PROCEEDINGS OF THE 25TH IEEE CONFERENCE ON DECISION AND CONTROL

DECEMBER 10—12, 1986

Atheneum Intercontinental Hotel
Athens, Greece

VOLUME 1 OF 3

86CH2344-0

WP8 - 7:00

**Proceedings of 25th Conference
on Decision and Control
Athens, Greece • December 1986**

NONLINEAR BOUNDARY VALUE CONTROL PROBLEMS

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Summary. This note deals with a general notion of controllability (called ℓ-controllability) of nonlinear control processes. Using topological degree arguments the author states sufficient conditions for local and global ℓ-controllability by means only of controls belonging to a finite dimensional subspace of $L^\infty([0,1],\mathbb{R}^m)$.

Introduction

We shall consider a general type of controllability of the nonlinear control process described by the differential system

$$\dot{x}(t) = f(t,x(t),u(t)) \quad \text{for almost all (a.a.) } t \in [0,1] \qquad (1)$$

where the function $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies Carathéodory type conditions that will be specified later.

The control law $t \to u(t)$ is a function belonging to the Banach space $L^\infty([0,1],\mathbb{R}^m)$ of the Lebesgue measurable essentially bounded functions $u : [0,1] \to \mathbb{R}^m$. The state $t \to x(t)$ of (1) is a function belonging to the Banach space $AC([0,1],\mathbb{R}^n)$ of absolutely continuous functions $x : [0,1] \to \mathbb{R}^n$. While $C([0,1],\mathbb{R}^n)$ will indicate the Banach space of continuous functions $x : [0,1] \to \mathbb{R}^n$.

In what follows $(AC)^s$, $(C)^s$ and $(L^p)^s$, $s \in N$, will denote the spaces $AC([0,1],\mathbb{R}^s)$, $C([0,1],\mathbb{R}^s)$ and $L^p([0,1],\mathbb{R}^s)$ with $1 \le p \le +\infty$ equipped with the usual norms.

In order to introduce the notion of ℓ-controllability of (1) we consider a (non necessarily linear) continuous "boundary" operator

$$\ell : (AC)^n \times (L^\infty)^m \to \mathbb{R}^k \qquad \text{with} \qquad k > n$$

mapping bounded sets into bounded sets, and for a given $y \in \mathbb{R}^k$ we will say that the system (1) is ℓ-controllable at y if the nonlinear boundary value control problem

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t)) & \text{for a.a. } t \in [0,1] \\ \ell(x,u) = y \end{cases} \qquad (2)$$

is solvable for some $u \in (L^\infty)^m$.
Furthermore, if there exists a neighborhood $N(y_0)$ of $y_0 \in \mathbb{R}^k$ such that for any $y \in N(y_0)$ problem (2) is solvable for some $u \in (L^\infty)^m$ then system (1) is said to be locally ℓ-controllable at $y_0 \in \mathbb{R}^k$. Finally, if (1) is ℓ-controllable for any $y \in \mathbb{R}^k$ then it is said to be globally ℓ-controllable. In particular, if $\ell(x,u) = (x(0),x(1)) \in \mathbb{R}^n \times \mathbb{R}^n$ where $x(1) = x(0) + \int_0^1 f(t,x(t),u(t))dt$, then the above definitions reduce to the usual definitions of controllability in $\mathbb{R}^n$.

The aim of this note is to provide sufficient conditions for local and global ℓ-controllability by means only of controls belonging to a subspace U of dimension $d = k-n$.

Tools and procedure

The approach to the previous problem is essentially based on Mawhin's coincidence topological degree (see[1] and the extensive references therein). Specifically, we rewrite the nonlinear boundary value control problem (2) as the following abstract equation

$$\Lambda(x,u) = \Psi(x,u) \qquad (3)$$

where $\Lambda,\Psi : (AC)^n \times U \to (L^1)^n \times \mathbb{R}^k$ are the operators defined by

$$\Lambda(x,u) = (L(x,u),\ell(x,u))$$

with $L(x,u)(t) = \dot{x}(t)$ for a.a. $t \in [0,1]$, and

$$\Psi(x,u) = (\Phi(x,u),y)$$

where $\Phi : (AC)^n \times U \to (L^1)^n$ is the Nemitskii operator generated by f. Ψ is a compact operators (i.e. continuous and sending bounded sets into relatively compact sets).

This abstract form of (2) is suitable for the application of topological degree arguments and, in particular, we will have the homotopy invariance property. In fact, under appropriate conditions on the asymptotic behaviour of the map $\ell|_{\mathrm{Ker}L}$ (i.e. the restriction of the boundary operator ℓ to the kernel of the operator L) which guarantee that its topological degree with respect to y is different from zero, and under certain "a priori bounds" for the solutions (x,u) of the family $\Lambda(x,u) = \lambda\Psi(x,u)$ depending on the parameter $\lambda \in [0,1]$ (homotopy), we get the solvability of (3) as

CH2344-0/86/0000-0600 $1.00 © 1986 IEEE

stated in theorem 1 below. Furthermore, in theorem 2 we state conditions for the global ℓ-controllability of (1). For a linear control process in $\mathbb{R}^n$ with a general linear boundary operator, controllability results were obtained by Marchiò[2] and for perturbed linear control processes by Denkowski and Traple[3].

Such topological arguments were previously used by the author[4] to treat periodic control problems. Moreover, by means of a suitable homotopy, Furi-Nistri-Pera and Zezza[5] obtained sufficient conditions for the global controllability of (1), using only controls belonging to an n-dimensional subspace of $(L^\infty)^m$.

