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Remarks on a Theorem of Fel'dbaum.

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Sunto. – *Si generalizza un risultato di Fel'dbaum sulla limitazione uniforme del numero degli scatti dei controlli t-ottimali per un processo di controllo lineare autonomo $\dot{x} = Ax + Bu$ al caso in cui esso non sia necessariamente normale (né completamente controllabile). Inoltre si prova che la condizione che gli autovalori della matrice A siano reali non è solo sufficiente ma anche necessaria per avere tale limitazione uniforme.*

Let us consider the minimum time problem for null controllability of the system:

$$L) \qquad \dot{x} = Ax + Bu$$

with $x \in E^n$; A, B respectively $n \times n$ and $n \times m$ real constant matrices,

$$u \in U = \{u \in L^\infty(\boldsymbol{R}^+, E^m) : u(t) \in \Omega \quad \text{a.e. } t \in \boldsymbol{R}^+\}$$

and the restraint set Ω is the unitary cube of E^m.

The minimum time problem consists in studying the existence of a control $u' \in U$ which steers the state $x_0 \in E^n$ to the origin in the minimum time T, along the solution $x(t, u')$ of $L)$ with $x(0, u') = x_0$.

That is for every $u \in U$ and the corresponding solution $x(t, u)$ of $L)$, with $x(0, u) = x_0$ one has $x(t, u) \neq 0$ for every $t \in [0, T[$ while $x(T, u') = 0$.

The control u' is called a t-optimal control.

Let b_i, $i = 1, 2, \ldots, n$, denote the columns of the matrix B and let us assume that the control process $L)$ is normal, that is:

$$\operatorname{rank}(b_i, Ab_i, \ldots, A^{n-1}b_i) = n$$

for $i = 1, 2, \ldots, m$.

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This is equivalent to assuming that for $y \in E^n$:

$$y^* \exp[-tA] b_i = 0 , \quad t \in \mathbf{R} \Rightarrow y = 0 , \quad i = 1, 2, \ldots, m.$$

(the star denotes transposition).

If the process L) is normal, then every t-optimal control has a finite number of switches on the interval $[0, T]$, whatever $T > 0$ finite.

In the general case it is impossible to find a uniform bound on the number of switches, which holds for every t-optimal control.

However, if the matrix A has only real eigenvalues, a theorem due to Fel'dbaum, gives a uniform bound on the number of switches for every t-optimal control of L) [see [1] pag. 106].

Also we know that this is impossible if A has no purely real eigenvalues even if $m = 1$ [see [6] pag. 143].

The assumption that A has no purely real eigenvalues is very strong. For instance, if the dimension of the real matrix A is odd, it is obvious that such an assumption fails.

In this paper first we generalize Fel'dbaum's theorem to a process L) which is not necessarily normal (and not even completely controllable).

Then assuming L) to be completely controllable, we can prove the existence of t-optimal controls with an arbitrary number of switches if A admits at least one eigenvalue with a non zero imaginary part.

Therefore we have shown that the condition: « the eigenvalues of the matrix A are real », is not only sufficient but also necessary to have a uniform bound on the number of switches.

We begin with the following lemma, which is easy to prove:

LEMMA. – *Let X be a convex, compact set in E^n, with a boundary ∂X. Then for any $y \in E^n$ there is at least an $x \in \partial X$ such that:*

$$(*) \qquad y^* x = \sup_{a \in X} y^* a .$$

Viceversa for any $x \in \partial X$ there is at least a vector $y \in E^n$, $y \neq 0$ which satisfies (*).

Let us now consider the control process:

$$L) \qquad \dot{x} = Ax + Bu$$

on the interval $[0, T]$, $T > 0$. The solution of L) with initial state x_0 is given by:

$$x(t) = \exp[tA]x_0 + \int_0^t \exp[(t-s)A]Bu(s)\,ds$$

Let $x(T)$ be null.

