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Erratum to "Existence of Periodic Solutions of an Autonomous Damped Wave Equation in Thin Domains" [*J. Dynam. Diff. Eq.* 10, 409–424 (1998)]

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In the title paper [2], we studied the autonomous damped wave equation with Neumann boundary condition

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = \Delta_x u + \frac{1}{\varepsilon^2}\frac{\partial^2 u}{\partial y^2} - \beta\frac{\partial u}{\partial t} - \alpha u + g(x, \varepsilon y, u) \\ &\frac{\partial u}{\partial v} = 0 \end{aligned} \tag{1}$$

for $(x, y) \in \Omega \times (0, 1) \subset \mathbf{R}^{N+1}$, $N \geqslant 1$. Here Ω is a C^2-smooth bounded domain in $\mathbf{R}^N$, and ε is a small positive parameter. The constant α and β are taken positive, and g is a C^1-smooth function to be specified below. The scaling transformation $Y = \varepsilon y$ takes (1) into a damped wave equation with Neumann boundary condition in the "thin domain" $\Omega \times (0, \varepsilon) \ni (x, Y)$.

The following limit problem is associated to (1):

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = \Delta_x u - \beta\frac{\partial u}{\partial t} - \alpha u + g(x, 0, u) \\ &\frac{\partial u}{\partial v} = 0 \end{aligned} \tag{2}$$

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for $x \in \Omega$. In [2], we used methods of Hale and Raugel [1] together with the theory of the topological degree to prove a theorem to the following effect: suppose that $t \to u_0(t, \cdot)$ is a T_0-periodic solution of (2), and suppose that hypotheses are satisfied which ensure that u_0 has nonzero topological index with respect to a certain nonlinear operator obtained by "functionalizing the parameter T_0" in (2). Then for all small $\varepsilon \neq 0$, problem (1) admits a T_ε-periodic solution $u_\varepsilon(t, \cdot)$ such that $T_\varepsilon \to T_0$ and $u_\varepsilon \to u_0$ in an H^1-type norm as $\varepsilon \to 0$.

Unfortunately, as Prof. P. Krecji has kindly pointed out to us, neither Eq. (1) nor Eq. (2) can admit nonstationary periodic solutions. One can see this for (2), for example, by multiplying the differential equation by $\partial u_0/\partial t$, integrating the result over $\Omega \times (0, T_0)$, then observing that $\int_0^{T_0} (\partial u_0/\partial t)^2 \, dt = 0$. Thus [2] contains in effect an overly complicated proof of the existence of a family u_ε of stationary solutions of (1) which branch from a stationary solution u_0 of (2) which has nonzero topological index.

This being said (or rather, admitted), we would like to sustain the thesis that problems (1) and (2) have a structure too degenerate to permit a good illustration of the power of the methods we used in (1). To illustrate this point, we discuss some examples of autonomous problems (1) and (2) which can be treated using the methods of [2] and which give rise to a family u_ε of T_ε-periodic solutions of (1) which converge to a T_0-periodic solution u_0 of (2) ($T_0 > 0$). The nonlinearity g will be time independent but will contain arguments other than x and u.

First, let $r \neq 0$, and suppose that $g = g(x, u, u, u_r)$, where $u_r(t, \cdot) = u(t - r, \cdot)$. In this case, as observed in [3] for ordinary differential equations, the presence of the term u_r permits the existence of nonconstant periodic solutions. Precisely, we can prove the following: let $\varepsilon_0 > 0$, and suppose that $g\colon \bar{\Omega} \times [0, \varepsilon_0] \times \mathbf{R}^2 \to \mathbf{R}$ is a function which is C^1 jointly in all its arguments and which satisfies the following estimates everywhere in its domain of definition:

$$
\begin{aligned}
|g_x(x, Y, u, v)| &\leqslant a(1 + |u|^{\theta+1} + |v|^{\theta+1}) \\
|g_Y(x, Y, u, v)| &\leqslant a(1 + |u|^{\theta+1} + |v|^{\theta+1}) \\
|g_u(x, Y, u, v)| &\leqslant a(1 + |u|^{\theta} + |v|^{\theta}) \\
|g_v(x, Y, u, v)| &\leqslant a(1 + |u|^{\theta} + |v|^{\theta})
\end{aligned}
\tag{3}
$$

where $\theta \in [0, \infty)$ if $N = 1$ and $\theta \in [0, 2/(N-1))$ if $N \geqslant 2$. Using the methods of (1) we can prove the following.

Theorem A. *Let u_0 be a T_0-periodic solution of the problem*

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u - \beta \frac{\partial u}{\partial t} - \alpha u + g(x, 0, u, u_r)$$
$$\frac{\partial u}{\partial \nu} = 0 \qquad (2)_{bis}$$

for $x \in \Omega$. Suppose that the hypotheses of Theorem 1 of [2] are satisfied. Then for all small $\varepsilon > 0$, the problem

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u + \frac{1}{\varepsilon^2} \frac{\partial^2 u}{\partial y^2} - \beta \frac{\partial u}{\partial t} - \alpha u + g(x, \varepsilon y, u, u_r)$$
$$\frac{\partial u}{\partial \nu} = 0 \qquad (1)_{bis}$$

for $(x, y) \in Q$ has a T_ε-periodic solution u_ε with the following properties. First, $T_\varepsilon \to T_0$. Second, if $\tilde{u}_\varepsilon(t, \cdot) = u_\varepsilon((T_\varepsilon/T_0), \cdot)$, then

$$\left\| \begin{pmatrix} \tilde{u}_\varepsilon \\ \dfrac{\partial \tilde{u}_\varepsilon}{\partial t} \end{pmatrix} - J \begin{pmatrix} u_0 \\ \dfrac{\partial u_0}{\partial t} \end{pmatrix} \right\|_{C_{T_0}(Y^1_\varepsilon)} \to 0$$

as $\varepsilon \to 0$.

