(0, 2\pi)$ is not appropriate, while $X_0 = C[0, 2\pi]$ (or $X_0 = L_6(0, 2\pi)$) is suitable, in the sense that $\mathcal{T}_\omega$ is well-defined on X. In fact we ask more that

$$\mathcal{T}_\omega : X \longrightarrow X$$

be compact. For F continuous, this will be the case for $X_0 = C[0, 2\pi]$ since in this case $\mathcal{T}_\omega[x]$ will lie in

$$Z = \{y \mid y(t+2\pi) = y(t) \text{ for all } t \in R,\ y \in W^{1,2}((0,2\pi), R^m)\}.$$

Here $W^{1,2}$ is the Sobolev space of functions with generalized derivative in L_2, which for functions of a single variable coincides with space of absolutely continuous functions with derivative in L_2. The natural embedding of this space into C or L_2 is compact.

Assuming that appropriate spaces X, Z have been defined, with $\mathcal{T}_\omega$ a compact mapping from X into $Z \subseteq X$, we note that for a given $\sigma \in [0, 2\pi)$, *any fixed point in X of $\mathcal{T}_\omega$ is a 2π-periodic solution of (3a), hence corresponds to a T-periodic solution of (1) with delay σ/ω*. Thus the search for periodic solutions of (1) has been reduced to a search for those values of the parameters ω and $\sigma = \omega\tau$, which yield fixed points of $\mathcal{T}_\omega$ different from $x = 0$.

3. The Method of Harmonic Balance/Reproducing Nonlinearities

We assume that functions in X have Fourier expansions

$$x \in X \Rightarrow x(t) \longleftrightarrow \sum_{-\infty}^{\infty} a_k\, e^{ikt}$$

with $a_0 \in \mathbb{R}^n$, $a_k \in \mathbb{C}^n$ for $k \neq 0$, $a_{-k} = \overline{a}_k$ (the conjugate of a_k). Define the standard Fourier projection

$$P_n : x \longmapsto x_n, \qquad x_n(t) = \sum_{-n}^{n} a_k\, e^{ikt}. \tag{6}$$

Note that $F(P_n[x](t)$ is 2π-periodic; we assume our space X has been defined in such a way that $x_n \in X$ and $F[x_n(t)] \in X$, so $F[x_n(t)]$ also has a Fourier expansion with

$$P_nF[x_n(t)] = \sum_{-n}^{n} f_k\, e^{ikt}, \qquad f_k = f_k(a_0, a_{\pm 1}, \ldots, a_{\pm n}).$$

The method of harmonic balance replaces the operator equation (5) by the following *finite-dimensional* (Galerkin) approximation:

$$x_n(t) = P_n\mathcal{T}_\omega[x_n](t) = \mathcal{G}_\omega\{BS_{-\sigma}[x_n] + P_n\mathcal{F}[x_n]\}(t). \tag{7}$$

Here we have used the facts $P_n\mathcal{G}_\omega = \mathcal{G}_\omega P_n$, $P_n^2 = P_n$, $P_nBS_{-\sigma}[x_n] = BS_{-\sigma}[x_n]$. For a given fixed n, (7) yields (after some easy calculations) a finite system of equations for the (vector) coefficients a_k, $k = 0, \pm 1, \ldots, \pm n$:

$$a_k - (i\omega kI - A)^{-1}\left[Be^{-ik\sigma}a_k + f_k(a_0, a_{\pm 1}, \ldots, a_{\pm n})\right] = 0. \tag{8}$$

These are called the *harmonic balance equations*; their derivation requires that $(i\omega kI - A)$ be invertible, which we will assume.

There is one small technical problem here. The original system in (1) is autonomous, hence if $x(t)$ is a periodic solution then so is $x(t - t_0)$. The Fourier coefficients of these two solutions are related by a rotation e^{-ikt_0}. We want only *one* of the family $\{x(t - t_0) \mid t_0 \in R\}$, so we *normalize* one component of some fixed a_{i_0}, say $a_{i_0 j_0}$, to have a fixed argument (usually arg $= 0$). This selects a single representative from this family (see [5] or [3] for a fuller discussion).

