Test of Discrete Event Systems - 07.01.2021

Additional exercises

Exercise 1

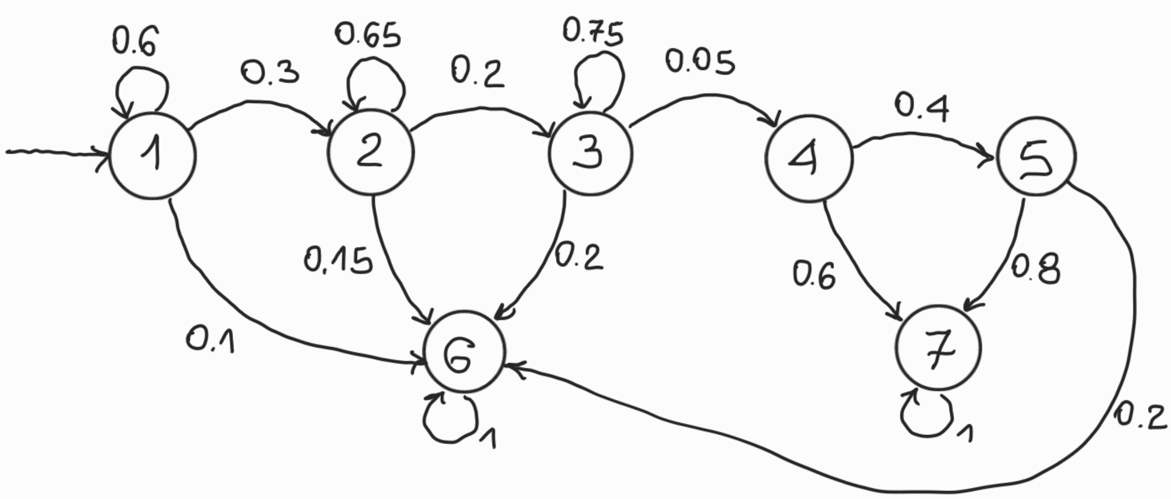
The students of an English School are organized into four classes, labeled 1 (starting class), 2, 3, and 4. At the end of a week, a student may get promoted from one class to the next, or decide to leave the school. A student in class 1 gets promoted to class 2 with probability 0.3, leaves the school with probability 0.1, or continues in the same class at the beginning of the next week. A student in class 2 gets promoted to class 3 with probability 0.2, leaves the school with probability 0.15, or continues in the same class at the beginning of the next week. A student in class 3 gets promoted to class 4 with probability 0.05, leaves the school with probability 0.2, or continues in the same class at the beginning of the next week. Finally, a student in class 4 for the first time either leaves the school successfully with probability 0.6 or continues an additional week in the same class. At the end of the second week, the student always leaves the school: the probability of success at the end of the second week is 0.8.

1. Model the dynamics of the student career in the school using a discrete-time homogeneous Markov chain.
2. Compute the probability that a new student in class 1 reaches class 4 in at most eight weeks.
3. Compute the probability that a student completes successfully the school.
4. Compute the average number of weeks spent by a generic student in the school.

EXERCISE 1

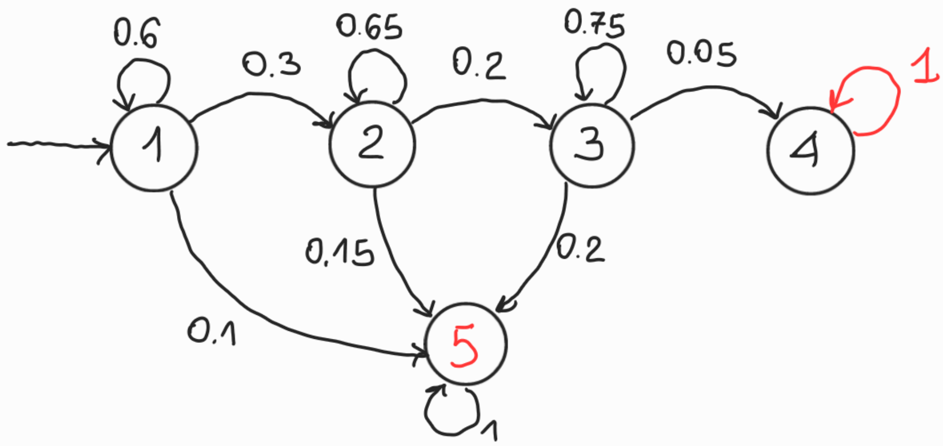
1. State:

