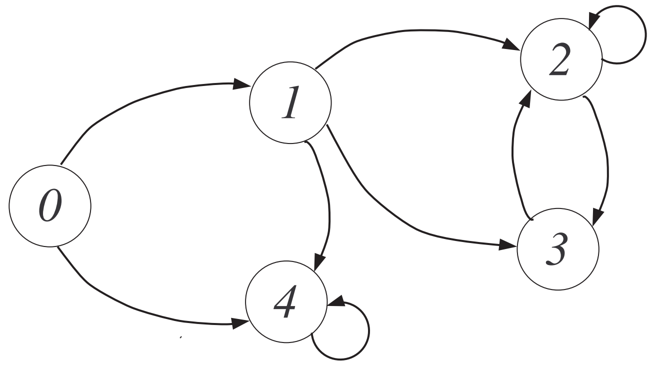
Test of Discrete Event Systems - 21.12.2020

Additional exercises

Exercise 1

Consider the discrete-time homogeneous Markov chain whose state transition diagram is represented in the figure, and with transition probabilities $p_{0,1} = 1/3$, $p_{1,2} = 1/8$, $p_{1,3} = 1/4$ and $p_{2,3} = 4/5$.



1. Compute the average recurrence time for each recurrent state.

2. Compute the stationary probabilities of all states, assuming the probability vector of the initial state $\pi(0) = [\ 1\ 0\ 0\ 0\ 0\ ]$.

Exercise 2

A simplified telephone call process in discrete time over a single telephone line works as follows.

- At most one telephone call can go through the line in a single time slot.
- The probability of an incoming call during a time slot is α. Assume that an incoming call may arrive only at the end of a time slot. If the line is busy, the call is lost; otherwise, the call is processed.
- The probability that a call in process terminates in any one time slot is β.

Assume that $\alpha = 1/3$ and $\beta = 1/2$.

1. Model the telephone call process through a discrete-time homogeneous Markov chain.

2. Compute the utilization of the telephone line at steady state.

3. Assume that the telephone line is idle. Compute the probability that the telephone line is busy at least 30% of the time over the next 10 time slots.

Exercise 3

A small warehouse may contain up to three pallets. Every hour a truck comes to collect stored pallets. Depending on the space available on it, the maximum number of pallets that the truck may collect is 0 with probability $p_0 = 0.1$, 1 with probability $p_1 = 0.2$, 2 with probability $p_2 = 0.4$ and 3 otherwise. The truck always collects as many pallets as possible compatibly with this constraint. The number of pallets shipped to the warehouse during one hour is a random variable taking values 0 with probability $q_0 = 0.3$, 1 with probability $q_1 = 0.4$ and 2 otherwise. Pallets arriving when the warehouse is full are rejected. Assume that pallet loading and unloading times are negligible.

1. Model the system through a discrete-time homogeneous Markov chain.
2. In steady state condition, compute the average number of pallets in the warehouse after a truck departure.
3. Compute the probability that the number of pallets in the warehouse after a truck departure is two for exactly three consecutive times.

In all the following questions, assume that the warehouse is initially empty.

4. Compute the probability that the number of pallets in the warehouse is never two after each of the first eight truck departures.
5. Compute the average number of hours to have the warehouse full after a truck departure.
6. Compute the probability that the warehouse will be full after a truck departure before it will be empty again.

Exercise 1

1. The recurrent states are 2,3 and 4. States 0 and 1 are transient.

In order to compute $E[T_{2,2}]$ and $E[T_{3,3}]$, we observe that states 2 and 3 form a closed subset which is irreducible, aperiodic and finite, with transition probability matrix:

$$\tilde{P} = \begin{bmatrix} p_{2,2} & p_{2,3} \\ p_{3,2} & p_{3,3} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ 1 & 0 \end{bmatrix}$$