The harmonic balance equations can be straightforward or difficult to solve, depending on the function F. In a different context, Bazley and collaborators (see [1], [2]) have identified an important class of nonlinearities for which Galerkin-type equations (of which (7) is an example) are far more tractable than in general. We briefly outline the connection with harmonic balance.

If H is a separable Hilbert space and if $\mathcal{B} = \{u_j\}$ is a countable subset of orthonormal elements (in practice, either $j = 0, 1, 2, \ldots$, or $j = 0, \pm 1, \pm 2, \ldots$), then Bazley defines a nonlinear mapping $F : D_F \subset H \to H$ $(D_F =$ domain of $F)$ to be a *reproducing nonlinearity for* $\mathcal{B}$ if for each natural number n there is a natural number $m(n)$ such that

$$F(\text{span}\,\{u_k\}_0^n) \subseteq \text{span}\{u_k\}_0^{m(n)} \tag{9}$$

$$(\text{respectively} \qquad F(\text{span}\{u_k\}_{-n}^n) \subseteq \text{span}\{u_k\}_{-m(n)}^{m(n)}).$$

For such nonlinearities, the computations associated with Galerkin approximations are simplified. The connection with harmonic balance is clear: harmonic balance uses $u_k = e^{ikt}$ and goes one step further by *truncating* $\text{span}\{u_k\}_0^{m(n)}$ to $\text{span}\{u_k\}_0^n$ whenever $m(n) > n$, in fact Bazley et al. often do the same thing in practice.

When one has $\mathcal{B} = \{e^{ikt}\}_{-\infty}^{\infty}$, and $F[x] = x^p$, p a natural number, an easy computation shows F is reproducing; with for example $m(2) = 2p$.

The connection with harmonic balance is even more transparent when one writes (9), for $\mathcal{B} = \{e^{ikt}\}_{-\infty}^{\infty}$, in the more explicit form

$$F\left(\sum_{-n}^{n} a_k\, e^{ikt}\right) = \sum_{-m(n)}^{m(n)} f_k(a_0, a_{\pm 1}, \ldots, a_{\pm n})e^{ikt}.$$

4. The Existence of Periodic Solutions

We now describe our existence theorem for periodic solutions of (1). For convenience we choose X to be the periodic extension of the subspace $\{x(t) \mid x(\cdot) \in C[0, 2\pi],\ x(0) = x(2\pi)\}$. In this theorem we consider the operator equation (5) with σ and ω *independent* parameters. The operator $\mathcal{T}_{\omega,\sigma}$ only depends on σ through the shift operator $\mathcal{S}_{-\sigma}$ and the solution $\widehat{x}_n(t;\sigma)$ of (8). We state the theorem in a way that sets up a procedure which exploits this fact.

Theorem . *Assume that there exists a natural number n such that*

1) $\exists\, \mathcal{S} \subset R_+,\ \mathcal{S} \neq \phi$, *such that* $\forall\, \sigma \in \mathcal{S},\ \exists\, (\widehat{\omega}(\sigma), \widehat{x}_n(t;\sigma))$ *solving (7), with* $\widehat{\omega} \neq 0,\ \widehat{x}_n(t;\sigma) \not\equiv 0$.

Let $Proj_\omega(\mathcal{S}) = \{\omega \mid \exists\, \sigma \in \mathcal{S}$ *such that* $\omega = \widehat{\omega}(\sigma)\}$ *be the projection of* $\mathcal{S}$ *onto the* ω*-axis. Assume that*

2) $\exists\, \mathcal{J} \subset Proj_\omega(\mathcal{S})$ *such that* $\forall\, \omega \in \mathcal{J}$:

   (a) $I - e^{2\pi A/\omega}$ *and* $i\omega kI - A$ *are invertible for* $k = 0, \pm 1, \ldots, \pm n$;