$$x = \begin{cases} 1: \text{student in } 1^{st} \text{ class} \\ 2: \text{student in } 2^{nd} \text{ class} \\ 3: \text{student in } 3^{rd} \text{ class} \\ 4: \text{student in } 4^{th} \text{ class (first week)} \\ 5: \text{student in } 5^{th} \text{ class (second week)} \\ 6: \text{school left unsuccessfully} \\ 7: \text{school completed successfully} \end{cases}$$



$$P = \begin{bmatrix} 0.6 & 0.3 & 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0.65 & 0.2 & 0 & 0 & 0.15 & 0 \\ 0 & 0 & 0.75 & 0.05 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \pi_0 = [1\ 0\ 0\ 0\ 0\ 0\ 0]$$

2. We modify the model by making state 4 absorbing.



Using this model, the answer is:

$$P(\tilde{X}(8)=4) = \tilde{\pi}_4(8) = \tilde{\pi}_0 \tilde{P}^8 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \simeq 0.0435$$

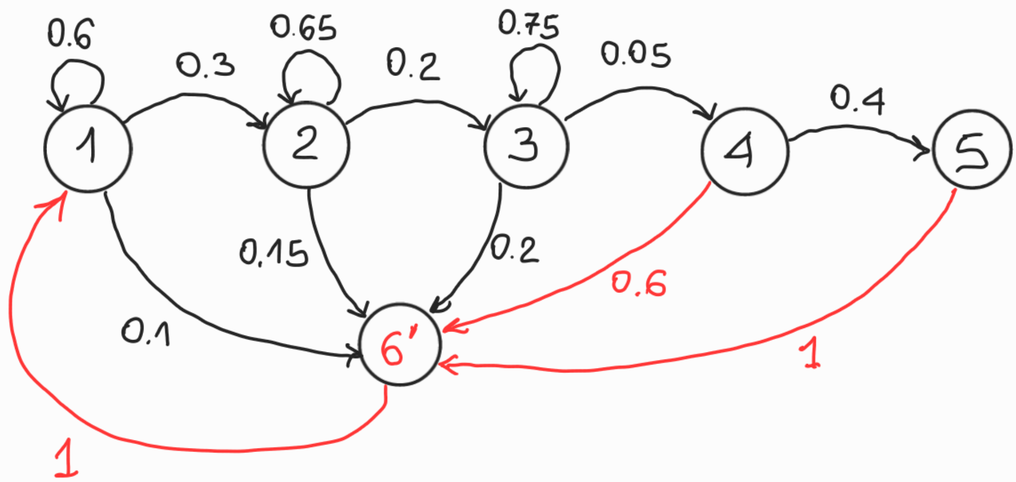
where

$$\tilde{P} = \begin{bmatrix} 0.6 & 0.3 & 0 & 0 & 0.1 \\ 0 & 0.65 & 0.2 & 0 & 0.15 \\ 0 & 0 & 0.75 & 0.05 & 0.2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{\pi}_0 = [1\ 0\ 0\ 0\ 0]$$

3. Using the model of point 1, the answer is

$$\pi_7 = \lim_{t\to\infty} \pi_7(t) = \lim_{t\to\infty} \pi_0 P^t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \simeq 0.0789$$

4. We modify the model of point 1 by merging states 6 and 7 into state $6'$, and adding a deterministic transition from state $6'$ to state 1.



The modified Markov chain is irreducible, finite and aperiodic.

Hence, using this model, the answer is:

$$E[T_{1,6'}] = \underbrace{E[T_{6',6'}]}_{= \frac{1}{\bar{\pi}_{6'}}} - 1 \simeq 6.4771$$

$$\text{where } \bar{\pi}_{6'} = \lim_{t\to\infty} \bar{\pi}_{6'}(t) = \lim_{t\to\infty} \bar{\pi}_0 \bar{P}^t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \simeq 0.1337$$

$$\bar{P} = \begin{bmatrix} 0.6 & 0.3 & 0 & 0 & 0 & 0.1 \\ 0 & 0.65 & 0.2 & 0 & 0 & 0.15 \\ 0 & 0 & 0.75 & 0.05 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(independent of $\bar{\pi}_0$)