Solving:

$$\begin{cases} \tilde{\pi} = \tilde{\pi}\tilde{P} \\ \tilde{\pi}_2 + \tilde{\pi}_3 = 1 \end{cases}$$

we obtain $\tilde{\pi} = \left[\frac{5}{9} \; \frac{4}{9} \right]$. Therefore:

$$E[T_{2,2}] = \frac{1}{\left(\frac{5}{9}\right)} = \frac{9}{5} = 1.80$$

$$E[T_{3,3}] = \frac{1}{\left(\frac{4}{9}\right)} = \frac{9}{4} = 2.25$$

State 4 is absorbing, therefore:

$E[T_{4,4}] = 1.$

2. The transition probability matrix of the Markov chain is

$$P = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{5}{8} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We are asked to compute

$$\pi = \lim_{t\to\infty} \pi(0) P^t, \quad \text{with } \pi(0) = [1\ 0\ 0\ 0\ 0].$$

Notice that the Markov chain is <u>non-irreducible!</u>

A possible way to circumvent the computation of the limit, is as follows:

$$\left.\begin{array}{l}\pi_0 = 0\\ \pi_1 = 0\end{array}\right\}\text{ states 0 and 1 are transient}$$

$$\pi_2 = \underbrace{P(\text{the chain enters the closed subset } \{2,3\})}_{\substack{P(0\to 1)\left[P(1\to 2)+P(1\to 3)\right] \\ = \frac{1}{3}\left(\frac{1}{8}+\frac{1}{4}\right)=\frac{1}{8}}} \cdot \tilde{\pi}_2 = \frac{1}{8}\cdot\frac{5}{9} = \frac{5}{72}$$

computed in item 1: $\tilde{\pi}_2 = \frac{5}{9}$

$$\pi_3 = P(\text{the chain enters the closed subset } \{2,3\}) \cdot \tilde{\pi}_3 = \frac{1}{8}\cdot\frac{4}{9} = \frac{1}{18}$$

computed in item 1: $\tilde{\pi}_3 = \frac{4}{9}$

$$\pi_4 = P(\text{the chain enters the closed subset } \{4\}) =$$

$$= P(0\to 1)P(1\to 4) + P(0\to 4) = \frac{1}{3}\cdot\frac{5}{8} + \frac{2}{3} = \frac{7}{8}.$$

Therefore,

$$\pi = \left[0 \quad 0 \quad \frac{5}{72} \quad \frac{1}{18} \quad \frac{7}{8}\right].$$

<u>Remark</u>: Since the Markov chain is non-irreducible, it was not possible to compute π by solving $\begin{cases}\pi = \pi P \\ \sum_{i=0}^{4} \pi_i = 1\end{cases}$. Indeed,

$$\begin{cases} 0 = \pi_0 \\ \frac{1}{3}\pi_0 = \pi_1 \\ \frac{1}{8}\pi_1 + \frac{1}{5}\pi_2 + \pi_3 = \pi_2 \\ \frac{1}{4}\pi_1 + \frac{4}{5}\pi_2 = \pi_3 \\ \frac{2}{3}\pi_0 + \frac{5}{8}\pi_1 + \pi_4 = \pi_4 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases} \Rightarrow \begin{cases} \pi_0 = 0 \\ \pi_1 = 0 \\ \frac{4}{5}\pi_2 = \pi_3 \\ \cancel{\frac{4}{5}\pi_2 = \pi_3} \quad \text{redundant} \\ \pi_4 = \pi_4 \quad \Rightarrow \pi_4 \text{ can be chosen arbitrarily: } \pi_4 = \gamma,\ \gamma\in[0,1] \\ \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

$$\Rightarrow \pi_2 + \frac{4}{5}\pi_2 + \gamma = 1 \Rightarrow \pi_2 = \frac{5}{9}(1-\gamma), \quad \pi_3 = \frac{4}{9}(1-\gamma)$$

It follows that the system of equations:

$$\begin{cases} \pi = \pi P \\ \sum_{i=0}^{4} \pi_i = 1 \end{cases}$$

has infinite solutions parameterized by $\gamma \in [0,1]$.

