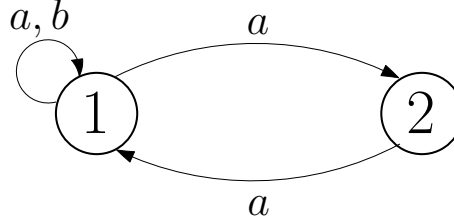


## Exercise on stochastic timed automata

Consider the stochastic timed automaton  $(\mathcal{E}, \mathcal{X}, \Gamma, p, p_{X_0}, F)$  in the figure, with  $\mathcal{E} = \{a, b\}$ ,  $\mathcal{X} = \{1, 2\}$ ,  $p(1|1, a) = 1/4$ ,  $p_{X_0}(1) = 2/3$  and  $F = \{F_a, F_b\}$ , where  $F_a$  and  $F_b$  are uniform distributions over the intervals  $[0, 4]$  and  $[0, 2]$ , respectively.



lifetime	<i>cdf</i>	<i>pdf</i>
$V_a$	$F_a(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t/4 & \text{if } 0 < t < 4 \\ 1 & \text{otherwise} \end{cases}$	$f_a(t) = \begin{cases} 1/4 & \text{if } 0 < t < 4 \\ 0 & \text{otherwise} \end{cases}$
$V_b$	$F_b(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t/2 & \text{if } 0 < t < 2 \\ 1 & \text{otherwise} \end{cases}$	$f_b(t) = \begin{cases} 1/2 & \text{if } 0 < t < 2 \\ 0 & \text{otherwise} \end{cases}$

### Problem

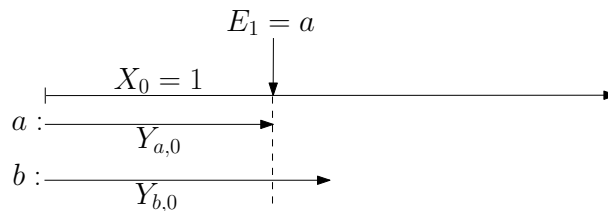
Compute  $p_{E_k}(e) \triangleq P(E_k = e)$  and  $p_{X_k}(x) \triangleq P(X_k = x)$  for all  $k = 1, 2, \dots$ ,  $e \in \mathcal{E}$  and  $x \in \mathcal{X}$ .

### Solution

We start by computing  $P(E_1 = a)$ . By applying the total probability rule with the partition of the sample space given by the possible values of the initial state, we have:

$$P(E_1 = a) = P(E_1 = a | X_0 = 1) \cdot P(X_0 = 1) + P(E_1 = a | X_0 = 2) \cdot P(X_0 = 2). \quad (1)$$

In the right-hand side of (1) some quantities are known. In particular,  $P(X_0 = 1) = p_{X_0}(1) = 2/3$  and  $P(X_0 = 2) = p_{X_0}(2) = 1 - p_{X_0}(1) = 1/3$ . Moreover,  $P(E_1 = a | X_0 = 2) = 1$ , because  $a$  is the only possible event in state 2. Hence, we have only to compute  $P(E_1 = a | X_0 = 1)$ . To do this, we represent the situation with the following sample path:



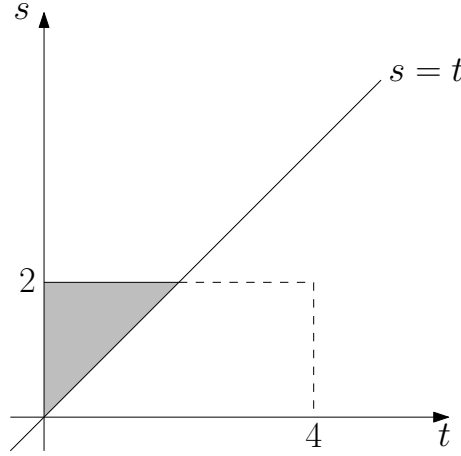
enabling us to write

$$\begin{aligned} P(E_1 = a | X_0 = 1) &= P(Y_{a,0} < Y_{b,0} | X_0 = 1) \\ &= P(V_{a,1} < V_{b,1}), \end{aligned} \quad (2)$$

where we used the fact that  $Y_{a,0} = V_{a,1}$  and  $Y_{b,0} = V_{b,1}$  at initialization starting from state  $X_0 = 1$ . Since the probability distributions of  $V_{a,1}$  and  $V_{b,1}$  are known, and lifetimes are assumed to be independent, the right-hand side of (2) can be computed as follows:

$$P(V_{a,1} < V_{b,1}) = \int \int_{\mathcal{A}} f_a(t) f_b(s) dt ds, \quad (3)$$

with  $\mathcal{A} = \{(t, s) \in \mathbb{R}^2 : t < s\}$  (the half-plane above the line  $s = t$  in the figure).