Here the spaces Y^1_ε and $C_{T_0}(Y^1_\varepsilon)$ are as defined in [2], as is the injection J: $H^1(\Omega) \times L^2(\Omega) \to Y^1_\varepsilon$. As in [2], the term "periodic solution" refers to a periodic solution of an appropriate integral equation. The proof of Theorem 1 of [2] carries over immediately to give a proof of Theorem A if we substitute for the quantity f_ε of [2] the functional

$$f_\varepsilon(T, w)(t)(x, y) = ((T/T_0)\, g(x, \varepsilon y, u(t, x, y), u_r(t, x, y)))$$

and for the quantity f_0 of [2] the functional

$$f_0(T, w)(t)(x) = ((T/T_0)\, g(x, 0, u(t, x), u_r(t, x))).$$

In fact, making these substitutions for f_ε and f_0, the proof of Theorem A is line-for-line the same as the proof of Theorem 1 in [2].

Of course, Theorem A can be generalized. For example, let

$$g = g(x, Y, u, v_1, ..., v_n) \colon \bar{\Omega} \times [0, \varepsilon_0] \times \mathbf{R}^{n+1} \to \mathbf{R}$$

be a function which is C^1 jointly in all its arguments. Suppose that the appropriate analogues of the estimates (3) are valid, in particular,

$$|g_{v_k}(x, Y, u, v_1, ..., v_n)| \leqslant a\left(1+|u|^\theta + \sum_{l=1}^{n} |v_l|^\theta\right) \qquad (k=1, 2, ..., n)$$

Let $r_1, ..., r_n$ be real numbers, and substitute $g(x, \varepsilon y, u, u_{r_1}, ..., u_{r_n})$ resp. $g(x, 0, u, u_{r_1}, ..., u_{r_n})$ in the right-hand side of $(1)_{\text{bis}}$ reps. $(2)_{\text{bis}}$. Then Theorem A is true as stated.

Other variations are possible. Let $T: H^1(Q) \to H^1(Q)$ be a bounded linear map which commutes with the orthogonal projection $P: H^1(Q) \to H^1(\Omega)$. Let $g: \bar{\Omega} \times [0, \varepsilon_0] \times \mathbf{R}^2 \to \mathbf{R}$ be a jointly C^1-function which satisfies the estimates (3). Put

$$\tilde{g}(x, Y, u) = g(x, Y, T(u))$$

then substituting $\tilde{g}(x, \varepsilon y, u)$ resp. $\tilde{g}(x, 0, u)$ in $(1)_{\text{bis}}$ resp. $(2)_{\text{bis}}$, one see that Theorem A is true as stated. One can also consider appropriate nonlinear, Fréchet-differentiable maps $T: H^1(Q) \to H^1(Q)$ in this context.

We make a final observation to the effect that our theory can be applied to systems of equations of type (1). We illustrate with the case of two equations in two unknown functions u_1, u_2. The reader will be able to generalize to the case of n equations in n unknown functions $u_1, ..., u_n$. Thus consider

$$\begin{aligned} \frac{\partial^2 u_1}{\partial t^2} &= \Delta_x u_1 + \frac{1}{\varepsilon^2}\frac{\partial^2 u_1}{\partial y^2} - \beta_1 \frac{\partial u_1}{\partial t} - \alpha_1 u_1 + g_1(x, \varepsilon y, u_1, u_2) \\ \frac{\partial^2 u_2}{\partial t^2} &= \Delta_x u_2 + \frac{1}{\varepsilon^2}\frac{\partial^2 u_2}{\partial y^2} - \beta_2 \frac{\partial u_2}{\partial t} - \alpha_2 u_2 + g_2(x, \varepsilon y, u_1, u_2) \qquad (1)_{\text{tris}} \\ \frac{\partial u_1}{\partial v} &= \frac{\partial u_2}{\partial v} = 0 \end{aligned}$$

for $(x, y) \in \Omega \times (0, \varepsilon) \subset \mathbf{R}^{N+1}$ and Ω as before. Suppose that g_1 and g_2 are of class C^1 jointly in all their arguments and that $g_i = g_i(x, Y, u_1, u_2)$ satisfy estimates of type (3) with respect to x, Y, u_1, and u_2, $(i=1, 2)$. Also suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$.

One introduces a limiting system $(2)_{\text{tris}}$ in Ω by setting $\varepsilon = 0$ in $(1)_{\text{tris}}$. One can show that, if $u^0 = \binom{u_1^0}{u_2^0}$ is a T_0-periodic solution of the limit problem which satisfies the hypotheses of Theorem 1 of [3] (when these are modified to take account of the vector nature of u^0), then $(1)_{\text{tris}}$ admits a

T_ε-periodic solution $u^\varepsilon = \binom{u_1^\varepsilon}{u_2^\varepsilon} \in Y_\varepsilon^1 \times Y_\varepsilon^1$. One has that $T_\varepsilon \to T_0$, and that the conclusion of Theorem A holds with $Y_\varepsilon^1 \times Y_\varepsilon^1$ in place of Y_ε^1.

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