Notice that γ can be interpreted as the probability that the chain enters the closed subset $\{4\}$, and therefore $1-\gamma$ is the probability that the chain enters the closed subset $\{2,3\}$. Some examples:

- if the initial state is 0, then $\gamma = \frac{7}{8}$, and therefore $1-\gamma = \frac{1}{8}$;
- if the initial state is either 2 or 3, then $\gamma = 0$, and therefore $1-\gamma = 1$,

ecc.

Exercise 2

1. state

$$x = \begin{cases} 0: & \text{line idle} \\ 1: & \text{line busy} \end{cases}$$

transition probabilities

$$p_{0,0} = P(X(t+1)=0 \mid X(t)=0)$$
$$= P(\text{no incoming call} \mid \text{line idle}) = 1-\alpha$$

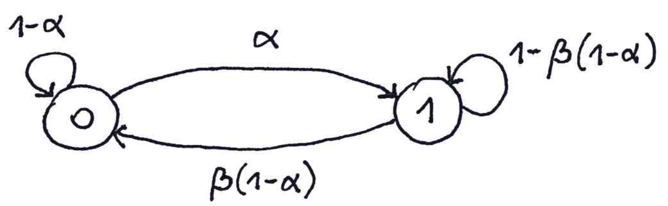
$$p_{0,1} = P(X(t+1)=1 \mid X(t)=0)$$
$$= P(\text{one incoming call} \mid \text{line idle}) = \alpha$$

$$p_{1,0} = P(X(t+1)=0 \mid X(t)=1)$$
$$= P(\text{termination of a call } \underline{\text{AND}} \text{ no incoming call} \mid \text{line busy}) = \beta \cdot (1-\alpha)$$

$$p_{1,1} = P(X(t+1)=1 \mid X(t)=1)$$
$$= P((\text{no termination of a call}) \ \underline{\text{OR}} \ (\text{termination of a call } \underline{\text{AND}} \text{ one incoming call}) \mid \text{line busy})$$
$$= (1-\beta) + \beta \cdot \alpha = 1 - \beta(1-\alpha)$$

$$\Rightarrow P = \begin{bmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & p_{1,1} \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta(1-\alpha) & 1-\beta(1-\alpha) \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

The initial state probability vector is unspecified.



2. The utilization of the telephone line at steady state is the stationary probability of state 1.

The Markov chain is irreducible, aperiodic and finite.

$$\Rightarrow \begin{cases} \pi = \pi P \\ \pi_0 + \pi_1 = 1 \end{cases} \quad \text{where } \pi = [\pi_0 \ \ \pi_1]$$

$$\begin{cases} \pi_0 = \frac{2}{3}\pi_0 + \frac{1}{3}\pi_1 \\ \cancel{\pi_1 = \frac{1}{3}\pi_0 + \frac{2}{3}\pi_1} \text{ redundant} \\ \pi_0 + \pi_1 = 1 \end{cases} \qquad \begin{cases} \pi_1 = \pi_0 \\ 2\pi_0 = 1 \end{cases} \Rightarrow \begin{cases} \pi_0 = \frac{1}{2} \\ \pi_1 = \frac{1}{2} \end{cases}$$

