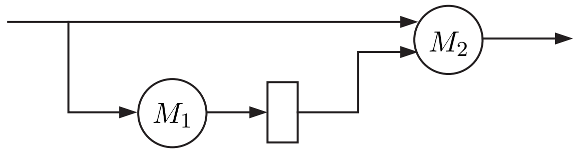
Test of Discrete Event Systems - 14.12.2020

Exercise 1

Consider the queueing network in the figure.



Arriving parts require preprocessing in M_1 with probability $p = 1/3$, otherwise they are routed directly to M_2. When a part arrives and the corresponding machine is unavailable, the part is rejected. There is a one-place buffer between M_1 and M_2. When M_1 terminates preprocessing of a part and M_2 is busy, the part is moved to the buffer, if the buffer is empty. Otherwise, the part is kept by M_1, that therefore remains unavailable for a new job until M_2 terminates the ongoing job. Parts arrive according to a Poisson process with expected interarrival time equal to 5 min, whereas service times in M_1 and M_2 follow exponential distributions with rates $\mu_1 = 0.5$ services/min and $\mu_2 = 0.8$ services/min, respectively.

1. Compute the expected number of parts in the system at steady state.
2. Verify the condition $\lambda_{eff} = \mu_{eff}$ for the whole system at steady state.
3. Compute the expected time spent by a part in M_1 at steady state.
4. Compute the utilization of M_1 and M_2 at steady state.
5. Compute the blocking probability of the system at steady state for those parts requiring preprocessing in M_1.

Exercise 1

STEP 1: stochastic timed automaton

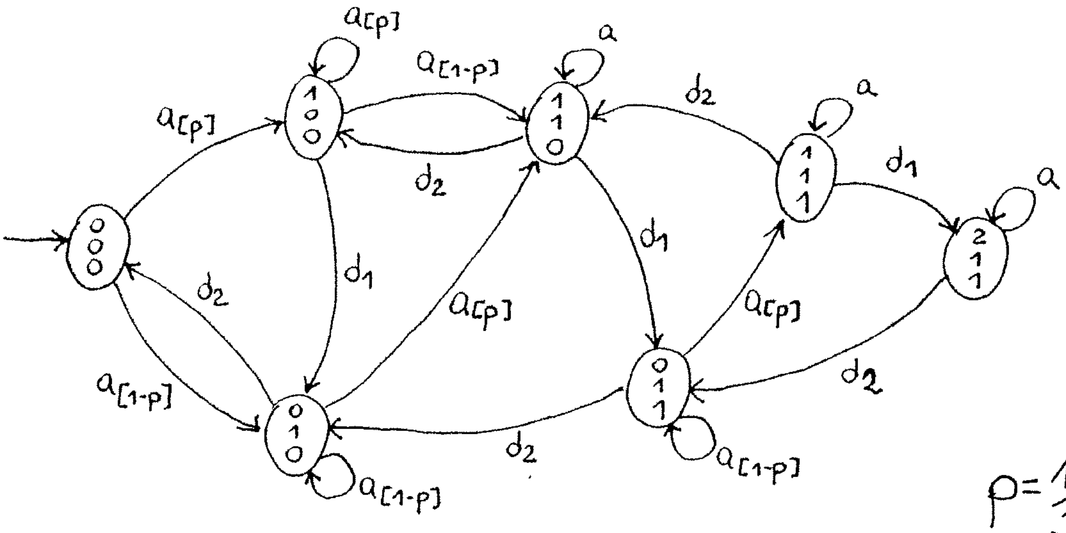
Definition of state:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{array}{l} \rightarrow M_1: \text{idle } (0), \text{ working } (1), \text{ blocked } (2) \\ \rightarrow M_2: \text{idle } (0), \text{ working } (1) \\ \rightarrow \text{buffer: empty } (0), \text{ full } (1) \end{array}$$

State space: $\mathcal{X} = \left\{ \begin{bmatrix} 0\\0\\0 \end{bmatrix}, \begin{bmatrix} 1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \begin{bmatrix} 1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1 \end{bmatrix}, \begin{bmatrix} 1\\1\\1 \end{bmatrix}, \begin{bmatrix} 2\\1\\1 \end{bmatrix} \right\}$ 7 states

Events: $\mathcal{E} = \{a, d_1, d_2\}$

- a: arrival of a new part
- d_1: termination of a job in M_1
- d_2: termination of a job in M_2



$p = \frac{1}{3}$

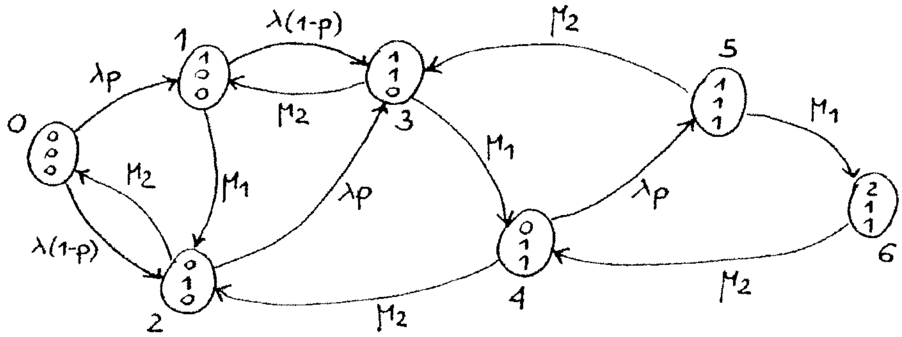
$F_a(t) = 1 - e^{-\lambda t}, \ t \geq 0$ where $\frac{1}{\lambda} = 5$ minutes $\Rightarrow \lambda = \frac{1}{5}$ arrivals/min