Then from the previous formula we have:

$$-x_0 = \int_0^T \exp[-sA]Bu(s)\,ds\ .$$

Let D_T denote the set of the points in E^n, which are null controllable in time $t \leqslant T$ by means of controls:

$$u \in U(T) = \{u \in L^\infty([0, T], \mathbf{R}^m) : u(t) \in \Omega \text{ a. e. } t \in [0, T]\}$$

Therefore we have:

$$D_T = \{x = -x_0 \in E^n : x = \int_0^T \exp[-sA]Bu(s)\,ds;\ u \in U(T)\}$$

D_T is compact, convex and symmetrical with respect to the origin of E^n for every $T > 0$ and if $T_1 \leqslant T_2$ then $D_{T_1} \subseteq D_{T_2}$.

Now for a fixed vector $y \in E^n$ and $T > 0$, according to the lemma, there exists at least one point $x \in \partial D_T$ for which $y^* x = \sup_{a \in D_T} y^* a$.

Every t-optimal control for such initial state is a solution of the equation:

$$y^* \exp[-tA]Bu(t) = \max_{\omega \in \Omega} y^* \exp[-tA]B\omega \qquad \text{a.e. } t \in [0, T]\ .$$

This equation is the form of the maximum principle for the process L).

The set of t-optimal controls will be denoted by $U(T, x, y)$.

We prove the following

THEOREM 1. – *Let the real constant matrix A have only real eigenvalues. Then for each point $x \in D_\infty = \bigcup_{T \geqslant 0} D_T$ there exists at least one piecewise constant control, which transfers the point x to the origin of E^n in the minimum time T with a number of switches less than an integer $N > 0$, independent of T and x.*

PROOF. – Let us consider the scalar processes:

$$L_i) \qquad \dot{x} = Ax + b_i u_i \qquad i = 1, 2, \ldots, m$$

on the interval $[0, T]$, whatever $T > 0$, where b_i is the i-th column of the matrix B and $u_i(t)$ is the i-th component of the control vector $u(t) \in U$.

Naturally $|u_i(t)| \leqslant 1$ a.e. $t \in [0, T]$. The set of functions $t \to u_i(t)$ such that $|u_i(t)| \leqslant 1$ a.e. $t \in [0, T]$ will be denoted by $U_i(T)$.

$D_T^i \subset E^n$ is the set of null controllable points in time $\leqslant T$ for $L_i)$. The properties already asserted for D_T hold for D_T^i and since $Bu = \sum_{i=1}^m b_i u_i$ we have:

$$D_T = \sum_{i=1}^m D_T^i$$

that is D_T is the algebraic sum of the sets D_T^i. Now let $y \in E^n$ be fixed and consider the sets D_T^i relative to those indices i for which:

$$(1) \qquad y^* \exp[-tA] b_i \neq 0$$

where y is the exterior normal to the supporting hyperplane for D_T^i, which has only one contact point x_i with D_T^i; the control $u_i(t)$ is uniquely defined by the maximum principle [see [1], [3], [5], [6], [7]] and we have:

$$u_i(t) = \operatorname{sgn}[y^* \exp[-tA] b_i] \,.$$

Since the eigenvalues of A are real, the piecewise constant control $u_i(t)$ has at most $n-1$ switches on $[0, T]$, for any T [see [7] pag 118].

Now let us consider the indices i for which:

$$(2) \qquad y^* \exp[-tA] b_i = 0$$

Hence D_T^i is contained in the subspace E_i of E^n spanned by the columns of the matrix $(b_i, \ldots, A^{n-1} b_i)$ and y is a normal vector to E_i.

The process $L_i)$ is completely controllable in E_i and so it is a normal process in E_i with respect to the set U_i of scalar controls. We recall that:

$$D_T^i = \bigcup_{T' \in [0, T]} D_{T'}^i \,.$$

Now we shall define a piecewise constant control for every point $x_i \in D_T^i$ on the interval $[0, T]$, in such a way that these controls have a bound for the number of switches independent of x_i.

For this purpose we fix a point $x_i \in D_T^i$. Then there exists a time $T' > 0$ and a vector $y' \in E_i$ such that the vector y' is the exterior normal to the supporting hyperplane to the set $D_{T'}^i$, [see the Lemma] which has a unique contact point x_i with $D_{T'}^i$:

The t-optimal control for this point is defined by:

$$(3) \qquad u_i(t) = \operatorname{sgn}[y'^* \exp[-tA] b_i] \qquad \text{a.e. } t \in [0, T'] .$$

Such a control is piecewise constant with no more than $n-1$ switches on $[0, T'] \subseteq [0, T]$ for any T and T'.