$$\Rightarrow U_{line} = \pi_1 = 0.5 = 50\%$$

utilization of the line

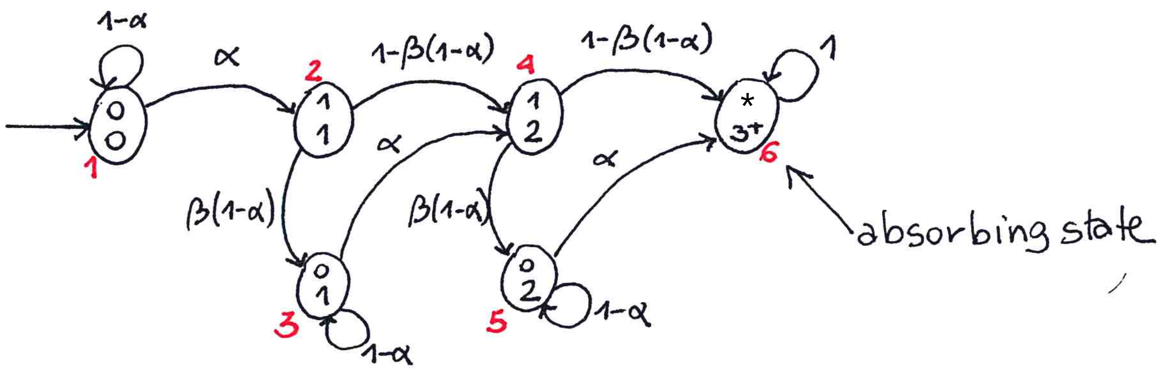
3. The telephone line is initially idle.

busy at least 30% of the time over the next 10 time slots

$\equiv$ busy at least 3 time slots over the next 10 time slots

We modify the previous model as follows:

state $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $x_1 \rightarrow$ line: 0 (idle), 1 (busy); $x_2 \rightarrow$ counter of time slots spent in state 'busy' $\in \{0, 1, 2, 3^+\}$ (3 or more)



$$\tilde{\pi}(0) = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$$

$$\tilde{P} = \begin{bmatrix} 1-\alpha & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta(1-\alpha) & 1-\beta(1-\alpha) & 0 & 0 \\ 0 & 0 & 1-\alpha & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta(1-\alpha) & 1-\beta(1-\alpha) \\ 0 & 0 & 0 & 0 & 1-\alpha & \alpha \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

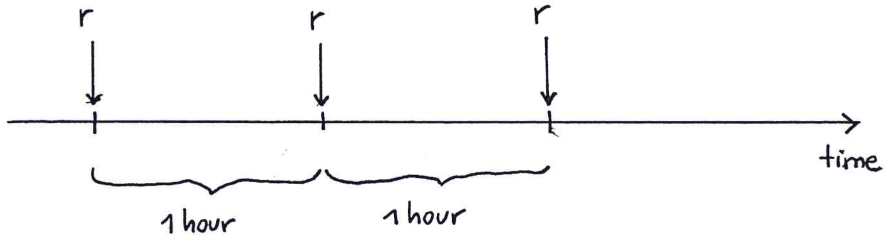
Using this model, the answer to the question is:

$$P(\tilde{X}(10) = 6) = \tilde{\pi}(0)\tilde{P}^{10}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \simeq 0.8439$$

because state 6 is absorbing.

EXERCISE 3

1.



Since pallet loading times are negligible, return and departure of the track coincide (event r). Let the state of the system be defined as

X = # pallets in the warehouse immediately after truck departure $\in \{0,1,2,3\}$

Transition probabilities are as follows:

$p_{0,0} = q_0 + q_1(p_1+p_2+p_3) + q_2(p_2+p_3) = 0.87$

$p_{0,1} = q_1 p_0 + q_2 p_1 = 0.1$

$p_{0,2} = q_2 p_0 = 0.03$

$p_{0,3} = 0$

$p_{1,0} = q_0(p_1+p_2+p_3) + q_1(p_2+p_3) + q_2 p_3 = 0.64$

$p_{1,1} = q_0 p_0 + q_1 p_1 + q_2 p_2 = 0.23$

$p_{1,2} = q_1 p_0 + q_2 p_1 = 0.1$

$p_{1,3} = q_2 p_0 = 0.03$

$p_{2,0} = q_0(p_2+p_3) + (q_1+q_2)p_3 = 0.42$

$p_{2,1} = q_0 p_1 + (q_1+q_2)p_2 = 0.34$

$p_{2,2} = q_0 p_0 + (q_1+q_2)p_1 = 0.17$

$p_{2,3} = (q_1+q_2)p_0 = 0.07$

$p_{3,0} = p_3 = 0.3$

$p_{3,1} = p_2 = 0.4$

$p_{3,2} = p_1 = 0.2$

$p_{3,3} = p_0 = 0.1$

$$\Rightarrow P = \begin{bmatrix} p_{0,0} & p_{0,1} & p_{0,2} & p_{0,3} \\ p_{1,0} & p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,0} & p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,0} & p_{3,1} & p_{3,2} & p_{3,3} \end{bmatrix}$$

