Test of Discrete Event Systems - 21.12.2020

Exercise 1

An electronic device driving the opening of a safe generates one of three numbers, 0, 1, or 2, according to the following rules:

i) if 0 was generated last, then the next number is 0 again with probability 1/2 or 1 with probability 1/2;

ii) if 1 was generated last, then the next number is 1 again with probability 2/5 or 2 with probability 3/5;

iii) if 2 was generated last, then the next number is either 0 with probability 7/10 or 1 with probability 3/10.

Moreover, the first generated number is 0 with probability 3/10, 1 with probability 3/10, and 2 with probability 2/5. The safe is opened the first time the sequence 120 takes place.

1. Define a discrete time Markov chain for the above described opening mechanism of the safe.
2. Compute the probability that the safe is opened with the fourth generated number.
3. Compute the average length of the sequence generated to open the safe.

Exercise 2

A study of the strengths of the basketball teams of the main American universities shows that, if a university has a strong team one year, it is equally likely to have a strong team or an average team next year; if it has an average team, it is average next year with 50% probability, and if it changes, it is just as likely to become strong as weak; if it is weak, it has 65% probability of remaining so, otherwise it becomes average.

1. What is the probability that a team is strong on the long run?
2. Assume that a team is strong. Compute the probability that it is strong for at least three consecutive years.
3. Assume that a team is weak. How many years are needed on average for it to become strong?
4. Assume that a team is weak. Compute the probability that, in the next ten years, it is strong at least two years.

Exercise 1

1. ## model #1

We define the following values for the state:

1 : last number is $\emptyset$, not preceded by the subsequence 12

2 : last number is 1

3 : last number is 2, not preceded by 1

4 : last two numbers are 1 and 2 (subsequence 12)

5 : last three numbers are 1, 2 and $\emptyset$ (subsequence 120) => the safe is opened

The problem description provides the following conditional probabilities:

$$p(0|0) = \frac{1}{2}, \quad p(1|0) = \frac{1}{2}, \quad p(1|1) = \frac{2}{5}, \quad p(2|1) = \frac{3}{5},$$

next number   last number

$$p(0|2) = \frac{7}{10}, \quad p(1|2) = \frac{3}{10}$$

Therefore, we have:

$$p_{1,1} = p(0|0) = \frac{1}{2}, \quad p_{1,2} = p(1|0) = \frac{1}{2}$$

$$p_{2,2} = p(1|1) = \frac{2}{5}, \quad p_{2,4} = p(2|1) = \frac{3}{5}$$

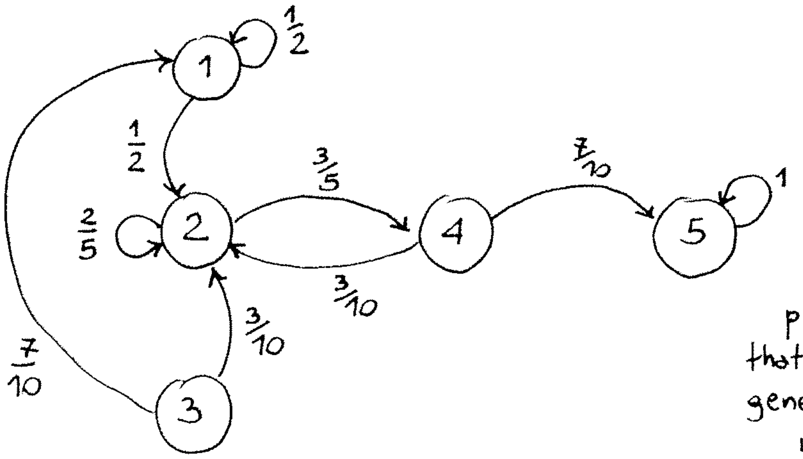
$$p_{3,1} = p(0|2) = \frac{7}{10}, \quad p_{3,2} = p(1|2) = \frac{3}{10}$$

$$p_{4,2} = p(1|2) = \frac{3}{10}, \quad p_{4,5} = p(0|2) = \frac{7}{10}$$

$p_{5,5} = 1$ (the safe is opened)

All other transition probabilities are $\emptyset$.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 \\ \frac{7}{10} & \frac{3}{10} & 0 & 0 & 0 \\ 0 & \frac{3}{10} & 0 & 0 & \frac{7}{10} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



The initial state probability vector is:

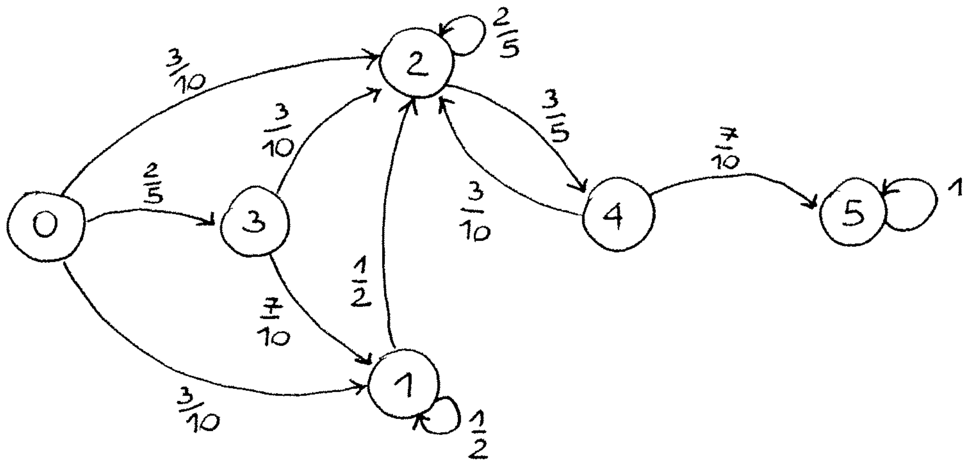
$$\Pi_0 = \left[\frac{3}{10} \;\; \frac{3}{10} \;\; \frac{2}{5} \;\; 0 \;\; 0 \right]$$

probability that the first generated number is $\emptyset$

etc.

model #2

Another model can be derived from model #1 by adding an initial state 0 corresponding to the fact that no number has been generated yet:



For this model, the matrix P is as follows:

$$P = \begin{bmatrix} 0 & \frac{3}{10} & \frac{3}{10} & \frac{2}{5} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 \\ 0 & \frac{7}{10} & \frac{3}{10} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{10} & 0 & 0 & \frac{7}{10} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the initial state probability vector is $\Pi_0 = [1\ 0\ 0\ 0\ 0\ 0]$.

2. Using model #1, the answer is

$$P(X(3)=5, X(2)\neq 5) = P(X(3)=5, X(2)=4) = P(X(3)=5 \mid X(2)=4)\, P(X(2)=4)$$

Note that, for model #1, time t = (number of generated numbers − 1)

$$= p_{4,5} \cdot \Pi_4(2)$$

where $p_{4,5} = \frac{7}{10}$ and $\Pi_4(2)$ can be computed through

$\Pi(2) = \Pi_0 P^2$, where $\Pi(2) = [\,\Pi_1(2)\ \Pi_2(2)\ \Pi_3(2)\ \Pi_4(2)\ \Pi_5(2)\,]$

It turns out that

$$\pi(2) = \left[\frac{43}{200} \quad \frac{17}{40} \quad 0 \quad \underbrace{\frac{117}{500}}_{\pi_4(2)} \quad \frac{63}{500} \right]$$

and therefore

$$P(X(3)=5 \mid X(2) \neq 5) = \frac{7}{10} \cdot \frac{117}{500} = \frac{819}{5000} \simeq 0.1638$$

Using model #2, the answer is:

$$P(X(4)=5, X(3)\neq 5) = P(X(4)=5, X(3)=4) = P(X(4)=5 \mid X(3)=4)\, P(X(3)=4)$$