Since the joint pdf of  $V_{a,1}$  and  $V_{b,1}$  is zero outside the box  $[0, 4] \times [0, 2]$ , it turns out that the integral in (3) is to be computed only over the grey region in the figure:

$$\begin{aligned} P(V_{a,1} < V_{b,1}) &= \int_0^2 \int_0^s \frac{1}{4} \cdot \frac{1}{2} ds dt \\ &= \frac{1}{8} \int_0^2 s ds = \frac{1}{8} \left[ \frac{s^2}{2} \right]_0^2 = \frac{1}{4}. \end{aligned} \quad (4)$$

Thus we found that  $P(E_1 = a | X_0 = 1) = 1/4$ . By replacing all the probabilities in the right-hand side of (1) with their values, we get:

$$P(E_1 = a) = \frac{1}{4} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{1}{2} = 0.5. \quad (5)$$

From (5) we straightforward obtain:

$$P(E_1 = b) = 1 - P(E_1 = a) = \frac{1}{2} = 0.5. \quad (6)$$

As the next step, we compute  $P(X_1 = 1)$ . By applying again the total probability rule with the partition of the sample space given by the possible values of the initial state, we have:

$$P(X_1 = 1) = P(X_1 = 1 | X_0 = 1) \cdot P(X_0 = 1) + P(X_1 = 1 | X_0 = 2) \cdot P(X_0 = 2). \quad (7)$$

In the right-hand side of (7),  $P(X_0 = 1)$  and  $P(X_0 = 2)$  are known. Moreover,  $P(X_1 = 1|X_0 = 2) = 1$ . Hence, we have only to compute  $P(X_1 = 1|X_0 = 1)$ . To do this, we apply again the total probability rule, but with the partition of the sample space now given by the possible values of the first event. We have:

$$\begin{aligned} P(X_1 = 1|X_0 = 1) &= P(X_1 = 1|X_0 = 1, E_1 = a) \cdot P(E_1 = a|X_0 = 1) \\ &\quad + P(X_1 = 1|X_0 = 1, E_1 = b) \cdot P(E_1 = b|X_0 = 1). \end{aligned} \quad (8)$$

In the right-hand side of (8) all the quantities are known. Indeed,  $P(E_1 = a|X_0 = 1) = 1/4$  from (2) and (4). Then,  $P(E_1 = b|X_0 = 1) = 1 - P(E_1 = a|X_0 = 1) = 3/4$ . Moreover,  $P(X_1 = 1|X_0 = 1, E_1 = a) = p(1|1, a) = 1/4$ , and  $P(X_1 = 1|X_0 = 1, E_1 = b) = p(1|1, b) = 1$ . By replacing all the probabilities in the right-hand side of (8) with their values, we get:

$$P(X_1 = 1|X_0 = 1) = \frac{1}{4} \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{13}{16}. \quad (9)$$

Then, from (7) we obtain:

$$P(X_1 = 1) = \frac{13}{16} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{7}{8} = 0.875. \quad (10)$$

and:

$$P(X_1 = 2) = 1 - P(X_1 = 1) = \frac{1}{8} = 0.125. \quad (11)$$

The next step is to compute  $P(E_2 = a)$ . One would be tempted to mimic the same approach used in (1):

$$P(E_2 = a) = P(E_2 = a|X_1 = 1) \cdot P(X_1 = 1) + P(E_2 = a|X_1 = 2) \cdot P(X_1 = 2). \quad (12)$$

Some quantities in (12) are known. In particular,  $P(X_1 = 1) = 7/8$  and  $P(X_1 = 2) = 1/8$ . Moreover,  $P(E_2 = a|X_1 = 2) = 1$ . Regarding  $P(E_2 = a|X_1 = 1)$ , similarly to (2) we can write:

$$P(E_2 = a|X_1 = 1) = P(Y_{a,1} < Y_{b,1}|X_1 = 1), \quad (13)$$

where  $Y_{a,1}$  and  $Y_{b,1}$  are the residual lifetimes of events  $a$  and  $b$  after the first event. However, to compute this probability we would need the joint distribution of the two random variables, which is actually unknown. This can be understood by considering that  $Y_{a,1} = V_{a,2}$  (that is,  $Y_{a,1}$  equals a total lifetime) if the first event was  $a$ , while  $Y_{a,1} = V_{a,1} - V_{b,1}$  if the first event was  $b$ . In other words, to compute  $P(E_2 = a)$  we need more information about the past history of the system. Indeed, in (12) we only used information about the state after the first event, and no information about the first event and the initial state. This suggests to consider all the history of the system starting from initialization by partitioning the event  $\{E_2 = a\}$  into all the possible state-event sequences  $(x_0, e_1, x_1, e_2)$  with  $e_2 = a$ :

$$(i) \quad x_0 = 1 \xrightarrow{e_1=a} x_1 = 1 \xrightarrow{e_2=a}$$

$$(ii) \quad x_0 = 1 \xrightarrow{e_1=a} x_1 = 2 \xrightarrow{e_2=a}$$

$$(iii) \quad x_0 = 1 \xrightarrow{e_1=b} x_1 = 1 \xrightarrow{e_2=a}$$

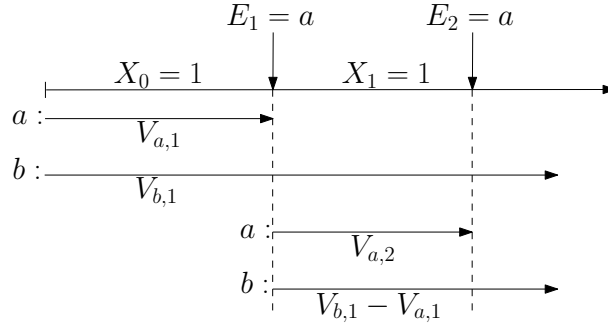
$$(iv) \quad x_0 = 2 \xrightarrow{e_1=a} x_1 = 1 \xrightarrow{e_2=a}$$

This enables us to write:

$$\begin{aligned}
P(E_2 = a) &= P(E_2 = a, X_1 = 1, E_1 = a, X_0 = 1) \\
&\quad + P(E_2 = a, X_1 = 2, E_1 = a, X_0 = 1) \\
&\quad + P(E_2 = a, X_1 = 1, E_1 = b, X_0 = 1) \\
&\quad + P(E_2 = a, X_1 = 1, E_1 = a, X_0 = 2) \\
&= P(E_2 = a, X_1 = 1, E_1 = a | X_0 = 1) \cdot P(X_0 = 1) \\
&\quad + P(E_2 = a, X_1 = 2, E_1 = a | X_0 = 1) \cdot P(X_0 = 1) \\
&\quad + P(E_2 = a, X_1 = 1, E_1 = b | X_0 = 1) \cdot P(X_0 = 1) \\
&\quad + P(E_2 = a, X_1 = 1, E_1 = a | X_0 = 2) \cdot P(X_0 = 2). \tag{14}
\end{aligned}$$

In (14),  $P(X_0 = 1)$  and  $P(X_0 = 2)$  are known. To compute the conditional probabilities, we follow the same approach based on sample paths used in (2).

- $P(E_2 = a, X_1 = 1, E_1 = a | X_0 = 1)$ . The sample path is as follows:

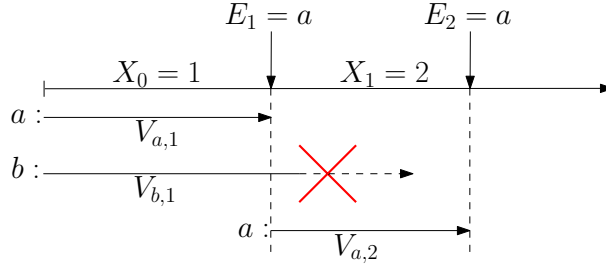


Hence,

$$\begin{aligned}
P(E_2 = a, X_1 = 1, E_1 = a | X_0 = 1) &= p(1|1, a) \cdot P(V_{a,1} < V_{b,1}, V_{a,2} < V_{b,1} - V_{a,1}) \\
&= \underbrace{p(1|1, a)}_{1/4} \cdot \underbrace{P(V_{a,1} + V_{a,2} < V_{b,1})}_{1/24} = \frac{1}{96}, \tag{15}
\end{aligned}$$

where we used the fact that, since lifetimes are nonnegative numbers,  $V_{a,1} + V_{a,2} < V_{b,1}$  implies  $V_{a,1} < V_{b,1}$ .