We define a piecewise constant t-optimal control $u_i(t)$ for the fixed point $x_i \in \partial D_{T'}^i$, on the interval $[0, T]$ by:

$$\tilde{u}_i(t) = \begin{cases} u_i(t) & \text{given by (3) for } t \in [0, T'] , \\ 0 & \text{for } t \in]T', T] . \end{cases}$$

So $\tilde{u}_i(t)$ has at most n switches on $[0, T]$, whatever $T > 0$. Define the vector:

$$x = \sum_i x_i \in \partial D_T$$

where the sum ranges over the set of indices for which relation (1) holds.

The supporting hyperplane, whose exterior normal is y, to the set D_T contains the set:

$$Q = x + \sum_i D_T^i$$

where the sum ranges over the set of indices for which (2) holds. Let $r(y)$ be their number; clearly $0 \leqslant r(y) \leqslant m$.

Every point in Q admits at least one t-optimal piecewise constant control, whose components have been defined before in the different cases (1) and (2).

Therefore it has no more than $[m - r(y)](n-1) + r(y)n = m(n-1) + r(y)$ switches on $[0, T]$, whatever $T > 0$.

Since $r(y) \leqslant m$ the theorem is proved, if we repeat the foregoing arguments for each vector $y \in E^n$, and the maximum number of switches is mn.

EXAMPLE. – Let us consider the control process in $\boldsymbol{R}^2$:

$$L)\qquad \dot{x} = Ax + Bu$$

with $u \in U$, $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for $L)$ we have in this case:

$$D_T^1 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \boldsymbol{R}^2 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\int_0^T s u_1(s)\, ds \\ \int_0^T u_1(s)\, ds \end{pmatrix} |u_1(s)| \leq 1, \text{ a.e. } s \in [0, T] \right\},$$

$$D_T^2 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \boldsymbol{R}^2 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \int_0^T u_2(s)\, ds \\ 0 \end{pmatrix} |u_2(s)| \leq 1, \text{ a.e. } s \in [0, T] \right\}.$$

By simple calculations we can determine these sets in the plane $x_1 x_2$ [see figure 1, below (for $T = 1$)].

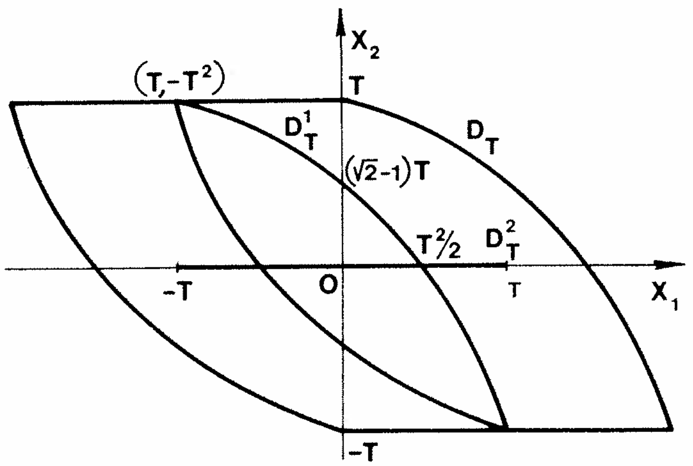

Fig. 1.

For such a control process Fel'dbaum's theorem fails, because $L)$ is not normal.

But according to theorem 1 for any $x \in \partial D_T$ and $T > 0$, there exists at least one t-optimal control with a number of switches less than a constant, independent of T and x.

We prove now the following:

THEOREM 2. – *Assume that the control process $L)$ is completely controllable. Then for every integer $N > 0$ there exists an initial state $x_0 = x_0(N)$, for which every t-optimal control has more than N switches if and only if the real constant matrix A has at least one pair of conjugate eigenvalues with a non zero imaginary part.*

PROOF. – *Necessity.* Necessity follows from *Theorem* 1.

Sufficiency. For every $T > 0$, from the lemma it follows that for every vector $y \in E^n$, $y \neq 0$, there exists at least one point $x \in \partial D_T$ which satisfies (*).