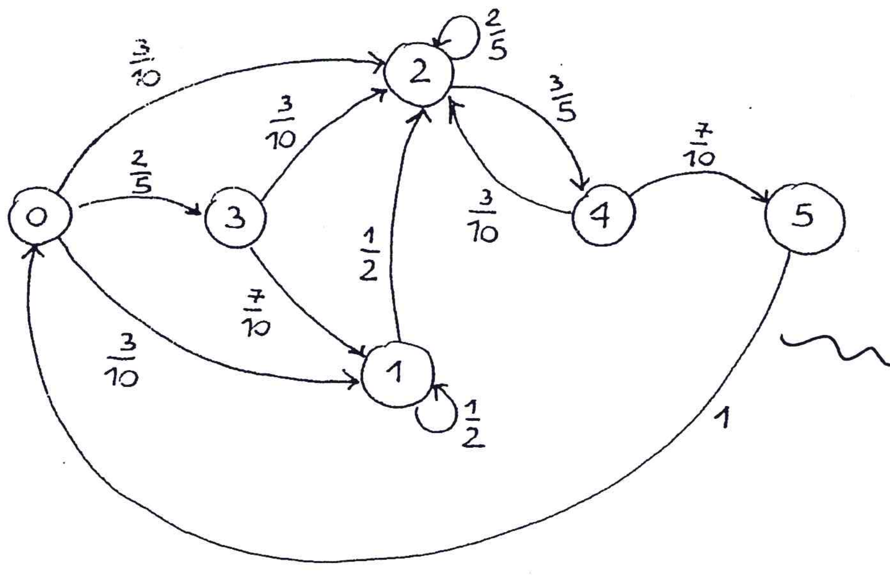
$$= p_{4,5} \cdot \pi_4(3)$$

$$= \frac{7}{10} \cdot \frac{117}{500} = \frac{819}{5000}$$

Note that, for model #2,
time t = number of generated numbers

Of course, the results obtained using the two models are equal.

3. We modify model #2 as follows:



We add a fictitious deterministic transition from state 5 to state 0.

In this way, the average length of the sequence generated to open the safe (denote it $E[N]$) is equal to the average recurrence time of state 5 ($M_5 = E[T_{5,5}]$) minus 1:

$$E[N] = E[T_{5,5}] - 1$$

Since the modified Markov chain is irreducible, aperiodic and finite, we know that we can compute M_5 as:

$$M_5 = \frac{1}{\pi_5}$$

where π_5 is the 6th element of the stationary state probability vector π obtained by solving:

$$\begin{cases} \pi = \pi \tilde{P} \\ \sum_{i=0}^{5} \pi_i = 1 \end{cases} \quad \text{where} \quad \tilde{P} = \begin{bmatrix} 0 & \frac{3}{10} & \frac{3}{10} & \frac{2}{5} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 \\ 0 & \frac{7}{10} & \frac{3}{10} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{10} & 0 & 0 & \frac{7}{10} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow !$$

$$\Rightarrow \pi = \left[\frac{525}{3869} \quad \frac{609}{3869} \quad \frac{1250}{3869} \quad \frac{210}{3869} \quad \frac{271}{1338} \quad \frac{525}{3869} \right]$$

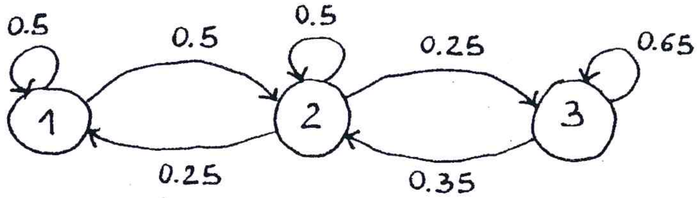
π_5

$$\Rightarrow E[N] = \frac{3869}{525} - 1 = \frac{3344}{525} \simeq 6.3695$$

EXERCISE #2

1. model #1

$$\text{state } x = \begin{cases} 1: \text{strong} \\ 2: \text{average} \\ 3: \text{weak} \end{cases}$$



$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.35 & 0.65 \end{bmatrix}$$

The Markov chain is irreducible, aperiodic and finite. This implies that stationary state probabilities can be computed by solving the set of linear equations:

$$\begin{cases} \pi = \pi P \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \pi \simeq [\underset{\pi_1}{0.2258} \quad \underset{\pi_2}{0.4516} \quad \underset{\pi_3}{0.3226}]$$

The answer to question #1 is $\pi_1 \simeq 0.2258$.

2. The current state is $X(t) = 1$.

We are asked to compute

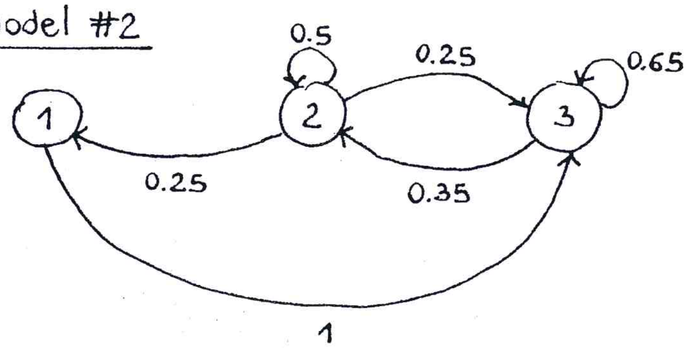
$$P(V(1) \geq 3) = 1 - P(V(1)=1) - P(V(1)=2)$$

state holding time

$$= 1 - (1 - p_{1,1}) - p_{1,1}(1 - p_{1,1}) = p_{1,1}^2 = 0.25$$

3. To answer question #3, we modify model #1 as follows:

model #2



$$\tilde{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.35 & 0.65 \end{bmatrix}$$