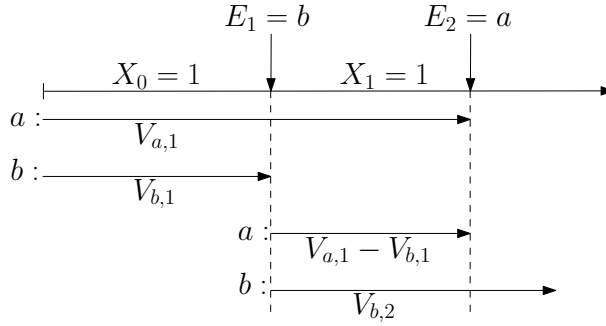
- $P(E_2 = a, X_1 = 2, E_1 = a | X_0 = 1)$ . The sample path is as follows:



Hence,

$$P(E_2 = a, X_1 = 2, E_1 = a | X_0 = 1) = \underbrace{p(2|1, a)}_{3/4} \cdot \underbrace{P(V_{a,1} < V_{b,1})}_{1/4} = \frac{3}{16}. \quad (16)$$

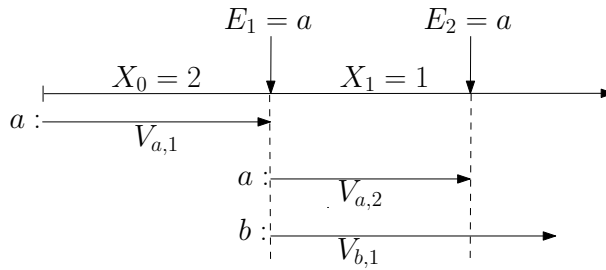
- $P(E_2 = a, X_1 = 1, E_1 = b | X_0 = 1)$ . The sample path is as follows:



Hence,

$$\begin{aligned} P(E_2 = a, X_1 = 1, E_1 = b | X_0 = 1) &= p(1|1, b) \cdot P(V_{b,1} < V_{a,1}, V_{a,1} - V_{b,1} < V_{b,2}) \\ &= \underbrace{p(1|1, b)}_1 \cdot \underbrace{P(V_{b,1} < V_{a,1} < V_{b,1} + V_{b,2})}_{1/4} = \frac{1}{4}. \end{aligned} \quad (17)$$

- $P(E_2 = a, X_1 = 1, E_1 = a | X_0 = 2)$ . The sample path is as follows:



Hence,

$$P(E_2 = a, X_1 = 1, E_1 = a | X_0 = 2) = \underbrace{p(1|2, a)}_1 \cdot \underbrace{P(V_{a,2} < V_{b,1})}_{1/4} = \frac{1}{4}. \quad (18)$$

Notice that, to compute the probabilities in (15)-(18), one has to solve multiple integrals in two or three dimensions. By replacing all the probabilities in the right-hand side of (14) with their values, we get:

$$P(E_2 = a) = \frac{1}{96} \cdot \frac{2}{3} + \frac{3}{16} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{3} = \frac{55}{144} \simeq 0.3819. \quad (19)$$

From (19) we straightforward obtain:

$$P(E_2 = b) = 1 - P(E_2 = a) = \frac{89}{144} \simeq 0.6181. \quad (20)$$

Then we should continue by computing  $P(X_2 = 1)$ ,  $P(X_2 = 2)$ ,  $P(E_3 = a)$ ,  $P(E_3 = b)$ , and so on.

**Remark 1** Notice that the approach based on the total probability rule used in (1) can be reinterpreted in view of what we did to obtain (14). By partitioning the event  $\{E_1 = a\}$  into all the possible state-event sequences  $(x_0, e_1)$  with  $e_1 = a$ :

$$(i) \quad x_0 = 1 \xrightarrow{e_1=a}$$

$$(ii) \quad x_0 = 2 \xrightarrow{e_1=a},$$

we can write

$$\begin{aligned} P(E_1 = a) &= P(E_1 = a, X_0 = 1) + P(E_1 = a, X_0 = 2) \\ &= P(E_1 = a|X_0 = 1) \cdot P(X_0 = 1) + P(E_1 = a|X_0 = 2) \cdot P(X_0 = 2). \end{aligned} \quad (21)$$

## Conclusion

It is easy to figure out from this example that the complexity of computing analytically the probabilities of the type  $P(E_k = e)$  and  $P(X_k = x)$  increases in general with the event index  $k$ , because we need to consider more and more cases, and compute integrals in larger and larger dimensions. The take-away message of this example is therefore that *analysis of stochastic timed automata* has to be tackled differently in practice. Possible approaches are:

- use Monte Carlo methods;
- find special cases for which analytical computations are easy.

We will see that the second approach is based on the use of exponential distributions for the lifetimes of the events.