   (b) $\exists\, \mathcal{S}_\omega \subset \mathcal{S},\ \overline{B}_n(\widehat{x}_n, \rho_\omega) \subset X$, *with* $\mathcal{S}_\omega \neq \phi,\ 0 < \rho_\omega < \|\widehat{x}_n\|$, *such that* $\forall\, \sigma \in \mathcal{S}_\omega,\ \mathcal{T}_{\omega,\sigma} : \overline{B} \to \overline{B}$ *is a compact map.*

Then $\forall\ \omega \in \mathcal{J}\ \exists$ a not-identically-zero periodic solution of (1) with period $2\pi/\omega$, for every delay

$$\tau \in \{\sigma/\omega \mid \sigma \in \mathcal{S}_\omega\}.$$

The proof is based on the Schauder Fixed-Point Theorem. Indeed, we look for a ball $\overline{B}(\widehat{x}_n, \rho)$ in X, with $\rho < \|\widehat{x}_n\|$, with $\omega \in \mathcal{J}$, $\sigma \in \mathcal{S}_\omega$. We require $\mathcal{T}_{\omega,\sigma} : \overline{B} \to \overline{B}$, so we write

$$\begin{aligned} \|\mathcal{T}_{\omega,\sigma}[x] - \widehat{x}_n\| &= \|\mathcal{G}_\omega[BS_{-\sigma}x + \mathcal{F}[x]] - \widehat{x}_n\| \\ &= \|\mathcal{G}_\omega BS_{-\sigma}(x - \widehat{x}_n) + \mathcal{G}_\omega \circ \mathcal{F}[x] - \mathcal{G}_\omega \circ \mathcal{F}[\widehat{x}_n] \\ &\qquad + \mathcal{G}_\omega(BS_{-\sigma}\widehat{x}_n + \mathcal{F}[\widehat{x}_n]) - \widehat{x}_n\| \\ &\leq \|\mathcal{G}_\omega BS_{-\sigma}\|\|x - \widehat{x}_n\| + \|\mathcal{G}_\omega \circ (\mathcal{F}[x] - \mathcal{F}[\widehat{x}_n])\| \\ &\qquad + \|\mathcal{G}_\omega BS_{-\sigma}\widehat{x}_n + \mathcal{G}_\omega \circ \mathcal{F}[\widehat{x}_n] - \widehat{x}_n\|\,. \end{aligned} \tag{10}$$

If $\mathcal{G}_\omega \circ \mathcal{F}[\cdot]$ is Lipschitz with constant $\lambda_\omega(\rho)$ on the ball $\overline{B}(\widehat{x}_n, \rho)$, then

$$\begin{aligned} \|\mathcal{T}_{\omega,\sigma}[x] - \widehat{x}_n\| &\leq [\|\mathcal{G}_\omega BS_{-\sigma}\| + \lambda_\omega(\rho)]\,\|x - \widehat{x}_n\| \\ &\qquad + \|\mathcal{G}_\omega BS_{-\sigma}\widehat{x}_n + \mathcal{G}_\omega \circ \mathcal{F}[\widehat{x}_n] - \widehat{x}_n\|_\infty \\ &\equiv M_\omega\|x - \widehat{x}_n\| + N_\omega. \end{aligned} \tag{11}$$

To guarantee a fixed point in $\overline{B}$ we require

$$M_\omega\rho + N_\omega < \rho < \|\widehat{x}_n\|. \tag{12}$$

Thus we can guarantee a solution of the operator equation (5) if there exists a $\rho > 0$ such that (ρ may depend on ω).

$$\begin{aligned} &\tfrac{N_\omega}{1 - M_\omega} < \rho < \|\widehat{x}_n\| \quad \text{and} \\ &M_\omega \equiv \|\mathcal{G}_\omega BS_{-\sigma}\| + \lambda_\omega(\rho) < 1. \end{aligned} \tag{13}$$

Each fixed point $x(t)$ will yield a periodic solution to (1) with delay $\tau = \sigma/\omega$ and period $T = 2\pi/\omega$.

5. References

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