2. The DTHMC is irreducible, aperiodic and finite.
The limit probabilities can be computed by solving

$$\begin{cases} \pi P = \pi \\ \sum_{i=0}^{3} \pi_i = 1 \end{cases}$$

where $\pi = [\pi_0 \ \pi_1 \ \pi_2 \ \pi_3]$. It turns out that:

$\pi \simeq [0.8142 \quad 0.1307 \quad 0.0471 \quad 0.0080]$ (with Matlab)

$\Rightarrow E[X] = 0 \cdot \pi_0 + 1 \cdot \pi_1 + 2 \cdot \pi_2 + 3 \cdot \pi_3 \simeq 0.2490$

3. $P(V(2)=3) = (1-p_{2,2})\,p_{2,2}^2 \simeq 0.0240$

4. We modify the model by making state 2 absorbing:

$$\tilde{P} = \begin{bmatrix} p_{0,0} & p_{0,1} & p_{0,2} & p_{0,3} \\ p_{1,0} & p_{1,1} & p_{1,2} & p_{1,3} \\ 0 & 0 & 1 & 0 \\ p_{3,0} & p_{3,1} & p_{3,2} & p_{3,3} \end{bmatrix} \quad \text{with } \tilde{\pi}(0) = [1 \ 0 \ 0 \ 0]$$

We want to compute

$$P(\tilde{X}(8) \neq 2) = 1 - P(\tilde{X}(8) = 2) = 1 - \tilde{\pi}(0)\,\tilde{P}^8 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \simeq 0.7349$$

5. We modify the model by adding a deterministic transition from state 3 to state 0:

$$\bar{P} = \begin{bmatrix} p_{0,0} & p_{0,1} & p_{0,2} & p_{0,3} \\ p_{1,0} & p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,0} & p_{2,1} & p_{2,2} & p_{2,3} \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The new model is irreducible, aperiodic and finite.

The limit probabilities can be computed by solving:

$$\begin{cases} \bar{\pi}\bar{P} = \bar{\pi} \\ \sum_{i=0}^{3} \bar{\pi}_i = 1 \end{cases} \Rightarrow \bar{\pi} \simeq [0.8216 \quad 0.1265 \quad 0.0449 \quad 0.0069]$$

where $\bar{\pi} = [\bar{\pi}_0 \ \bar{\pi}_1 \ \bar{\pi}_2 \ \bar{\pi}_3]$. Moreover, $E[\bar{T}_{3,3}] = \frac{1}{\bar{\pi}_3}$.

recurrence time of state 3 in the new model

Since $\bar{T}_{3,3} = 1 + T_{0,3}$,

number of steps from state 0 to state 3 in both models

we have: $E[T_{0,3}] = E[\bar{T}_{3,3}] - 1 = \frac{1}{\bar{\pi}_3} - 1 \simeq 143.0471$.

6. We modify the model by duplicating state 0 (new state 4) and by making states 3 and 4 absorbing:

$$\hat{P} = \left[\begin{array}{cccc|c} 0 & p_{0,1} & p_{0,2} & p_{0,3} & p_{0,0} \\ 0 & p_{1,1} & p_{1,2} & p_{1,3} & p_{1,0} \\ 0 & p_{2,1} & p_{2,2} & p_{2,3} & p_{2,0} \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

We want to compute

$$\lim_{t\to\infty} P(\hat{X}(t) = 3) \qquad \text{with } \hat{\pi}(0) = [1 \ 0 \ 0 \ 0 \ 0].$$

$$\Rightarrow \lim_{t\to\infty} \hat{\pi}(0)\hat{P}^t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \simeq 0.0084$$