The components of the t-optimal control for this point are given by:

$$u_i(t) = \operatorname{sgn}[y^*(t) b_i] \qquad \text{a.e. } t \in [0, T]\,, \quad i = 1, 2, \ldots, m$$

where $y(t) = \exp[-tA^*]y$ is the solution of the adjoint system:

$$\dot{y}(t) + A^* y(t) = 0 \tag{4}$$

with $y(0) = y$.

From the assumption of complete controllability it follows that $\operatorname{int} D_T \neq \emptyset$, for any T.

Therefore there exists at least one index $\bar{i}$ such that the analytic function $t \to y^*(t) b_{\bar{i}}$ is not identically zero. Suppose now that the real matrix A has at least one eigenvalue with a non zero imaginary part. In this case we define a control $u(t)$, which satisfies the maximum principle and which has an infinite number of switches on $[0, +\infty)$.

In order that $u(t)$ has an infinite number of switches on $[0, +\infty)$ it suffices that the analytic function $t \to y^*(t) b_{\bar{i}}$ changes sign infinitely many times on $[0, +\infty)$. Therefore in this case the $\bar{i}$-th component will be a piecewise scalar function with an infinite number of switches.

The vector function $y(t)$ is the solution of the adjoint system (4) with $y(0) = y$.

In this way we have to show that there is a vector $y_0 \in E^n$, such that $y^*(t) b_{\bar{i}}$ has the aforementioned property if $y(0) = y_0$.

Our assumption implies that there exists at least a pair of conjugate eigenvalues of the real matrix $-A^*$ with non zero imaginary part, since the eigenvalues of $-A^*$ are those of $-A$.

Let λ, $\bar{\lambda}$ denote this pair, and let k be their common multiplicity. The linear equation:

$$(-A^* - \lambda E)^k x = 0$$

has k linearly independent solutions $x_1, x_2, \ldots, x_k$ X is the linear subspace of $\boldsymbol{C}^n$ spanned by the vectors x_i, $i = 1, 2 \ldots k$.

In the same way the linear equation:

$$(-A^* - \bar{\lambda} E)^k x = 0$$

has k linearly independent solutions $\bar{x}_i$, $i = 1, 2 \ldots k$, which are the conjugate vectors of x_i. They span the conjugate subspace of X in $\boldsymbol{C}^n$. Let it be $\bar{X}$.

X and $\bar{X}$ are invariant subspaces of $\boldsymbol{C}^n$ with respect to the transformation identified by $-A^*$, and the restriction of the transformation identified by $(-A^* - \lambda E)$ to the subspace X is nilpotent of index k.

Also the restriction of the transformation identified by $(-A^* - \bar{\lambda} E)$ to the subspace $\bar{X}$ is nilpotent of index k [see, [2], [4]]. Let us consider now the solution of (4) with non zero initial condition:

$$y_0 = x + \bar{x}\,, \qquad \text{with } x \in X\,, \quad \bar{x} \in \bar{X}\,. \tag{5}$$

y_0 is a real vector. (Note that the real vectors of the form (5) span a real space of dimension $2k$).

With respect to the decomposition (5) of the initial condition, the solution of (4) for which $y(0) = y_0$ is given by:

$$\begin{aligned} y(t) &= \exp[-tA^*]y_0 = \exp[\lambda t]\exp[-tA^* - t\lambda E]x + \\ &+ \exp[\bar{\lambda} t]\exp[-tA^* - t\bar{\lambda} E]\bar{x} = \exp[\lambda t]\left[\sum_{i=0}^{k-1}\frac{t^i}{i!}(-A^* - \lambda E)^i\right]x + \\ &+ \exp[\bar{\lambda} t]\left[\sum_{i=0}^{k-1}\frac{t^i}{i!}(-A^* - \bar{\lambda} E)^i\right]\bar{x}\,. \end{aligned} \tag{6}$$

This solution is real, for $\exp[-tA^*]$ and y_0 are real.