$F_{d_1}(t) = 1 - e^{-\mu_1 t}, \ t \geq 0$ where $\mu_1 = \frac{1}{2}$ services/min

$F_{d_2}(t) = 1 - e^{-\mu_2 t}, \ t \geq 0$ where $\mu_2 = \frac{4}{5}$ services/min

STEP 2: equivalent continuous-time homogeneous Markov chain

(possible because the stochastic clock structure is a Poisson one)



$$Q=\begin{bmatrix} -\lambda & \lambda p & \lambda(1-p) & 0 & 0 & 0 & 0 \\ 0 & -[\lambda(1-p)+\mu_1] & \mu_1 & \lambda(1-p) & 0 & 0 & 0 \\ \mu_2 & 0 & -(\lambda p+\mu_2) & \lambda p & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & -(\mu_1+\mu_2) & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & -(\lambda p+\mu_2) & \lambda p & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & -(\mu_1+\mu_2) & \mu_1 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & -\mu_2 \end{bmatrix}$$

STEP 3: computation of stationary state probabilities

The Markov chain is irreducible and finite; the stationary probabilities can be computed by solving:

$$\begin{cases} \pi Q = 0 \\ \sum \pi_i = 1 \end{cases}$$

However, using Matlab, it is more convenient to approximate $\lim_{t\to\infty} \pi_0 e^{Qt}$ by evaluating $\pi_0 e^{Qt}$ with t very large:

```
>> lambda = 1/5;
>> mu1 = 1/2;
>> mu2 = 4/5;
>> p = 1/3;
>> Q = [ -lambda lambda*p lambda*(1-p) 0 0 0 0 ; ...
0 -(lambda*(1-p)+mu1) mu1 lambda*(1-p) 0 0 0 ; ...
mu2 0 -(lambda*p+mu2) lambda*p 0 0 0 ; ...
0 mu2 0 -(mu1+mu2) mu1 0 0 ; ...
0 0 mu2 0 -(lambda*p+mu2) lambda*p 0 ; ...
0 0 0 mu2 0 -(mu1+mu2) mu1 ; ...
0 0 0 0 mu2 0 -mu2 ];
>> pi0 = [ 1 0 0 0 0 0 0 ];
>> T = 1e6; % take it very large
>> pi = pi0*expm(Q*T)

pi =

    0.6964    0.0977    0.1741    0.0193    0.0115    0.0006    0.0004
```

π_0	π_1	π_2	π_3	π_4	π_5	π_6
0.6964	0.0977	0.1741	0.0193	0.0115	0.0006	0.0004

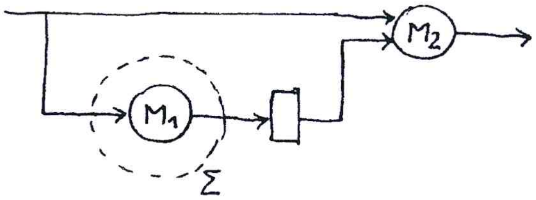
1. $E[X] = 0 \cdot \pi_0 + 1 \cdot (\pi_1 + \pi_2) + 2 \cdot (\pi_3 + \pi_4) + 3 \cdot (\pi_5 + \pi_6) \simeq 0.3363$

number of parts in the system at steady state
$X \in \{0,1,2,3\}$

2. $\lambda_{eff} = \lambda p (\pi_0 + \pi_2 + \pi_4) + \lambda (1-p)(\pi_0 + \pi_1) \simeq 0.1647$

$\mu_{eff} = \mu_2 (\pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6) \simeq 0.1647$

3. Consider a closed curve surrounding M_1 only:



$\Rightarrow$

$\lambda_\Sigma = \lambda p (\pi_0 + \pi_2 + \pi_4) \simeq 0.0588$

$E[X_\Sigma] = 0 \cdot (\pi_0 + \pi_2 + \pi_4) + 1 (\pi_1 + \pi_3 + \pi_5 + \pi_6) \simeq 0.1180$

number of parts in Σ at steady state
$X_\Sigma \in \{0,1\}$

and apply the Little's law to Σ:

$$E[S_\Sigma] = \frac{E[X_\Sigma]}{\lambda_\Sigma} \simeq 2.0063$$

time spent by a part in M_1 at steady state

Notice that $E[S_\Sigma] > \frac{1}{\mu_1} = 2.0$.
Indeed, the time spent by a part in M_1 may include also the waiting time that the buffer is empty.

4. $U_1 = \pi_1 + \pi_3 + \pi_5 \simeq 0.1176$ $\qquad U_2 = \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 \simeq 0.2059$

utilization of M_1 at steady state

utilization of M_2 at steady state

5. $P_{B,preproc} = \pi_1 + \pi_3 + \pi_5 + \pi_6 \simeq 0.1180$

blocking probability at steady state for those parts requiring preprocessing in M_1

We apply the PASTA property