ℓ-Controllability results

Assume that the function $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies the following Carathéodory conditions

(f_1) for any $(p,q) \in \mathbb{R}^n \times \mathbb{R}^m$, the mapping $t \to f(t,p,q)$ is Lebesgue measurable on $[0,1]$; for a.a. $t \in [0,1]$ the mapping $(p,q) \to f(t,p,q)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m$;

(f_2) for each $\rho > 0$ there exists $\gamma_\rho \in (L^1)^1$ such that for a.a. $t \in [0,1]$ and every (p,q) with $|p| + |q| \le \rho$ one has $|f(t,p,q)| \le \gamma_\rho(t)$.

Let U be a finite dimensional subspace of $(L^\infty)^m$ with $\dim U = k-n = d$. Choose, for simplicity, $y = 0$ in problem (2). The following theorem states the ℓ-controllability at 0 of (1).

Theorem 1. Suppose that

(i) the set

$$S = \{(x,u) \in (AC)^n \times U : (x,u) \text{ is a solution of } \dot{x}(t) = \lambda f(t,x(t),u(t)) \text{ for a.a. } t \in [0,1],\ \ell(x,u) = 0 \text{ for some } \lambda \in [0,1]\}$$

is bounded in $(C)^n \times (L^\infty)^m$.

(ii) $\deg(\ell_o,\Omega_o,0) \neq 0$, where $\ell_o = \ell\big|_{\mathbb{R}^n \times U}$ and Ω_o is any bounded open set containing $S_o = \ell_o^{-1}(0)$.
($\deg(\ell_o,\Omega_o,0)$ denotes the topological degree of the map $\ell_o : \Omega_o \to \mathbb{R}^k$ with respect to 0).
Then the nonlinear boundary value control problem (2), with $y = 0$, has a solution in $(AC)^n \times U$.

As a consequence of the above theorem we obtain the following

Corollary. Suppose that the assumptions (i)-(ii) of theorem 1 are satisfied. Then there exists $\varepsilon > 0$ such that for every $y \in \mathbb{R}^k$ with $|y| < \varepsilon$, problem (2) has a solution in $(AC)^n \times U$.
Furthermore, assume that
(iii) given $\bar{y} \in \mathbb{R}^k$, the set

$$\{(x,u) \in (AC)^n \times U : (x,u) \text{ is a solution of } \dot{x}(t) = f(t,x(t),u(t)),\ \ell(x,u) = \mu\bar{y} \text{ for some } \mu \in [0,1]\}$$

is bounded in $(C)^n \times (L^\infty)^m$.
Then (2) is solvable in $(AC)^n \times U$ for $y = \bar{y}$.

Finally the following theorem states the global ℓ-controllability of (1).

Theorem 2. Assume that assumptions (i)-(ii) of theorem 1 are satisfied. Moreover assume that

(iii) for every $M \ge 0$ the set

$$\{(x,u) \in (AC)^n \times U : (x,u) \text{ is a solution of (2) with } |y| \le M\}$$

is bounded in $(C)^n \times (L^\infty)^m$.

Then system (1) has a connected component Σ of solution (x,u) such that, for any $y \in \mathbb{R}^k$ there exists $(x,u)_y \in \Sigma$ which solves (2).

Comments

Since in the previous results we are always concerned with the assumptions involving a priori bound type conditions, it is not hard to convince oneself that it is of foundamental importance to investigate how the solutions depend on the parameters. Specifically, Lemma below states a result which enables us to deduce the boundedness of the set S from that of the parameters.

For this, let A,B be bounded open sets of $\mathbb{R}^n$ and $\mathbb{R}^d$ respectively and let $\bar{A},\bar{B}$ be their respective closures. Consider the following condition

(H) for every $c = (a,b) \in \bar{A} \times \bar{B}$ and every $\lambda \in [0,1]$ the set $X(c,\lambda) = \{x \in (AC)^n : \dot{x}(t) = \lambda f(t,x(t),u(t))$ for a.a. $t \in [0,1]$, $x(t) = a$ for some $t \in [0,1]$ and $u = J(b)\}$ is bounded in $(C)^n$. ($J : \mathbb{R}^d \to U$ being an isomorphism).

Lemma. Assume that for every pair $(x,u) \in S$ we have $x(t) \in A$ for some $t \in [0,1]$ and $J^{-1}(u) \in B$ then condition (H) implies assumption (i) of theorem 1.

Note that condition (H) is satisfied, for instance, if for any $\rho > 0$ there exist two functions $\alpha_\rho,\beta_\rho \in (L^1)^1$ such that $|f(t,p,q)| \le \alpha_\rho(t) + \beta_\rho(t)|p|$ for all $|q| \le \rho$, all $p \in \mathbb{R}^n$ and a.a. $t \in [0,1]$. Obviously, condition (H) is also verified if the set $X(c,\lambda)$ is a singleton for any $c \in \bar{A} \times \bar{B}$ and $\lambda \in [0,1]$.

Detailed proofs of the results presented in this note can be found in a more extensive paper of the author[6]. Furthermore, in that paper several conditions to ensure that assumption (ii) of theorem 1 holds are presented. Examples illustrating the theory are also included; in particular, we pointed out conditions on f and ℓ and the role of an appropriate choice of the space U of controls in order to satisfy the assumptions in the previous abstract results.

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