Put $\lambda = \alpha + i\beta$. Then the right part of (6) has components given by:

$$y_j(t) = \exp[\alpha t](P_j(t)\cos\beta t + Q_j(t)\operatorname{sen}\beta t)\,; \qquad j = 1, 2, \ldots n$$

with $P_j(t)$ and $Q_j(t)$ real polynomials of at most degree $k-1$. Let (b_i^j) $i=1,2,\dots,m$; $j=1,2,\dots,n$, be the entries of the matrix B.

So we have:

$$(7)\quad \sum_{j=1}^{n} y_j(t) b_i^j = \exp[\alpha t](P_i(t)\cos\beta t + Q_i(t)\,\mathrm{sen}\,\beta t); \qquad i=1,2\dots m\,.$$

We want to prove that the scalar function in (7), with $i=\bar{i}$, changes sign infinitely many times on $[0,+\infty)$.

In fact

$$\exp[\alpha t](P_{\bar{i}}(t)\cos\beta t + Q_{\bar{i}}(t)\,\mathrm{sen}\,\beta t) = \\ = \exp[\alpha t]\,t^h(a\cos\beta t + b\,\mathrm{sen}\,\beta t) + R(t)$$

where $h\geqslant 0$ is the greatest degree between those of $P_{\bar{i}}(t)$ and $Q_{\bar{i}}(t)$.

The function

$$T(t) = a\cos\beta t + b\,\mathrm{sen}\,\beta t$$

is not identically zero.

The function $R(t)$ is such that

$$\lim_{t\to+\infty} \exp[-\alpha t]\,t^{-h}R(t) = 0\,.$$

$T(t)$ is a periodic function with period equal to 2π.

Moreover $T(t)$ has zero mean value on $[0,+\infty)$ and so there exists a number $\varepsilon>0$ such that $T(t)$ has values greater than $\varepsilon>0$ and smaller than $-\varepsilon<0$ on every interval of lenght 2π.

Now let $\bar{t}>0$ such that

$$|\exp[-\alpha t]\,t^{-h}R(t)| < \varepsilon/2 \qquad \text{for } t\geqslant\bar{t}$$

Then the function

$$\exp[-\alpha t]\,t^{-h}\Big(\sum_{j=1}^{n} y_j(t)\,b_i^j\Big) = T(t) + \exp[-\alpha t]\,t^{-h}R(t)$$

must have a root in each intrval of length 2π, starting from the instant $\bar{t}$.

Therefore (7) changes sign infinitely many times on $[0,+\infty)$.

Let x_0 denote an intersection point of D_T with the supporting hyperplane to D_T, whose exterior normal is y_0. Then the t-optimal control for this point, whose components are given by:

$$(8) \qquad u_i(t) = \operatorname{sgn}[y^*(t)b_i] \qquad \text{a.e. } t \in [0, T],\ i = 1, 2, \ldots, m$$

where $y(t) = \exp[-tA^*]y_0$, has as its $\bar{i}$-th component, the one defined above. If follows that they have an infinite number of switches on $[0, +\infty)$. Hence for any $N > 0$, there exists a vector $y_0 \in E^n$, $y_0 \neq 0$ as in (5) and a time $T > 0$ such that every t-optimal control for the point x_0, whose components are given by (8) has more than N switches on $[0, T]$.

NOTE 1. The normality assumption on L) implies that $\forall y \in E^n$; $y \neq 0$ and $i = 1, 2, \ldots, m$

$$y^* \exp[-tA]b_i \neq 0 \qquad \text{a.e. on } [0, T], \text{ whatever } T > 0\,.$$

In this case the t-optimal controls are bang-bang controls; that is they are piecewise constant and take as values only the vertices of the unitary cube Ω. Furthermore, under such assumption, for any $x_0 \in D_\infty$, there exists one and only one t-optimal control.

NOTE 2. Remind the definition of $U(T, x, y)$.

In theorem 2 we proved that the number of switches of all the controls of the set $U(T, x_0, y_0)$, for a convenient T, is greater than an arbitrary fixed integer $N > 0$.

If the point x_0 is a corner point for the set D_T (that is, if there are more supporting hyperplanes to the set D_T through x_0) it may happen that there exists a vector y_0' which is not expressible as in (5), representing the exterior normal to one of the hyperplanes, such that the controls of the set $U(T, x_0, y_0')$ have a bounded number of switches, for any T.

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