Using model #2, we can write

$$\underbrace{T_{1,1}}_{\text{recurrence time of state 1}} = 1 + \underbrace{T_{3,1}}_{\text{time to reach state 1 from state 3}}$$

Taking expectations of both sides, we have:

$$\underbrace{E[T_{3,1}]}_{\text{answer to question \#2}} = \underbrace{E[T_{1,1}]}_{} - 1$$

Since the Markov chain is irreducible, aperiodic and finite, $E[T_{1,1}] = \dfrac{1}{\tilde{\pi}_1}$, where $\tilde{\pi}_1$ is the stationary probability of state 1.

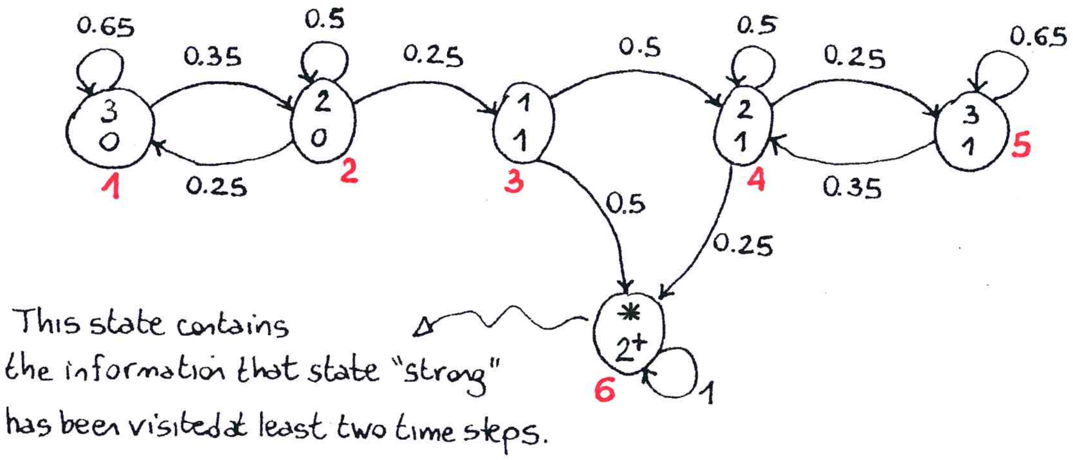
$$\begin{cases} \tilde{\pi} = \tilde{\pi}\tilde{P} \\ \tilde{\pi}_1 + \tilde{\pi}_2 + \tilde{\pi}_3 = 1 \end{cases} \Rightarrow \tilde{\pi} \simeq [\underbrace{0.0933}_{\tilde{\pi}_1} \quad \underbrace{0.3733}_{\tilde{\pi}_2} \quad \underbrace{0.5333}_{\tilde{\pi}_3}]$$

$$\Rightarrow E[T_{3,1}] = \frac{1}{\tilde{\pi}_1} - 1 \simeq 9.7143$$

4. model #3

Now the state must take into account the number of steps spent as "strong".

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{matrix} \to \text{current strength of the basketball team} \in \{1, 2, 3\} \\ \to \text{number of steps spent as "strong"} \in \{0, 1, \underbrace{2^+}_{\text{2 or more}}\} \end{matrix}$$



· Enumerate the states of model #3 from 1 to 6 as reported in the figure. Then:

$$\hat{P} = \begin{bmatrix} 0.65 & 0.35 & 0 & 0 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.25 & 0.25 \\ 0 & 0 & 0 & 0.35 & 0.65 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The answer to question #3 is:

$$P(\hat{X}(10)=6) = \underbrace{[1\ 0\ 0\ 0\ 0\ 0]}_{\hat{\pi}(0)} \hat{P}^{10} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \simeq 0.4593$$

we start from the state where the team is weak and was never strong in the past.

Note that $\hat{\pi}(10) = \hat{\pi}(0)\hat{P}^{10}$ is a row vector and we need the last component $\hat{\pi}_6(10)$. This is obtained by multiplying $\hat{\pi}(10)$ on the right by a suitable vector.